P_4-Colorings and P_4-Bipartite Graphs

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P_4-Free Colorings and P_4-Bipartite Graphs

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A vertex partition of a graph into disjoint subsets V_is is said to be a P_4-free coloring if each color class V_i induces a subgraph without a chordless path on four vertices (denoted by P_4). Examples of P_4-free 2-colorable graphs (also called P_4-bipartite graphs) include parity graphs and graphs with "few" P_4s like P_4-reducible and P_4-sparse graphs. We prove that, given $k \geq 2$, P_4-FREE k-COLORABILITY is NP-complete even for comparability graphs, and for P_5-free graphs. We then discuss the recognition, perfection and the Strong Perfect Graph Conjecture (SPGC) for P_4-bipartite graphs with special P_4-structure. In particular, we show that the SPGC is true for P_4-bipartite graphs with one P_3-free color class meeting every P_4 at a midpoint.

Keywords: Perfect graph, the Strong Perfect Graph Conjectrue, graph partition, cograph, NP-completeness

1 Introduction

A graph G is *perfect* if, for each induced subgraph H of G, the chromatic number of H is equal to the clique number of H. Claude Berge introdued perfect graphs and conjectured around 1960's that a graph is perfect if and only if it has no induced cycle of odd length at least five or the complement of such a cycle. Nowadays this conjecture is known as the Strong Perfect Graph Conjecture (SPGC) and is still open. We refer to [4] for more information on perfect graphs.

A measure of a graph's imperfection has been considered by Brown and Corneil [8] as follows. Given a graph G and a positive integer k, a map $\pi : V(G) \to \{1, \ldots, k\}$ is a *perfect k-coloring* of G if the subgraphs induced by each color class $\pi^{-1}(i)$ is perfect. Thus, a graph is perfect if and only if it is perfect 1-colorable. Note also that, by the Perfect Graph Theorem [33], a graph G is perfect k-colorable if and only if its complement $\overline{G}$ is perfect k-colorable. In this paper we consider a particular example of perfect colorings. Our discussion is motivated by the fact that the perfection of a graph depends only on the structure of its induced paths on four vertices (denoted by P_4); see [36]. In this sense, graphs with empty P_4-structure (P_4-free graphs) form a somewhat based graph class in discussing graph's perfection; they are indeed perfect by a result due to Seinsche [38] (see also Jung [31]).

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Now, we call a perfect k-coloring of a graph *P_4-free k-coloring* if the subgraphs of that graph induced by the color classes are P_4-free. Note that the P_4 is self-complementary, hence G is P_4-free k-colorable if and only if $\overline{G}$ is P_4-free k-colorable. For general graphs, Brown [6] proved that P_4-FREE k-COLORABILITY is NP-complete for $k \geq 3$, and in [1], Achlioptas proved a more general result implying the NP-completeness of P_4-FREE k-COLORABILITY for $k \geq 2$. In the next section we shall prove that, for any integer $k \geq 2$, P_4-FREE k-COLORABILITY is NP-complete even for (particular) perfect graphs, and for P_5-free graphs. In Section 3 we shall give some examples of P_4-free 2-colorable graphs, which we also call *P_4-bipartite graphs*. Many well understood classes of perfect graphs consists of P_4-bipartite graphs only. In Sections 4 and 5, perfect P_4-bipartite graphs and the SPGC for P_4-bipartite graphs with special P_4-structure will be discussed.

The complement of a graph G is denoted by $\overline{G}$. Graphs having no induced subgraphs isomorphic to a given graph H are called *H-free*. If X is a set of vertices in G, $G[X]$ is the subgraph of G induced by X, and $N_G(X)$ is the *neighborhood* of X in G; that is, the set of all vertices outside X adjacent to some vertex in X. If the context is clear, we simply write $N(X)$. The path on m vertices v_1, v_2, $\ldots$, v_m with edges v_iv_{i+1} $(1 \leq i < m)$ is denoted by $P_m = v_1v_2 \cdots v_m$. The vertices v_1 and v_m are the *endpoints* of that path, the other vertices are the *midpoints*. The cycle on m vertices v_1, v_2, $\ldots$, v_m with edges v_iv_{i+1} $(1 \leq i < m)$ and v_1v_m is denoted by $C_m = v_1v_2 \cdots v_m$. C_{2k+1} and $\overline{C_{2k+1}}$, $k \geq 2$, are also called *odd holes*, respectively, *odd antiholes*. Graphs without odd holes and odd antiholes are called *Berge graphs*.

2 NP-completeness results

We now consider the following problem for fixed positive integer k.

P_4-FREE k-COLORABILITY *Is a given graph P_4-free k-colorable?*

We show in this section that, for fixed $k \geq 2$, P_4-FREE k-COLORABILITY is NP-complete for perfect graphs. Notice that P_4-free 1-colorability (that is, recognizing P_4-free graphs) is solvable in linear time [14]. We shall reduce the following NP-complete problem ([37], see also [16]) to P_4-FREE k-COLORABILITY.

NOT-ALL-EQUAL 3SAT *Given a collection $\mathcal{C}$ of clauses over set V of Boolean variables such that each clause has exactly three literals. Is there a truth assigment for V such that each clause in $\mathcal{C}$ has at least one true literal and at least one false literal?*

A comparability graph G is one which admits a transitive orientation $\vec{G}$: If (x,y) and (y,z) are arcs of $\vec{G}$, then (x,z) is also an arc of $\vec{G}$. It is well known that comparability graphs are perfect. A typical example of comparability graphs are P_4-free graphs, as proved by Jung [31].

Lemma 1 *Given a comparability graph G, it is NP-complete to decide whether G is P_4-bipartite.*

Proof. The problem is clearly in NP. We shall reduce NOT-ALL-EQUAL 3SAT to our problem. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be any set of clauses $C_i = (c_{i1}, c_{i2}, c_{i3})$ given as input for NOT-ALL-EQUAL 3SAT, where the literals c_{ik} $(1 \leq i \leq m, 1 \leq k \leq 3)$ are taken from the set of variables V. We shall construct a comparability graph G which has a partition into two P_4-free graphs if and only if $\mathcal{C}$ is satisfiable. For convenience, we call a vertex partition of a graph into two P_4-free graphs a *good partition* of that graph. For each variable $v \in V$ let $G(v, \overline{v})$ be the graph shown in Figure 1 (left).

Observation 1 *$G(v, \overline{v})$ has a good partition. Every good partition of $G(v, \overline{v})$ must contain the labelled vertex v in one part and the labelled vertex $\overline{v}$ in the other part.* $\diamond$

For each clause C_i, let $G(C_i)$ be the graph shown in Figure 1 (right).

Observation 2 *$G(C_i)$ has a good partition. Every good partition of $G(C_i)$ must contain two of the labelled vertices c_{i1}, c_{i2}, c_{i3} in one part and the other labelled vertex in the other part. Moreover, every partition of $\{c_{i1}, c_{i2}, c_{i3}\}$ into two non-empty subsets can be extended to a good partition of $G(C_i)$.* $\diamond$

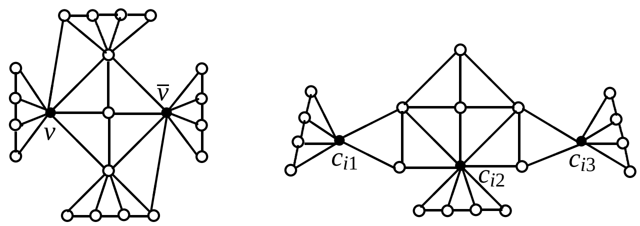

Fig. 1: The graphs $G(v,\overline{v})$ (left) and $G(C_i)$ (right)

The proofs of the observations will follow by inspection, hence are omitted. We now create the graph $G = G(\mathcal{C})$ from the graphs $G(v,\overline{v})$ $(v \in V)$ and the graphs $G(C_i)$ $(1 \le i \le m)$ as follows: For each $v \in V$ and each $1 \le i \le m$, connect the vertex $x \in \{v,\overline{v}\}$ in $G(v,\overline{v})$ with the vertex c_{ik} in $G(C_i)$ by an edge if, and only if, x is the literal c_{ik} in the clause C_i. Thus, in G, every c_{ik} $(1 \le k \le 3)$ has exactly one neighbor outside $G(C_i)$ which is one of the labelled vertices $v,\overline{v}$ in a graph $G(v,\overline{v})$ (with $c_{ik} \in \{v,\overline{v}\}$ in the given NOT-ALL-EQUAL 3SAT instance).

Suppose that G has a good partition into two P_4-free graphs A and B. Then it is easy to see that, for all $v \in V$, if $x \in \{v,\overline{v}\}$ is adjacent to c_{ik}, then x and c_{ik} are in different parts A, B. We define a truth assigment for NOT-ALL-EQUAL 3SAT as follows:

v is true if and only if the labelled vertex v in $G(v,\overline{v})$ belongs to A.

By Observation 1, this assignment is well-defined. By Observation 2, it is clear that each clause C_i has at least one but not all true literals.

Conversely, suppose that there is a truth assigment satisfying NOT-ALL-EQUAL 3SAT. Then let $A(v,\overline{v})$, $B(v,\overline{v})$ be a good partition of $G(v,\overline{v})$ such that $A(v,\overline{v})$ contains the true vertex in $\{v,\overline{v}\}$ and $B(v,\overline{v})$ contains the false vertex of them. Such a good partition exists by Observation 1. Let A_i, B_i be a good partition of $G(C_i)$ such that A_i contains the false literals vertices in $\{c_{i1}, c_{i2}, c_{i3}\}$ and B_i contains the true vertices of them. Such a good partition exists by Observation 2, and the fact that every C_i has at least one but not all true literals. Set

$$A = \bigcup_{v \in V} A(v,\overline{v}) \cup \bigcup_{1 \le i \le m} A_i, \qquad B = \bigcup_{v \in V} B(v,\overline{v}) \cup \bigcup_{1 \le i \le m} B_i.$$

Clearly, $V(G) = A \cup B$. Now, each $A(v,\overline{v})$ and each A_i is a P_4-free graph, and no edge exists between two parts of the $A(v,\overline{v})$'s and A_i's, hence A is a P_4-free subgraph of G. similarly, B is P_4-free. Thus, G is P_4-bipartite.

To complete the proof, note that each $G(v,\overline{v})$ and each $G(C_i)$ admits a transitive orientation such that the labelled vertices $v,\overline{v}$ are sinks and the labelled vertices c_{i1}, c_{i2}, c_{i3} are sources. To obtain a transitive orientation of G, direct the edges xy, $x \in \{v,\overline{v}\}$ and $y \in \{c_{i1}, c_{i2}, c_{i3}\}$ with $x = y$ in the given instance of NOT-ALL-EQUAL 3SAT, from y to x. $\square$

Theorem 1 *Given a comparability graph G and an integer $k \geq 2$, it is NP-complete to decide whether G is P_4-free k-colorable.*

Proof. The case $k = 2$ is settled by Lemma 1. We shall make use of a construction for vertex-critical P_4-free k-colorable graphs in [7] to reduce the case $k = 2$ to the case $k \geq 3$. Let H be a comparability graph, and let G be the graph obtained from an induced P_4 by substituting three (arbitrary) vertices by the graph H. Then G is clearly a comparability graph, and it can easily be seen that G is P_4-free k-colorable if and only if H is P_4-free $(k-1)$-colorable. □

We shall remark that Brown [6] and Achlioptas [1] showed the NP-completeness of P_4-FREE k-COLORABILITY for fixed $k \geq 3$ by reducing k-COLORABILITY to P_4-FREE k-COLORABILITY. Since k-COLORABILITY can be decided in polynomial time when considering perfect graphs (see [17]), Brown's and Achlioptas's reduction cannot be used in proving NP-completeness of P_4-FREE k-COLORABILITY for perfect graphs.

Since a graph is P_4-free k-colorable if and only if its complement is, P_4-FREE k-COLORABILITY is NP-complete for cocomparability graphs as well. Graphs which are both comparability graphs and cocomparability graphs are called *permutation graphs*. We do not know the complexity of P_4-FREE COLORABILITY on permutation graphs.

Problem 1 *Find a polynomial time algorithm for solving P_4-FREE k-COLORABILITY on permutation graphs, or prove that the problem is NP-complete for the class of permutation graphs.*

Notice that, using the construction mentioned in the proof of Theorem 1, one can show that for every fixed $k \geq 1$ there are P_4-free k-colorable permutation graphs which are not P_4-free $(k-1)$-colorable.

We now are going to show that P_4-FREE k-COLORABILITY is NP-complete for (C_4, C_5)-free graphs. As a consequence, P_4-FREE k-COLORABILITY is also NP-complete for P_5-free graphs. This is best possible in the sense that the problem is trivial for P_4-free graphs.

Lemma 2 *Given a (C_4, C_5)-free graph G, it is NP-complete to decide whether G is P_4-bipartite.*

Proof. We shall reduce NOT-ALL-EQUAL 4SAT to our problem (the NP-completeness of NOT-ALL-EQUAL 4SAT follows easily from that of NOT-ALL-EQUAL 3SAT). Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be any set of clauses $C_i = (c_{i1}, c_{i2}, c_{i3}, c_{i4})$ given as input for NOT-ALL-EQUAL 4SAT, where the literals c_{ik} $(1 \leq i \leq m, 1 \leq k \leq 4)$ are taken from the set of variables V. We may assume that,

$$\text{for every } v \in V, \text{ no clause } C_i \text{ contains both } v \text{ and } \overline{v}. \tag{1}$$

We now construct a (C_4, C_5)-free graph G which has a partition into two P_4-free graphs if and only if $\mathcal{C}$ is satisfiable. For each variable $v \in V$ let $G(v, \overline{v})$ be the graph shown in Figure 2 (left). For each clause C_i, let $G(C_i)$ be the P_4 shown in Figure 2 (right). We create the graph $G = G(\mathcal{C})$ from the graphs $G(v, \overline{v})$ $(v \in V)$ and the graphs $G(C_i)$ $(1 \leq i \leq m)$ as follows: For each $v \in V$ and each $1 \leq i \leq m$, connect the vertex $x \in \{v, \overline{v}\}$ in $G(v, \overline{v})$ with the vertex c_{ik} in $G(C_i)$ by an edge if, and only if, x is the literal c_{ik} in the clause C_i. Clearly, the construction and assumption (1) guarantee that G cannot contain an induced C_4 or C_5.

Now, we can show, similar to Lemma 1, that G is P_4-bipartite if and only if $\mathcal{C}$ is satisfiable. □

Theorem 2 *Given a (C_4, C_5)-free graph G and an integer $k \geq 2$, it is NP-complete to decide whether G is P_4-free k-colorable.*

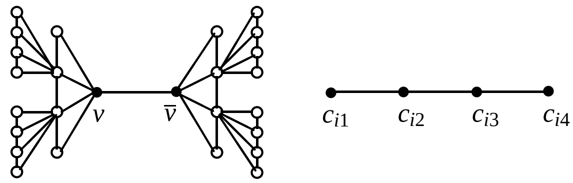

Fig. 2: The graphs $G(v,\overline{v})$ (left) and $G(C_i)$ (right)

Proof. The case $k = 2$ is settled by Lemma 2. Let $k \geq 3$. Let H be a (C_4, C_5)-free graph. Construct a graph G as follows: Take $k+2$ disjoint copies $G_1, \ldots, G_{k+2}$ of H and $k+2$ new vertices $v_1, \ldots, v_{k+2}$, and connect every pair v_i, v_j $(1 \leq i \neq j \leq k+2)$ by an edge and connect every vertex in G_i with v_i $(1 \leq i \leq k+2)$ by an edge. Clearly, G is also (C_4, C_5)-free.

Suppose that H is P_4-free k-colorable. Then G is P_4-free $(k+1)$-colorable by coloring the vertices v_i's with one new color.

Suppose, conversely, that G is P_4-free $(k+1)$-colorable. Then H is P_4-free k-colorable. If not, consider two distinct vertices $v_i, v_j \in \{v_1, \ldots, v_{k+2}\}$ with the same color c in a P_4-free $(k+1)$-coloring of G. Since H is not P_4-free k-colorable, the color c must appear in every copy of H. Say, for some $i \neq j$, $x \in G_i$ and $y \in G_j$ are colored by c. But then xv_iv_jy is a monochromatic P_4 in G, a contradiction. Thus, H must be P_4-free k-colorable, as claimed. $\square$

Since C_4-free graphs are $\overline{P_5}$-free, Theorem 2 implies that P_4-FREE k-COLORABILITY is NP-complete for $\overline{P_5}$-free graphs, and, by considering complementation, for P_5-free graphs as well. This is best possible in the sense that P_4-FREE k-COLORABILITY is trivial for P_4-free graphs.

Also, Theorem 2 implies that P_4-FREE k-COLORABILITY is NP-complete for $(C_5, \overline{C_4})$-free graphs as well. Notice that graphs which are both (C_5, C_4)-free and $(C_5, \overline{C_4})$-free, i.e., split graphs, are P_4-free 2-colorable.

3 Examples of P_4-bipartite graphs

P_4-bipartite graphs generalize in a very natural way the well understood bipartite graphs, split graphs and cographs. Below we are going to list other well structured (perfect) graph classes that contain P_4-bipartite graphs only. See [5] for a survey on these and related graph classes.

PROPER INTERVAL GRAPHS. Interval graphs without induced $K_{1,3}$ are called proper interval graphs. In [2], it was shown that every proper interval graph can be partitioned into two P_3-free subgraphs. In particular, proper interval graphs are P_4-bipartite. Notice that, for every k, there exists an interval graph that is P_4-free k-colorable, but not P_4-free $(k-1)$-colorable.

DISTANCE-HEREDITARY AND PARITY GRAPHS. Distance-hereditary graphs are those graphs in which for all vertices u, v, all induced paths connecting u and v have equal length [24]. In [9], Burlet and Uhry introduced the bigger class of parity graphs; these graphs are defined by the condition that all induced paths connecting u and v have equal parity. Let G be a parity graph, and let v be a vertex in G. In [9, Lemma 4] (see also [35]) it was shown that, for each i, the set $N^i(v)$ of vertices at distance exactly i from v induces a P_4-free subgraph in G. Thus, $\bigcup N^{2i}(v)$ and $\bigcup N^{2i+1}(v)$ is a P_4-free bipartition of G. We thank Stephan Olariu and Luitpold Babel for their hint to this fact on parity graphs.

In order to give other well known classes that consist of P_4-bipartite graphs only we need the term of p-connectedness introduced by Jamison and Olariu [30]. A graph is called *p-connected* if, for every partition of its vertex set into two nonempty, disjoint subsets, there is an induced P_4 with vertices in both parts. A *p-component* of a graph is a maximal p-connected subgraph of that graph. Clearly, a graph is a P_4-bipartite graph if and only if each of its p-components is a P_4-bipartite graph.

P_4-REDUCIBLE AND P_4-SPARSE GRAPHS. P_4-reducible graphs are those graphs in which each vertex belongs to at most one induced P_4 [26]. In [20], Hoàng introduced the bigger class of P_4-sparse graphs; these are defined by the condition that each set of at most five vertices induces at most one P_4. It was shown in [29] that every p-component of a P_4-sparse graph is a split graph. Since split graphs are P_4-bipartite, all P_4-sparse graphs are P_4-bipartite.

P_4-EXTENDIBLE AND P_4-LITE GRAPHS. P_4-extendible graphs [28] are those graphs in which each p-component has at most five vertices. P_4-lite graphs [27] are those graphs in which every induced subgraph with at most six vertices either has at most two P_4s or is a (special) split graph. It was shown in [3] that every p-component of a P_4-lite graph is a split graph or has at most six vertices. Notice that all graphs with at most six vertices are P_4-bipartite, hence P_4-lite and P_4-extendible graphs are P_4-bipartite.

COGRAPH CONTRACTIONS. In [25] Hujter and Tuza introduced the graphs called *cograph contractions*. These are graphs obtained from a cograph by contracting some pairwise disjoint stable sets and then making the 'contracted vertices' pairwise adjacent. It was shown in [32] that a graph is a cograph contraction if and only if it admits a clique meeting each P_4 in a midpint and each $\overline{P_5}$ in both endpoints of the P_5. In particular, cograph contractions are P_4-bipartite graphs.

Notice that the complements of the graphs mentioned above are also P_4-bipartite graphs.

4 Which P_4-bipartite graphs are perfect?

Let G be a graph whose vertices are colored red and white (each vertex receives only one color). A P_4 *abcd* of G is said to be of type

1 (or RRRR) if a,b,c,d are red,

2 (or WRRR) if a is white and b,c,d are red,

3 (or RWRR) if a,c,d are red and b is white,

4 (or RRWW) if a,b are red and c,d are white,

5 (or RWRW) if a,c are red and b,d are white,

6 (or RWWR) if a,d are red and b,c are white,

7 (or WRRW) if a,d are white and b,c are red,

8 (or RWWW) if a is red and b,c,d are white,

9 (or WRWW) if a,c,d are white and b is red,

10 (or WWWW) if a,b,c,d are white.

Clearly, G is P_4-bipartite if and only if its vertices can be colored red and white in such a way that no P_4 is of type 1 or 10. We also write $G = (R,W,E)$ for P_4-bipartite graph $G = (V,E)$ with partition $V = R \cup W$ such that $G[R]$ and $G[W]$ are P_4-free subgraphs in G.

For non-empty subset $S \subseteq \{2,3,\dots,9\}$, we call a graph G a *S-graph* if the vertices of G can be colored red and white such that every P_4 of G is of type $t \in S$. Thus S-graphs are P_4-bipartite. Bipartite graphs (respectively, complements of bipartite graphs) are, for instance, $\{5\}$-graphs (respectively, $\{4\}$-graphs).

Many classes of perfect P_4-bipartite graphs have been described in terms of types of P_4s. In [21], Hoàng proved that "odd P_4-bipartite graphs" are perfect; here the P_4-bipartite graph $G = (R,W,E)$ is *odd* if every P_4 of G has odd number of vertices in R (hence in W). Thus, odd P_4-bipartite graphs are exactly the $\{2,3,8,9\}$-graphs. Chvátal, Lenhart and Sbihi [13, Theorem 2], and independently Gurvich [19] extended odd P_4-bipartite graphs to a larger class of perfect P_4-bipartite graphs; they proved that all $\{2,3,4,5,8,9\}$-graphs are perfect. These results and more related results in [12, 13] motivate the following question:

> What are the maximal subsets $S \subset \{2,3,\dots,9\}$ with the property that all S-graphs are perfect?

We shall point out that the complete answer to this question already follows by the results in [12, 13].

Theorem 3 *Let S be a maximal subset of* $\{2,3,\dots,9\}$ *such that all S-graphs are perfect. Then S is exactly one of the follwing sets:* $S_1 = \{4,5,6,7\}, S_2 = \{2,3,4,5,8,9\}, S_3 = \{3,4,5,6,8\},$ *and* $S_4 = \{2,4,5,7,9\}$.

Proof. First, color the odd hole C_9 in the way RRWRRWRRW. Then every P_4 of this C_9 is of type 3 or 7, and every P_4 of the complement of this C_9 is of type 2 or 6. Second, color the odd hole C_9 in the way WWRWWRWWR. Then every P_4 of this C_9 is of type 6 or 9, and every P_4 of the complement of this C_9 is of type 7 or 8. Therefore, as odd holes and odd antiholes are imperfect,

> none of $\{3,7\}, \{2,6\}, \{6,9\}$ and $\{7,8\}$ is a subset of S.

Now, it is straightforward to show that S must be contained in one of the sets S_1, S_2, S_3, or S_4.

Finally, all S_1-graphs are perfect [12], all S_2-graphs are perfect [13, Theorem 2] (see also [19]), all S_3-graphs and all S_4-graphs are perfect [13, Theorem 6]. $\square$

We now turn to the recognition problem for P_4-bipartite graphs addressed in Theorem 3. Given a graph G, we consider the system of linear equations

$$w + x + y + z = 2 \qquad (w,x,y,z \text{ induce a } P_4 \text{ in } G).$$

It is easy to see the G is a S_1-graph if and only if this system of linear equations has a $0/1$-solution. Thus, S_1-graphs can be recognized in polynomial time. Also, S_3-graphs can be recognized in polynomial time; the task reduces to the 2SAT problem as follows.

> For each P_4 $wxyz$ in G, let $(x \vee y) \wedge (\overline{w} \vee \overline{z})$ be a Boolean formula.

The 2SAT formula for G is the product of such all formulas corresponding to the P_4s in G. Now, if G is a S_3-graph with a P_4-free coloring $V(G) = R \cup W$, then the truth assigment $v :=$ true $\Leftrightarrow v \in W$ satisfies our 2SAT formula. If, conversely, our 2SAT formula is satisfied, then $W := \{v : v \text{ is true}\}, R := \{v : v \text{ is false}\}$ is a P_4-free 2-coloring of G such that every P_4 of G is of type $t \in S_3$. Since a graph is a S_4-graph if and only if its complement is a S_3-graph, S_4-graphs can be recognized in polynomial time, too.

The recognition problem of S_2-graphs remains open; see also [10].

Problem 2 *Given a graph G. Can you find in polynomial time a P_4-free 2-coloring of G such that every P_4 of G is of type $t \in S_2$, or prove that such a coloring does not exist?*

We remark that it can be shown that Problem 2 is NP-complete if S_2 is replaced by $S_2 \cup \{6\}$, or replaced by $S_2 \cup \{7\}$.

5 P_4-bipartite graphs and the SPGC

The results in [21, 13, 19] mentioned in previous section will be implied by the truth of the following

Conjecture 1 *The SPGC is true for P_4-bipatite graphs.*

Conjecture 1 has been proved for some particular cases. The following theorem is a consequence of previously known results (see also [23]). It proves Conjecture 1 for P_4-free graphs with one color class being a stable set or a clique.

Theorem 4 *Let G have a stable set (or a clique) T such that T meets every P_4 of G. If G has no odd hole (respectively, no odd antihole), then G is perfect.* □

Also, in [23], Conjecture 1 is proved for P_4-bipartite graphs with one color class inducing a $(P_4, C_4, \overline{C_4})$-free graph and meeting every P_4 in certain way as follows:

Theorem 5 *Let G have a subset $T \subseteq V(G)$ such that*

(i) *T induces a threshold graph,*

(ii) *T meets every P_4 in an endpoint, or meets every P_4 in a midpoint.*

If G is Berge, then G is perfect. □

Therorem 4 suggests the following weaker conjecture for P_4-bipartite graphs with one color class consisting of vertex-disjoint cliques.

Conjecture 2 *The SPGC is true for P_4-bipatite graphs with one P_3-free color class.*

The main result of this section is the following theorem which is related to Therorem 5 and proves Conjecture 2 for the case when the P_3-free color class meets the P_4s in a certain way.

Theorem 6 *Let G have a subset $T \subseteq V(G)$ such that*

(i) *T induces a P_3-free graph,*

(ii) *T meets every P_4 in an endpoint, or meets every P_4 in a midpoint.*

If G is Berge, then G is perfect.

The proof of Theorem 6 relies on several known results on P_4-free graphs and minimal imperfect graphs. First, Seinsche [38] proved that

a P_4-free graph or its complement is disconnected. (2)

Two vertices x, y are *twins* if, for all other vertices z, z is adjacent to x if and only if z is adjacent to y. The next property of P_4-free graphs is well known and can be derived from (2).

Every P_4-free graph with at least two vertices has a pair of twins. (3)

A graph is *minimal imperfect* if it is not perfect but each of its proper induced subgraphs is. The well known Perfect Graph Theorem due to Lovász implies that

the complement of a minimal imperfect graph is also minimal imperfect. (4)

Two (nonadjacent) vertices x and y form an *even-pair* if every induced path connecting x to y has even length. Meyniel [34] showed that

no minimal imperfect graph has an even-pair. (5)

In particular, no minimal imperfect graph has a *two-pair* which is a pair of vertices x, y such that every induced path connecting x to y has exactly two edges.

A cutset S of G is called a *star-cutset*, respectively, a *stable-cutset*, respectively, a *complete multipartite-cutset* if $G[S]$ has a universal vertex, respectively, has no edge, respectively, is a complete multipartite graph (a complete multipartite graph is one whose vertex set can be partitioned into stable sets $S_1, \ldots, S_m$ such that, for $i \neq j$, every vertex in S_i is adjacent to every vertex in S_j). Chvátal [11] showed that

no minimal imperfect graph has a star-cutset. (6)

In particular,

no minimal imperfect graph has a clique-cutset, (7)

and

in a minimal imperfect graph, no vertex dominates another vertex. (8)

Here, the vertex x *dominates* the vertex y if $N(y) \subseteq N(x) \cup x$. The next property of minimal imperfect graphs was found by Tucker [39] saying that

no minimal imperfect graph has a stable-cutset, unless it is is an odd hole. (9)

Finally, Cornuéjols and Reed [15] showed that

no minimal imperfect graph has a complete multipartite-cutset. (10)

Proof of Theorem 6. Suppose that T meets every P_4 in an endpoint. Color the verices in T with color red and vertices outside T with color white. Then G has only P_4s of types 3, 4, 5, 6, or 8. In particular, G is an S_3-graph, hence perfect (see Theorem 3).

Let us consider the case when T meets every P_4 in a midpoint, and assume that G is a minimal imperfect Berge graph. Further, we may assume that

$G - T$ is disconnected.

Otherwise, by (2), $\overline{G} - T$ is disconnected and so T would be a stable-cutset or a complete multipartite-cutset of $\overline{G}$, contradicting (4) and (9) or (10). In particular, by (7),

T consists of $m \geq 2$ vertex-disjoint cliques.

For convenience, we call a P_4 *bad* if its both midpoints are outside T. By our hypothesis, no P_4 in G is bad.

CASE 1. *$G-T$ has two adjacent twins x,y.*
In this case, we claim that

x,y form an even-pair in $\overline{G}$.

To see this, consider an induced path $P = xx_1\cdots x_ky$, $k \geq 2$, in $\overline{G}$ connecting x and y. As x,y are twins in $\overline{G}-T$, x_1 must belong to T. Furthermore,

P has no edge in $\overline{G}[T]$.

For if P has an edge in $\overline{G}[T]$, then, since $\overline{G}[T]$ is a complete multipartite graph and $x_1 \in T$, this edge must be x_1x_2, and P is the P_4 xx_1x_2y. But then x_1yxx_2 is a bad P_4 in G, a contradiction.

P has no edge in $\overline{G}-T$.

Otherwise, let i be minimal such that x_ix_{i+1} is an edge in $\overline{G}-T$. Note that $i>1$. Set $x_0 := x$. Then $x_{i-1} \in T$ and $x_{i-2} \in \overline{G}-T$. But then $x_{i-1}x_{i+1}x_{i-2}x_i$ is a bad P_4 in G, a contradiction.

Thus, P has even number of edges, as claimed. This contradicts (4) and (5), and Case 1 is settled.

CASE 2. *$G-T$ has no adjacent twins.*
Write $G[T] = C_1 \cup C_2 \cup \cdots \cup C_m$ with vertex-disjoint cliques $C_1, C_2, \ldots, C_m$. Recall that $m \geq 2$.

Observation 3 *For all cliques $C = C_i$, $1 \leq i \leq m$, and all component H of $G-T$, if $N(C) \cap H \neq \emptyset$, then $H \subseteq N(C)$.*

Proof of Observation 3 Assume the contrary, and let H be a component of $G-T$ and let C be a clique of T such that $N(C)\cap H \neq \emptyset$ and $H - N(C) \neq \emptyset$. Let $x \in N(C)\cap H$ having a neighbor y in $H-N(C)$, and let $v \in C$ be a neighbor of x.

By (8), there exists a vertex z adjacent to y but not to x. $z \in N(C)\cap H$, otherwise $zyxv$ would be a bad P_4. The same argument shows that x and z have the same neighbors in C. Moereover,

$$\text{for all } u \in T-C, \text{ if } u \text{ is adjacent to } y, \text{ then } u \text{ is adjacent to both } x \text{ and } z. \tag{11}$$

(Else $uyxv$ or $uyzv$ would be a bad P_4), and

$$\text{for all } u \in T,\ u \text{ is adjacent to } x \text{ if and only if } u \text{ is adjacent to } z. \tag{12}$$

This is clear for $u \in C$. Suppose $u \in T-C$ is adjacent to x but not to z, then (11) implies that u is nonadjacent to y and so $uxyz$ is a bad P_4, a contradiction. The case when u is adjacent to z but not x can be settled in a similar manner. Thus, (12) holds.

We now show that x,z form a two-pair. Let $P = xx_1x_2\cdots x_kz$ be a chordless path connecting x and z, and assume that $k \geq 2$. By (12), $x_1 \in H$, hence x_1 is adjacent to y (because H is P_4-free). x_2 also belongs to H, otherwise, by (11), x_2 and y are nonadjacent and, by (12), x_2 and z are nonadjacent. But then x_2x_1yz is a bad P_4.

Thus, $x_1, x_2 \in H$. But then $x_3x_2x_1x$ (or zx_2x_1x if $k=2$) is a bad P_4. This contradiction proves Observation 3.$\diamond$

By (9) and $m \geq 2$, $G - T$ has a nontrivial component H. By (3), H has twins x, y which are nonadjacent by the hypothesis in this case. Write

$$N = N_H(x) = N_H(y),\ R = H - N - \{x, y\}.$$

Since H is connected, N is nonempty.

Observation 4 *For all vertices $v \in T$, if v is adjacent to x or y but not both, then v is adjacent to all vertices in N.*

Proof of Observation 4 Otherwise, there would be a bad P_4.$\diamond$

By (8), there exists a vertex x' adjacent to x but nonadjacent to y, and a vertex y' adjacent to y but nonadjacent to x. As x, y are twins in $G - T$, x' and y' belong to T.

Observation 5 *Such vertices x' and y' can be chosen in different cliques C_i, C_j.*

Proof of Observation 5 Assume that there are vertices a, b in a clique C of T such that a is adjacent to x but not to y, and b is adjacent to y but not to x. As C is not a clique-cutset of G (see (7)), some vertex of H has a neighbor in another clique $C' \neq C$ of T. By Observation 3, x has a neighbor $c \in C'$. c cannot be adjacent to y, otherwise $cxabyc$ would be a C_5, contradicting the minimality of G. Now, Observation 5 follows by setting $C_i = C'$, $C_j = C$, $x' = c$, and $y' = b$.$\diamond$

From now on, let $x' \in C_i$, $y' \in C_j$ with $i \neq j$. By Observation 4, x' and y' are adjacent to all vertices in N.

Observation 6 *For all $C \in \{C_1, C_2, \ldots, C_m\}$, $C \neq C_i$ or $C \neq C_j$, and for all $z \in N$, $N_C(x) \subseteq N_C(z)$ and $N_C(y) \subseteq N_C(z)$.*

Proof of Observation 6 If there is a vertex $v \in N_C(x) - N_C(z)$, then, by Observation 4, v must be adjacent to y. But then $vyzx'$ (if $C \neq C_i$) or $vxzy'$ (if $C \neq C_j$) is a bad P_4. Thus, $N_C(x) \subseteq N_C(z)$. By symmetry, $N_C(y) \subseteq N_C(z)$. $\diamond$

Observation 7 *N cannot have a vertex z^* that is adjacent to all vertices in $N - z^*$.*

Proof of Observation 7 Such a vertex z^* would dominate x (contradicting (8)): If v is a neighbor of x in T, and $v \in C$ for a clique $C \in \{C_1, C_2, \ldots, C_m\}$, then, as C_i and C_j are different cliques, $C \neq C_i$ or $C \neq C_j$, hence, by Observation 6, v must be adjacent to z^*. $\diamond$

By Observation 7, there exist two nonadjacent vertices z_1, z_2 in N. We are going to show that z_1, z_2 form a two-pair. This contradiction to (5) settles Case 2.

Consider an induced path $P = z_1 t_1 t_2 \cdots t_k z_2$ in G, and assume that $k \geq 2$. Then

t_1 must belong to N.

For, if $t_1 \in R$, then $t_1 z_1 x z_2$ would be a bad P_4; if $t_1 \in C$, say $C \neq C_j$, then $t_1 z_1 x z_2$ (if t_1 is not adjacent to x) or $t_1 x z_2 y'$ (if t_1 is adjacent to x) is a bad P_4, a contradiction. The case $t_1 \in C_j$ is similar. Now,

t_2 must belong to T,

otherwise, $z_1 t_1 t_2 t_3$ would be a bad P_4 (set $t_{k+1} := z_2$). Thus, $t_2 \in C$ for a clique C of T, say $C \neq C_j$. Moreover,

t_2 is adjacent to x and y,

otherwise, $t_2z_2xz_1$ or $t_2z_2yz_1$ (if t_2 and z_2 are adjacent), or $t_2t_1xz_2$ or $t_2t_1yz_2$ (if t_2 and z_2 are nonadjacent) would be a bad P_4, a contradiction.

But then t_2xz_1y' is a bad P_4. The case $t_2 \in C_j$ is similar. Thus, there is no induced path of length > 2 connecting x and y, and so x, y form a two-pair. The proof of Theorem 6 is complete. □

The class of perfect graphs described in Theorem 6 contains all P_4-free graphs, split graphs, cograph contractions, complements of cograph contractions, strongly P_4-stable graphs, complements of strongly P_4-stable graphs ([23]), bipartite graphs, and complements of bipartite graphs. In particular, this new class is not contained in BIP* ([11]), not in the class of strict-quasi parity graphs ([34]). We do not know whether there is a perfect graph described in Theorem 6 that is not quasi-parity ([34]). Also, we shall remark that these new perfect P_4-bipartite graphs do not belong to any class of the classes of S_i-graphs, $i = 1, \ldots, 4$, described in Theorem 3. This can be seen as follows. Let G be the graph obtained from the $\overline{C_6}$ by subdividing the three edges not belonging to a triangle (thus G has nine vertices). Then G satisfies the conditions of Theorem 6 with T consisting of the two disjoint triangles, but G is not an S_i-graph, for any $i = 1, \ldots, 4$.

To conclude the paper, we remark that Fonlupt (see [22]) conjectures that no minimal imperfect Berge graph contains a cutset that induces a P_3-free graph. Clearly, Conjecture 2 is implied by Fonlupt's conjecture together with (2) and (10).

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