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# Permutations Containing and Avoiding 123 and 132 Patterns

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We prove that the number of permutations which avoid 132-patterns and have exactly one 123-pattern, equals  $(n - 2)2^{n-3}$ , for  $n \geq 3$ . We then give a bijection onto the set of permutations which avoid 123-patterns and have exactly one 132-pattern. Finally, we show that the number of permutations which contain exactly one 123-pattern and exactly one 132-pattern is  $(n - 3)(n - 4)2^{n-5}$ , for  $n \geq 5$ .

**Keywords:** Patterns, Words

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## 1 Introduction

In 1990, Herb Wilf asked the following: How many permutations of length  $n$  avoid a given pattern,  $p$ ? By pattern-avoiding we mean the following: Let  $\pi$  be a permutation of length  $n$  and let  $p = (p_1, p_2, \dots, p_k)$  be a permutation of length  $k \leq n$  (we will call this a pattern of length  $k$ ). Let  $J$  be a set of  $r$  integers, and let  $j \in J$ . Define  $place(j, J)$  to be 1 if  $j$  is the smallest element in  $J$ , 2 if it is the second smallest, ..., and  $r$  if it is the largest. The permutation  $\pi$  avoids the pattern  $p$  if and only if there does not exist a set of indices  $I = (i_1, i_2, \dots, i_k)$ , such that  $p = (place(\pi(i_1), I), place(\pi(i_2), I), \dots, place(\pi(i_k), I))$ .

In two beautiful papers ([B1] and [N]), the number of subsequences containing exactly one 132-pattern and exactly one 123-pattern are enumerated. Noonan shows in [N] that the number of permutations containing exactly one 123-pattern is the simple formula  $\frac{3}{n} \binom{2n}{n+3}$ . Bóna proves that the even simpler formula  $\binom{2n-3}{n-3}$  enumerates the number of permutations containing exactly one 132-pattern. Bóna's result proved a conjecture first made by Noonan and Zeilberger in [NZ].

Noonan and Zeilberger considered in [NZ] the number of permutations of length  $n$  which contain exactly  $r$  p-patterns, for  $r \geq 1$ . Bóna, in [B2], made further progress concerning the number of permutations with exactly  $r$  132-patterns. In this article we work towards the following generalization: How many permutations of length  $n$  avoid patterns  $p_i$ , for  $i \geq 0$ , and contain  $r_j$   $p_j$ -patterns, for  $j \geq 1$ ,  $r_j \geq 1$ ? We will first consider the permutations of length  $n$  which avoid 132-patterns, but contain exactly one 123-pattern. We then define a natural bijection between these permutations and the permutations of length  $n$  which avoid 123-patterns, but contain exactly one 132-pattern. Finally, we will calculate the number of permutations which contain one 123-pattern and one 132-pattern. These results address questions first raised in [NZ].

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## 2 Known Results

For completeness, two results which are already known are given below.

**Lemma 1:** *The number of permutations of length  $n$  with one 12-pattern is  $n - 1$ .*

**Proof:** Induct on  $n$ . The base case is trivial. A permutation,  $\phi$ , of length  $n$  with one 12-pattern must have  $n = \phi(1)$  or  $n = \phi(2)$ . If  $n = \phi(1)$ , by induction we get  $n - 2$  permutation. If  $n = \phi(2)$ , then we must have  $n - 1 = \phi(1)$  (or we would have more than one 12-pattern). The rest of the entries of  $\phi$  must be decreasing. Hence we get 1 more permutation from this second case, for a total of  $n - 1$ .

**Lemma 2:** *The number of permutations which avoid both the pattern 123 and 132 is  $2^{n-1}$ .*

**Proof:** Let  $f_n$  denote the number of permutations we are interested in. Then  $f_n = \sum_{i=1}^n f_{n-i}$  with  $f_0 = 1$ . To see this, let  $\rho$  be a permutation of length  $n - 1$ . Insert the element  $n$  into the  $i^{\text{th}}$  position of  $\rho$ . Let  $\pi$  be this new permutation of length  $n$ . To assure that  $\pi$  avoids the 132-pattern, we must have all entries preceding  $n$  in  $\pi$  be larger than the entries following  $n$ . To assure that  $\pi$  avoids the 123-pattern, the entries preceding  $n$  must be in decreasing order. This argument gives the sum in the recursion. The recursion holds by noting that if  $n = 1$ , there is one permutation which avoids both patterns. To complete the proof note that  $f_n = 2^{n-1}$ .

## 3 One 123-pattern, but no 132-pattern

**Theorem 1:** *The number of permutations of length  $n$  which have exactly one 123-pattern, and avoid the 132-pattern is  $(n - 2)2^{n-3}$ .*

**Proof:** Call a permutation *good* if it has exactly one 123-pattern and avoids the 132-pattern, and let  $g_n$  denote the number of good permutations of length  $n$ . Let  $\gamma$  be a permutation of length  $n - 1$ . Insert the element  $n$  into the  $i^{\text{th}}$  position of  $\gamma$ . Call this newly constructed permutation of length  $n$ ,  $\pi$ . To assure that  $\pi$  avoids the 132 pattern, we must have all elements preceding  $n$  in  $\pi$  be larger than the elements following  $n$  in  $\pi$ . For  $\pi$  to be a good permutation, we must consider two disjoint cases.

**Case I:** The pattern 123 appears in the elements following  $n$  in  $\pi$ . This forces the elements preceding  $n$  to be in decreasing order. Summing over  $i$ , this case accounts for  $\sum_{i=1}^n g_{n-i}$  permutations.

**Case II:** The pattern 123 appears in the elements preceding and including  $n$  in  $\pi$ . This forces the 3 in the pattern to be  $n$ . Hence the elements preceding  $n$  must contain exactly one 12-pattern. (Further there must be at least 2 elements. Hence  $i$  must be at least 3). From Lemma 1, this number is  $i - 2$ . We are also forced to avoid both patterns in the elements following  $n$ . Lemma 2 implies that there are  $2^{n-i-1}$  such permutations. Summing over  $i$ , this case accounts for  $\sum_{i=3}^{n-1} (i - 2)2^{n-i-1} + n - 2$  permutations.

We have established that the recurrence relation

$$g_n = \sum_{i=1}^n g_{n-i} + \sum_{i=3}^{n-1} (i - 2)2^{n-i-1} + n - 2,$$

which holds for  $n \geq 3$  ( $g_0 = 0, g_1 = 0, g_2 = 0$ ), enumerates the permutations of length  $n$  which avoid the pattern 132 and contain exactly one 123-pattern.

One easy way to proceed would be to find the generating function of  $g_n$ . However, in this article we would like to employ a different, and in many circumstances more powerful, tool. We will use the Maple procedure `findrec` in Doron Zeilberger's Maple package `EKHAD`<sup>†</sup>. (The Maple shareware

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<sup>†</sup> Available for download at [www.math.temple.edu/~zeilberg/](http://www.math.temple.edu/~zeilberg/)

package `gfun` could have also been used.) Instructions for its use are available online. To use `find-rec` we compute the first few terms of  $g_n$ . These are (for  $n \geq 4$ ) 4, 12, 32, 80, 192, 448, 1024. We type `findrec([4,12,32,80,192,448,1024],0,2,n,N)` and are given the recurrence  $h_n = 4(h_{n-1} - h_{n-2})$  for  $n \geq 4$ . Define  $h_0 = 0, h_1 = 0, h_2 = 0$ , and  $h_3 = 1$ , and it is routine to verify that  $g_n = h_n$  for  $n \geq 0$ . Another routine calculation shows us that  $h_n = (n-2)2^{n-3}$  for  $n \geq 3$ , thereby proving the statement of the theorem.

## 4 One 132-pattern, but no 123-pattern

**Theorem 2:** *The number of permutations of length  $n$  which have exactly one 132-pattern, and avoid the 123-pattern is  $(n-2)2^{n-3}$ .*

**Proof:** We prove this by exhibiting a (natural) bijection from the permutations counted in Theorem 1 to the permutations counted in this theorem. Define  $S := \{\pi : \pi \text{ avoids 132-pattern and contains one 123-pattern}\}$  and  $T := \{\pi : \pi \text{ avoids 123-pattern and contains one 132-pattern}\}$ . We will show that  $|S| = |T|$ , by using the following bijection:

Let  $\phi : S \rightarrow T$ . Let  $s \in S$ , and let  $abc$  be the 123-pattern in  $s$ . Then  $\phi$  acts on the elements of  $s$  as follows:  $\phi(x) = x$  if  $x \notin \{b, c\}$ ,  $\phi(b) = c$ , and  $\phi(c) = b$ . In other words, all elements keep their positions except  $b$  and  $c$  switch places. An easy examination of several cases shows that this is a bijection, thereby proving the theorem.

## 5 One 132-pattern and one 123-pattern

**Theorem 3:** *The number of permutations of length  $n$  which have exactly one 132-pattern and one 123-pattern is  $(n-3)(n-4)2^{n-5}$ .*

**Proof:** We use the same insertion technique as in the proof of Theorem 1. Call a permutation *good* if it has exactly one 123-pattern and exactly one 132-pattern and let  $g_n$  denote the number of good permutations of length  $n$ . Let  $\gamma$  be a permutation of length  $n-1$ . Insert the element  $n$  into the  $i^{\text{th}}$  position of  $\gamma$ . Let  $\pi$  be this newly constructed permutation of length  $n$ . We note that the 132-pattern cannot consist of elements only preceding  $n$ . If this were the case, we would have two 123-patterns ending with  $n$ . For  $\pi$  to be a good permutation, we must consider the following disjoint cases.

**Case I:** The 132-pattern consists of elements following  $n$ . In this case all elements preceding  $n$  must be larger than the elements following  $n$ .

*Subcase A:* The 123-pattern consists of elements following  $n$ . Summing over  $i$  we get  $\sum_{i=1}^n g_{n-i}$  good permutations in this subcase.

*Subcase B:* The elements preceding  $n$  have exactly one 12-pattern. This gives a 123-pattern where the 3 in the pattern is  $n$ . We must also avoid the 123-pattern in the elements following  $n$ . Summing over  $i$  and using Lemma 1 and Theorem 1, we get  $\sum_{i=3}^{n-3} (i-2)(n-i-3)2^{n-i-2}$  good permutations in this subcase.

**Case II:** The 132-pattern has the first element preceding  $n$ , the last element following  $n$ , and  $n$  as the middle element. The elements preceding  $n$  must be  $n-1, n-2, \dots, n-1+2, n-i$ , where  $n-i$  immediately precedes  $n$  in  $\pi$ . See [B1] for a more detailed argument as to why this must be true.

*Subcase A:* The elements preceding  $n$  have exactly one 12-pattern. This gives a 123-pattern where the last element of the pattern is  $n$ . We must also avoid both the 123 and the 132 pattern in the elements following

$n$ . Summing over  $i$  and using Lemma 1 and Lemma 2 we have  $\sum_{i=4}^{n-1} (i-3)2^{n-i-1}$  good permutations in this subcase.

*Subcase B:* The 123-pattern consists of elements following  $n$ . We must have the elements preceding  $n$  in  $\pi$  be decreasing to avoid another 123-pattern. Further, the elements following  $n$  must not contain a 132-pattern. Using Theorem 1 and summing over  $i$ , we get a total of  $\sum_{i=2}^{n-3} (n-i-2)2^{n-i-3}$  good permutations in this subcase.

In total, we find that the following recurrence enumerates the permutations of length  $n$  which contain exactly one 123-pattern and one 132-pattern.

$$g_n = \sum_{i=1}^n g_{n-i} + \sum_{i=1}^{n-4} (2i(n-i-4) + n-3)2^{n-i-4}$$

for  $n \geq 5$  and  $g_1 = g_2 = g_3 = g_4 = 0$ .

Using `findrec` again by typing `findrec([2,12,48,160,480,1344,3584],1,1,n,N)` (where the list is the first few terms of our recurrence for  $n \geq 5$ ) we get the recurrence  $f_{n+1} = \frac{2(n+2)}{n}f_n$ , with  $f_1 = 2$ . After reindexing, another routine calculation shows that  $f_n = g_n$ . Solving  $f_n$  for an explicit answer, we find that  $g_n = (n-3)(n-4)2^{n-5}$ .

We conjecture that the generating function for the number of permutations with exactly zero or exactly one 132-pattern and exactly  $r$  123-patterns is  $P(z)/(1-2z)^{r+1}$ , where  $P(z)$  is a polynomial. For more evidence, and further extensions see [RWZ].

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