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Paths of specified length in a random k -partite graph

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Fix positive integers k and l . Consider a random k -partite graph on n vertices obtained by partitioning the vertex set into V_i , ($i = 1, \dots, k$) each having size $\Omega(n)$ and choosing each possible edge with probability p . Consider any vertex x in any V_i and any vertex y . We show that the expected number of simple paths of even length l between x and y differ significantly depending on whether y belongs to the same V_i (as x does) or not. A similar phenomenon occurs when l is odd. This result holds even when k, l vary slowly with n . This fact has implications to coloring random graphs. The proof is based on establishing bijections between sets of paths.

Keywords: random graphs, paths, bijections

1 Motivation

This problem arose in the analysis of algorithms for coloring random k -colorable graphs [2, 3]. Consider a random graph drawn as explained in the abstract. To separate a color class, we fix a vertex x in the largest (or smallest) V_i and compute the number of l -paths (paths of length l), $n(x, y, l)$, between x and an arbitrary vertex y . Depending on whether y belongs to the same class as x belongs to, the expectation of this quantity differs significantly. If we can show that $n(x, y, l)$ is close to its expected value almost surely, this gives us a way of separating the class containing x . Repeating this $k - 2$ times, one gets a k -coloring. The expectation of $n(x, y, l)$ is $N(x, y, l)p^l$, where $N(x, y, l)$ is the total number of l -paths in the complete k -partite graph formed by V_i s. The result stated in the abstract shows that the expectations differ significantly as required.

We do not discuss the algorithmic issues here since they have been outlined in [2]. We only prove the results stated in the abstract using only counting arguments. Even though the results are obviously true for bipartite graphs, for $k \geq 3$, it is not so straightforward. We believe the arguments used here would be of interest to know. The basic idea is to partition (for each pair of start-end vertices) the corresponding set of l -paths into groups (based on the color classes of intermediate vertices). Then, for two different

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pairs, we establish an (almost) bijection between the corresponding groups. For any such mapped pair of groups, we also establish an (almost) bijection between the l -paths in them. This establishes the required statement.

2 Paths of specified length

Definition 2.1 By an l -path between two vertices x and y , we mean a simple path of length l between x and y . A simple path is one in which no vertex appears more than once. An l -path is represented as a $(l+1)$ -tuple $\langle x, v_1, \dots, v_{l-1}, y \rangle$ of vertices such that successive vertices in this sequence belong to different partite sets V_i .

Notations : G is a complete k -partite graph on the partite sets V_1, \dots, V_k with each $|V_i| \geq n/C$ for some constant $C \geq k$. For each i , n_i denotes the size $|V_i|$. For each i , let $W_i = V_i \cup \dots \cup V_k$. For all i ($1 \leq i \leq k-1$), for all $x \in V_i$, for all $y \in W_i$ such that $y \neq x$, let $N(x, y, l, i)$ denote the number of l -paths between x and y involving only vertices from W_i . Given a tuple σ with integral component values and an integer j , $c(\sigma, j)$ denotes the number of times j appears in σ .

We obtain the following results.

Theorem 2.1 Assume that $n_1 \leq \dots \leq n_k$. Let l be any fixed even integer ≥ 2 . For all i , $1 \leq i \leq k-1$, for all $x, y \in V_i$, for all $z \in W_i - V_i$, we have

$$\begin{aligned} N(x, z, l, i) &= \Theta(n^{l-1}) \text{ if } i \leq k-2 \\ N(x, y, l, i), N(x, y, l, i) - N(x, z, l, i) &= \Theta(n^{l-1}) \end{aligned}$$

Proof: Consider any i ($i = 1, \dots, k-1$) and any $x, y \in V_i$ and $z \in V_r, r > i$ and fix these parameters. We use the factorial functions defined as follows : $(n)_0 = 1$. $(n)_l = n(n-1) \dots (n-l+1)$, $l \geq 1$. Let $P(x, y)$ denote the set of all l -paths between x and y involving only vertices from the $k-i+1$ partite sets V_j ($i \leq j \leq k$). $P(x, z)$ is defined similarly. That is,

$$P(x, y) = \{ \langle x, v_1, \dots, v_{l-1}, y \rangle \mid \text{the sequence is an } l\text{-path between } x \text{ and } y \}.$$

$$P(x, z) = \{ \langle x, v_1, \dots, v_{l-1}, z \rangle \mid \text{the sequence is an } l\text{-path between } x \text{ and } z \}.$$

Clearly, we have $|P(x, y)| = O(n^{l-1})$ and $|P(x, z)| = O(n^{l-1})$. Also if $i = k-1$, then there are only two partite sets, namely, V_{k-1} and V_k and hence $P(x, z) = \emptyset$ and $N(x, z, l, i) = 0$. Define

$$B_i^s = \{ \langle \sigma_1, \dots, \sigma_{l-1} \rangle \mid \sigma_1 \neq i, \sigma_{l-1} \neq i, i \leq \sigma_j \leq k, \sigma_j \neq \sigma_{j+1} \forall j \}$$

$$B_i^d = \{ \langle \sigma_1, \dots, \sigma_{l-1} \rangle \mid \sigma_1 \neq i, \sigma_{l-1} \neq r, i \leq \sigma_j \leq k, \sigma_j \neq \sigma_{j+1} \forall j \}$$

In the above, the superscript s (or d) is a short notation for the word ‘‘same’’ (or ‘‘different’’). We have $|B_i^s|, |B_i^d| \leq k^{l-1}$.

Now $f : P(x, y) \rightarrow B_i^s$ is a mapping which identifies each l -path $\langle x, v_1, \dots, v_{l-1}, y \rangle$ with the unique $(l-1)$ -tuple $\langle \sigma_1, \dots, \sigma_{l-1} \rangle$ in B_i^s where if $v_m \in V_j$ then $\sigma_m = j$. Similarly, we can define a mapping $g : P(x, z) \rightarrow B_i^d$ which identifies each l -path in $P(x, z)$ with a unique $(l-1)$ -tuple in B_i^d . We use the elements of B_i^s (or B_i^d) to partition the set $P(x, y)$ (or $P(x, z)$) as follows.

$$P(x, y) = \bigcup_{\sigma \in B_i^s} P_\sigma \text{ where } P_\sigma = \{ \tau \in P(x, y) \mid f(\tau) = \sigma \}.$$

$$P(x, z) = \bigcup_{\sigma \in B_l^d} P_\sigma \text{ where } P_\sigma = \{\tau \in P(x, z) \mid g(\tau) = \sigma\}.$$

Now, for each $\sigma \in B_l^s \cup B_l^d$, $|P_\sigma| = (\prod_{i \leq j \leq k} (n_j)_{c(\sigma, j)}) = (\prod_{i \leq j \leq k} (n_j)^{c(\sigma, j)}) \cdot [1 - o(1)]$. As a result, for each $\sigma \in B_l^s \cup B_l^d$, $|P_\sigma| = \Theta(n^{l-1})$. The $[1 - o(1)]$ factor arises not only because of factorials, but also because x, y and z have to be excluded from consideration.

Also B_l^s is non-empty and it contains at least one element, namely, the tuple $\langle r, i, r, i, \dots, r \rangle$. Hence $N(x, y, l, i) = |P(x, y)| = \Theta(n^{l-1})$. Also, if $i \leq k - 2$, then there are at least 3 partite sets to be considered and hence B_l^d is non-empty. Hence $N(x, z, l, i) = |P(x, z)| = \Theta(n^{l-1})$ if $i \leq k - 2$.

We need to prove that $|P(x, y)| - |P(x, z)| = \Theta(n^{l-1})$. In order to prove this, it is enough to prove that the following two assertions are true.

1. $|B_l^s| \geq |B_l^d| + 1$ and
2. There exists a *one-to-one* mapping $h : B_l^d \rightarrow B_l^s$ such that for each $\tau \in B_l^d$, we have $|P_{h(\tau)}| \geq |P_\tau| [1 - o(1)]$.

We prove that the two assertions are true as follows. Now, partition B_l^s, B_l^d into

$$B_l^s = B_{l,1}^s \cup \dots \cup B_{l,l-1}^s \cup B_{l,l}^s$$

$$B_l^d = B_{l,2}^d \cup \dots \cup B_{l,l-1}^d \cup B_{l,l}^d$$

where

$$B_{l,l}^s = \{\sigma \in B_l^s \mid \sigma_{l-1} \neq i, \sigma_{l-1} \neq r\}.$$

$$B_{l,l}^d = \{\sigma \in B_l^d \mid \sigma_{l-1} \neq i, \sigma_{l-1} \neq r\}.$$

$$B_{l,j}^s = \{\sigma \in B_l^s \mid \sigma_{j-1} \neq i, \sigma_{j-1} \neq r, \sigma_m = i, r \text{ for } m \geq j\}, \text{ for } 2 \leq j \leq l-1.$$

$$B_{l,j}^d = \{\sigma \in B_l^d \mid \sigma_{j-1} \neq i, \sigma_{j-1} \neq r, \sigma_m = i, r \text{ for } m \geq j\}, \text{ for } 2 \leq j \leq l-1.$$

$$B_{l,1}^s = \{\langle r, i, r, i, \dots, r \rangle\}$$

Now $B_{l,1}^d$ cannot be defined similarly since l is even. It is easy to see that the definitions form a well-defined partition of B_l^s and B_l^d . In other words, for each $\sigma \in B_l^s$, there exists a unique value of j between 1 and l such that $\sigma \in B_{l,j}^s$. Similarly, for each $\tau \in B_l^d$, there exists a unique value of j between 2 and l such that $\tau \in B_{l,j}^d$.

Now we claim that for all j such that $2 \leq j \leq l$, $|B_{l,j}^s| = |B_{l,j}^d|$. For $j = l$, this follows from $B_{l,l}^s = B_{l,l}^d$. For $j < l$, consider the mapping $h_j : B_{l,j}^d \rightarrow B_{l,j}^s$ defined as follows. Let $\tau \in B_{l,j}^d$ be any tuple. Then, $h_j(\tau) = \sigma$ where σ is defined as

- For all m ($1 \leq m \leq j-1$), $\sigma_m = \tau_m$.
- For all m such that $j \leq m \leq l-1$, $\sigma_m = i$ if $\tau_m = r$ and $\sigma_m = r$ if $\tau_m = i$.

Clearly $\sigma \in B_{l,j}^s$. Also it can be verified that h_j is a one-to-one and onto mapping. Since $B_{l,j}^s$ and $B_{l,j}^d$ are finite sets, it follows that $|B_{l,j}^s| = |B_{l,j}^d|$.

Thus, we have $|B_l^s| \geq |B_l^d| + 1$ and the first assertion is true.

To prove the second assertion, define the mapping $h : B_l^d \rightarrow B_l^s$ to be as follows. For each $\tau \in B_l^d$, define $h(\tau) = h_j(\tau)$ where j is such that $\tau \in B_{l,j}^d$. Clearly, h is a one-to-one mapping since each h_j is a one-to-one mapping.

We prove that for each $\tau \in B_l^d$, we have $|P_{h(\tau)}| \geq |P_\tau|[1 - o(1)]$. Let $\tau \in B_l^d$ be any tuple and let σ denote the tuple $h(\tau)$. We know $\tau \in B_{l,j}^d$ for some j , $2 \leq j \leq l$.

If $j = l$, then we have $\sigma = \tau$ and hence $|P_\sigma| \geq |P_\tau|[1 - o(1)]$.

If $j = l-2, l-4, \dots, 2$, then clearly, $c(\tau, m) = c(\sigma, m)$ for all values of m ($i \leq m \leq k$) and hence $|P_\sigma| \geq |P_\tau|[1 - o(1)]$.

If $j = l-1, l-3, \dots, 3$, then clearly, $c(\tau, m) = c(\sigma, m)$ for all values of m ($i \leq m \leq k$) such that $m \neq i$, $m \neq r$. Also, $c(\sigma, r) = c(\tau, r) + 1$ and $c(\tau, i) = c(\sigma, i) + 1$. Since $n_i \leq n_r$ ($r > i$) by assumption, we have $|P_\sigma| \geq |P_\tau|[1 - o(1)]$.

Thus, we have

$$\begin{aligned}
N(x, y, l, i) - N(x, z, l, i) &= |P(x, y)| - |P(x, z)| \\
&= \left| \bigcup_{\sigma \in B_l^s} P_\sigma \right| - \left| \bigcup_{\tau \in B_l^d} P_\tau \right| \\
&= \sum_{j=l-1, \dots, 3} \left(\sum_{\sigma \in B_{l,j}^s} |P_\sigma| \right) + \sum_{j=l, l-2, \dots, 2} \left(\sum_{\sigma \in B_{l,j}^s} |P_\sigma| \right) + \sum_{\sigma = \langle r, i, \dots, r \rangle} |P_\sigma| \\
&\quad - \sum_{j=l-1, \dots, 3} \left(\sum_{\tau \in B_{l,j}^d} |P_\tau| \right) - \sum_{j=l, l-2, \dots, 2} \left(\sum_{\tau \in B_{l,j}^d} |P_\tau| \right) \\
&\geq |P_\sigma| - o(|P_\sigma|) \text{ where } \sigma = \langle r, i, \dots, r \rangle.
\end{aligned} \tag{1}$$

Thus,

$$\begin{aligned}
N(x, y, l, i) - N(x, z, l, i) &= \Theta((n_r)_{l/2} (n_i)_{l/2-1}) \\
&= \Theta(n^{l-1})
\end{aligned}$$

Hence,

$$\begin{aligned}
N(x, z, l, i) &= \Theta(n^{l-1}) \text{ if } i \leq k-2 \\
N(x, y, l, i), N(x, y, l, i) - N(x, z, l, i) &= \Theta(n^{l-1}).
\end{aligned}$$

This completes the proof of the theorem. ■

Using similar arguments, we can prove the following theorem also.

Theorem 2.2 Assume that $n_1 \geq \dots \geq n_k$. Let l be any fixed odd integer ≥ 3 . For all i ($1 \leq i \leq k-1$), for all $x, y \in V_i$, for all $z \in W_i - V_i$, we have

$$\begin{aligned} N(x, y, l, i) &= \Theta(n^{l-1}) \text{ if } i \leq k-2 \\ N(x, z, l, i), N(x, z, l, i) - N(x, y, l, i) &= \Theta(n^{l-1}) \end{aligned}$$

3 Conclusions

1. The main result of the paper is that the number of l -paths joining a vertex x (in the largest or smallest V_i depending on the parity of l) and a vertex y differs significantly depending on where y comes from. A close look at the proof (particularly, derivation of (1)) shows that this holds even if we allow k, l and C to vary with n , provided $lk^l C^l = o(n)$.

2. It would be interesting to extend these results to structures other than simple paths. Such results can be applied to the design and analysis of efficient algorithms for random graphs (see [1] for a survey).

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