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# Overlap-Free Symmetric D0L words<sup>†</sup>

Anna E. Frid

Sobolev Institute of Mathematics, pr. Koptyuga, 4, 630090, Novosibirsk, Russia  
e-mail: frid@math.nsc.ru

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A D0L word on an alphabet  $\Sigma = \{0, 1, \dots, q-1\}$  is called symmetric if it is a fixed point  $w = \varphi(w)$  of a morphism  $\varphi : \Sigma^* \rightarrow \Sigma^*$  defined by  $\varphi(i) = \overline{t_1 + i} \overline{t_2 + i} \dots \overline{t_m + i}$  for some word  $t_1 t_2 \dots t_m$  (equal to  $\varphi(0)$ ) and every  $i \in \Sigma$ ; here  $\bar{a}$  means  $a \bmod q$ .

We prove a result conjectured by J. Shallit: if all the symbols in  $\varphi(0)$  are distinct (i.e., if  $t_i \neq t_j$  for  $i \neq j$ ), then the symmetric D0L word  $w$  is overlap-free, i.e., contains no factor of the form  $axaxa$  for any  $x \in \Sigma^*$  and  $a \in \Sigma$ .

**Keywords:** overlap-free word, D0L word, symmetric morphism

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## 1 Introduction

In his classical 1912 paper [15] (see also [3]), A. Thue gave the first example of an overlap-free infinite word, i. e., of a word which contains no subword of the form  $axaxa$  for any symbol  $a$  and word  $x$ . Thue's example is known now as the *Thue-Morse word*

$$w_{TM} = 01101001100101101001011001101001\dots$$

It was rediscovered several times, can be constructed in many alternative ways and occurs in various fields of mathematics (see the survey [1]).

The set of all overlap-free words was studied e. g. by E. D. Fife [8] who described all binary overlap-free infinite words and P. Séébold [13] who proved that the Thue-Morse word is essentially the only binary overlap-free word which is a fixed point of a morphism. Nowadays the theory of overlap-free words is a part of a more general theory of pattern avoidance [5].

J.-P. Allouche and J. Shallit [2] asked if the initial Thue's construction of an overlap-free word could be generalized and found a whole family of overlap-free infinite words built by a similar principle. This paper contains a further generalization of that result; its main theorem was conjectured by J. Shallit [14].

Let us give all the necessary definitions and state the main theorem. Consider a finite alphabet  $\Sigma = \Sigma_q = \{0, 1, \dots, q-1\}$ . For an integer  $i$ , let  $\bar{i}$  denote the residue of  $i$  modulo  $q$ . A morphism  $\varphi : \Sigma_q^* \rightarrow \Sigma_q^*$  is called *symmetric* if for all  $i \in \Sigma_q$  the equality holds

$$\varphi(i) = \overline{t_1 + i} \overline{t_2 + i} \dots \overline{t_m + i},$$

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where  $t_1 t_2 \dots t_m$  is an arbitrary word (equal to  $\varphi(0)$ ). Clearly, if  $t_1 = 0$ , then  $\varphi$  has a fixed point, i. e., a (right) infinite word  $w = w(\varphi)$  satisfying

$$w = \varphi(w).$$

Without loss of generality we assume that  $w$  starts with 0.

A symmetric morphism is *growing* if  $|\varphi(0)| \geq 2$ . We shall call a fixed point of a growing symmetric morphism a *symmetric DOL word*. For example, the Thue-Morse word  $w_{TM}$  is a fixed point of a symmetric morphism  $\varphi_{TM}$ :

$$\begin{cases} \varphi_{TM}(0) = 01, \\ \varphi_{TM}(1) = 10. \end{cases}$$

Symmetric DOL words include also other useful examples, such as the Dejean word [7], the Keränen word [11] and others (see Section 10.5 in [12], where in particular the term “symmetric” is introduced). Note that the class of symmetric DOL words is included in a wider class of uniform marked DOL words whose properties were studied e. g. in [10].

Note that an infinite word  $w = w_1 w_2 \dots w_n \dots$ , where  $w_i \in \Sigma$ , is the fixed point of the symmetric morphism  $\varphi$  if and only if

$$\forall k \geq 0 \forall i \in \{1, \dots, m\} \quad w_{km+i} = \overline{w_{k+1} + t_i}. \quad (1)$$

Indeed, this equality means that  $w_{km+i}$  is equal to the  $i$ th symbol of  $\varphi(w_{k+1})$ .

For every  $m > 1$ , let  $\varphi_{m,q} : \Sigma_q^* \rightarrow \Sigma_q^*$  be the symmetric morphism defined by  $\varphi_{m,q}(0) = 0\overline{1} \overline{2} \dots \overline{m-1}$ . Note that  $\varphi_{TM} = \varphi_{2,2}$ . Let  $w_{m,q}$  be the fixed point of  $\varphi_{m,q}$  starting with 0; then the  $i$ th symbol of  $w_{m,q}$  for each  $i$  can also be defined as  $s_m(i)$ , where  $s_m(i)$  is the sum of the digits in the base- $m$  representation of  $i$ .

J.-P. Allouche and J. Shallit proved the following generalization of Thue’s result:

**Theorem 1 ([2])** *The word  $w_{m,q}$  is overlap-free if and only if  $m \leq q$ .*

J. Shallit conjectured also that symmetric DOL words of a much wider class are overlap-free. We turn this conjecture into

**Theorem 2** *If  $\varphi : \Sigma_q^* \rightarrow \Sigma_q^*$  is a growing symmetric morphism, and if all symbols occurring in  $\varphi(0)$  are distinct, then the fixed point  $w = w(\varphi)$  is overlap-free.*

The remaining part of the paper is devoted to the proof of this result.

## 2 Proof of Theorem 2

Let us start with introducing some more notions and citing a result by J. Berstel and L. Boasson [4] which we shall need later.

A *partial word* is a word on the alphabet  $\Sigma \cup \{\diamond\}$ , where the symbol  $\diamond \notin \Sigma$  is called the *hole*<sup>‡</sup>. Each hole means an unknown symbol of  $\Sigma$ . A (partial) word  $u = u_1 \dots u_n$ , where  $u_i$  are symbols, is called (*locally*) *p*-periodic if  $u_i = u_{i+p}$  for all  $i \in \{1, \dots, n-p\}$  such that  $u_i \neq \diamond$  and  $u_{i+p} \neq \diamond$ .

The following result is a generalization of the classical Fine and Wilf’s theorem [9, 6]:

**Theorem 3 ([4])** *Let  $u$  be a partial word of length  $n$  which is  $p$ -periodic and  $q$ -periodic. If  $u$  contains only one hole, and if  $n \geq p+q$ , then  $u$  is  $\gcd(p, q)$ -periodic.*

Now let us start the proof of Theorem 2 and first consider the easiest case:

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<sup>‡</sup> This definition slightly differs from the one given in [4].

**Lemma 1** *If the symmetric morphism  $\varphi$  is defined by  $\varphi(0) = 0\bar{c}2\bar{c}\dots(m-1)\bar{c}$  for some integer  $c > 0$ , and if all the symbols of  $\varphi(0)$  are distinct, then the fixed point  $w$  of  $\varphi$  is overlap-free.*

**Proof.** Let  $S \subset \Sigma$  be the set of symbols occurring in  $w$  and  $q'$  be its cardinality. Denote  $\Sigma' = \{0, \dots, q' - 1\}$  and define  $h : (\Sigma')^* \rightarrow S^*$  as the symbol-to symbol morphism transforming each symbol  $i \in \Sigma'$  to  $h(i) = \bar{c}i$ . Since the cardinalities of  $S$  and  $\Sigma'$  coincide, and since each symbol of  $S$  can be represented as  $\bar{c}i$  for some  $i$ ,  $h$  is a one-to-one mapping. But it can be easily checked that  $\varphi h = h\varphi_{m,q'}$ . Since  $w_{m,q'} = \varphi_{m,q'}(w_{m,q'})$ , we have  $h(w_{m,q'}) = h(\varphi_{m,q'}(w_{m,q'})) = \varphi(h(w_{m,q'}))$ , so  $h(w_{m,q'})$  is the fixed point of  $\varphi$ ; it starts with 0 since  $h(0) = 0$ . We see that  $h(w_{m,q'}) = w$ , that is,  $w$  is obtained from  $w_{m,q'}$  by renaming symbols. It is overlap-free due to Theorem 1.  $\square$

A *block* is an image of symbol under a morphism. Let  $S(m)$  denote the class of all symmetric morphisms on  $\Sigma$  of block length  $m$  with all the symbols in a block distinct. We assume also that the image of 0 always starts with 0, so that all the morphisms of  $S(m)$  admit fixed points. Clearly, the class  $S(m)$  is non-empty only if  $m \leq q$ .

Our goal is to prove that, for any fixed  $m$ , all the fixed points of morphisms of  $S(m)$  are overlap-free. Suppose the opposite and consider the minimal counter-example, i. e., a morphism  $\varphi \in S(m)$  and its fixed point  $w$  containing an overlap  $v = axaxa$  of minimal length (so that overlaps occurring in other fixed points of morphisms of  $S(m)$  are not shorter). Here  $a \in \Sigma$  and  $x \in \Sigma^*$ ; we denote the length  $|ax|$  by  $l$ , and thus have  $|v| = 2l + 1$ . Let us fix an occurrence of  $v$  to  $w$  and its position with respect to blocks of  $\varphi$ : we consider  $v$  as a word obtained from  $\varphi(s)$ , where  $s$  is a factor of  $w$ , by erasing  $\alpha - 1$  symbols from the left and  $m - \beta$  symbols from the right, where  $1 \leq \alpha, \beta \leq m$ . So,  $v$  starts with the symbol numbered  $\alpha$  of a block and ends with the symbol numbered  $\beta$ .

**Claim 1** *The inequality  $l \geq m$  holds.*

**Proof.** Suppose that  $l < m$ . The 1st,  $(l + 1)$ th, and  $(2l + 1)$ th symbols of  $v$  are equal and thus must lie in three different blocks. So,  $v$  contains a complete block. But this block must be  $l$ -periodic since  $v$  is  $l$ -periodic; hence it must contain two equal symbols since  $l < m$ . A contradiction.  $\square$

**Claim 2** *The block length  $m$  does not divide  $l$ .*

**Proof.** Suppose the opposite: let  $l = mk$ . Then the length of the “inverse image”  $s$  of  $v$  is equal to  $2k + 1$ . Since  $v$  is an overlap, its  $(mi + 1)$ th symbol is equal to the  $(m(i + k) + 1)$ th one for any  $i \in \{0, \dots, k\}$ ; they are symbols numbered  $\alpha$  of respectively the  $(i + 1)$ th and the  $(i + k + 1)$ th blocks of  $\varphi(s)$ . Since the morphism  $\varphi$  is symmetric, each block is uniquely determined by its  $\alpha$ th symbol, so  $(i + 1)$ th and  $(i + k + 1)$ th symbols of  $s$  are equal. Thus,  $s$  is an overlap in  $w$  shorter than  $v$ , a contradiction.  $\square$

For every word  $u = u_1u_2\dots u_{n+1} \in \Sigma^{n+1}$ , where  $u_1, \dots, u_{n+1} \in \Sigma$ , let us define the word  $r(u) \in \Sigma^n$  as obtained from  $u$  by subtraction of consecutive symbols:

$$r(u) = \overline{u_2 - u_1} \overline{u_3 - u_2} \dots \overline{u_{n+1} - u_n}.$$

Clearly,  $u$  can be reconstructed from its first symbol  $u_1$  and the word  $r(u) = r_1 \dots r_n$ , where  $r_1, \dots, r_n \in \Sigma$ :

$$u = u_1 \overline{u_1 + r_1} \overline{u_1 + r_1 + r_2} \dots \overline{u_1 + r_1 + \dots + r_n}. \tag{2}$$

Let us consider the word  $r(v) = r(axaxa)$ . Its length is equal to  $2l$ , and it is  $l$ -periodic as well as  $v$ . Since  $\varphi$  is symmetric, the word  $r(\varphi(i))$  does not depend on the symbol  $i \in \Sigma$ ; we denote  $r(\varphi(i)) = b = b_1 \dots b_{m-1}$ ,

where  $b_1, \dots, b_{m-1} \in \Sigma$ . Since  $v$  starts with the symbol number  $\alpha$  of a block and ends with the symbol number  $\beta$ , we have

$$r(v) = b_\alpha \dots b_{m-1} c_1 b c_2 b \dots b c_n b_1 \dots b_{\beta-1},$$

where  $|s| = n + 1$  and  $c_1 \dots c_n$  are symbols of  $\Sigma$  depending on pairs of consecutive blocks in  $\varphi(s)$ ; if  $\alpha = m$ , then  $r(v)$  just starts with  $c_1$ , and if  $\beta = 1$ ,  $r(v)$  just ends with  $c_n$ . Let  $n'$  be the last number such that  $c_{n'}$  is situated in the first occurrence of  $r(axa)$  in  $r(v)$ . Since  $r(v)$  is  $l$ -periodic, for all  $i \in \{1, \dots, n'\}$  the symbol  $c_i$  is equal to the symbol of  $r(v)$  situated at distance  $l$  from it. Due to Claim 2,  $l \not\equiv 0 \pmod{m}$ , and thus all these symbols are equal to  $b_{l'}$ , where  $l \equiv l' \pmod{m}$ . So, the word  $r(axa)$  (equal to the prefix of length  $l$  of  $r(v)$ ) is  $m$ -periodic:

$$r(axa) = b_\alpha \dots b_{m-1} (b_{l'} b)^{n'-1} b_{l'} b_1 \dots b_{\gamma-1},$$

where  $\gamma - \alpha \equiv l \pmod{m}$ ,  $\gamma \in \{1, \dots, m\}$ .

Let us consider the prefix of  $r(v)$  of length  $m + l$ . It exists due to Claim 1 and is equal to

$$r(axa) b_\gamma \dots b_{m-1} c_{n'+1} b_1 \dots b_{\gamma-1}.$$

Substituting the unknown symbol  $c_{n'+1}$  by a hole  $\diamond$ , we obtain a partial word

$$b_\alpha \dots b_{m-1} (b_{l'} b)^{n'} \diamond b_1 \dots b_{\gamma-1},$$

which is  $l$ -periodic as well as  $r(v)$ . But at the same time, it is  $m$ -periodic; thus, due to Theorem 3 it is  $p$ -periodic, where  $p = \gcd(l, m)$ . Consequently,  $b = r(\varphi(0))$  is also  $p$ -periodic:  $b = \overline{(b_1 \dots b_p)^{m'-1} b_1 \dots b_{p-1}}$ , where  $m' = m/p$ . Let us return to  $\varphi(0)$  and denote  $g_1 = 0$ ,  $g_k = \overline{b_1 + b_2 + \dots + b_{k-1}}$  for  $k \in \{2, \dots, p\}$ , and  $c = \overline{b_1 + b_2 + \dots + b_p}$ ; due to (2), we see that  $\varphi(0)$  is of the form

$$\varphi(0) = g_1 \dots g_p \overline{g_1 + c} \dots \overline{g_p + c} \dots \overline{g_1 + (m' - 1)c} \dots \overline{g_p + (m' - 1)c}. \quad (3)$$

Here  $g_1 = 0$  since  $\varphi$  has a fixed point, and  $m' = m/p$ . The words of the form  $\overline{g_1 + ic} \dots \overline{g_p + ic}$ , where  $i \in \{0, \dots, m' - 1\}$ , will be called *subblocks*. Note that for all  $k \in \{1, \dots, p\}$ , a subblock is uniquely determined by its  $k$ th symbol, and that  $w$  consists of consecutive subblocks.

Let  $w_i$  denote the  $i$ th symbol of the fixed point  $w$  of  $\varphi$ , i. e., let  $w = w_1 \dots w_n \dots$ , where  $w_i \in \Sigma$ . Consider the arithmetical subsequence

$$w' = w_1 w_{p+1} w_{2p+1} \dots w_{np+1} \dots$$

**Claim 3** *The word  $w'$  is the fixed point of a morphism  $\varphi' \in S(m)$ .*

**Proof.** Let us define the symmetric morphism  $\varphi'$  by

$$\varphi'(0) = g_1 \overline{g_1 + c} \dots \overline{g_1 + (m' - 1)c} \overline{g_2 g_2 + c} \dots \overline{g_2 + (m' - 1)c} \dots g_p \overline{g_p + c} \dots \overline{g_p + (m' - 1)c}.$$

Since  $\varphi'(0)$  is obtained from  $\varphi(0)$  by permuting symbols, and all the symbols of  $\varphi(0)$  are distinct, so are the symbols of  $\varphi'(0)$ . Since  $g_1 = 0$ , and  $\varphi'$  is symmetric by definition,  $\varphi' \in S(m)$ . So we must prove only that  $w'$  is its fixed point, i. e., that

$$\forall k \geq 0 \forall i \in \{1, \dots, m\} \quad w'_{km+i} \text{ is equal to the } i\text{th symbol of } \varphi'(w'_{k+1}), \quad (4)$$

where  $w'_k$  is the  $k$ th symbol of  $w' = w'_1 w'_2 \dots w'_n \dots$

Clearly, each  $i \in \{1, \dots, m\}$  can be uniquely represented as  $i = jm' + \delta$ , where  $j \in \{0, \dots, p-1\}$  and  $\delta \in \{1, \dots, m'\}$ . Since by definition of  $w'$  for all  $v$  we have  $w'_v = w_{p(v-1)+1}$ , for any  $k \geq 0$

$$w'_{km+i} = w'_{km+jm'+\delta} = w_{p(km+jm'+\delta-1)+1} = w_{(pk+j)m+p(\delta-1)+1}.$$

By Equality (1),  $w_{(pk+j)m+p(\delta-1)+1}$  is equal to the  $(p(\delta-1)+1)$ th symbol of  $\varphi(w_{pk+j+1})$ , that is, to  $(\delta-1)c + w_{pk+j+1}$  (recall that  $g_1 = 0$ ). In its turn,  $w_{pk+j+1}$  is the  $(j+1)$ th symbol of the subblock starting with  $w_{pk+1} = w'_{k+1}$ . It is equal to  $w'_{k+1} + g_{j+1}$ , and thus,  $w'_{km+i} = w'_{k+1} + (\delta-1)c + g_{j+1}$ . By the definition of  $\varphi'$ , it is equal to the symbol numbered  $jm' + \delta = i$  of  $\varphi'(w'_{k+1})$ . We have proved (4) and Claim 3.  $\square$

**Claim 4** *The word  $w'$  contains an overlap of length  $2l' + 1$ , where  $l' = l/p$ .*

**Proof.** Let our occurrence of the overlap  $v$  to  $w$  start with the  $k$ th symbol of a subblock, i. e., let  $\alpha \equiv k \pmod{p}$ , where  $k \in \{1, \dots, p\}$ . It means that  $v = w_{jp+k}w_{jp+k+1} \dots w_{(j+2l')p+k}$  for some  $j \geq 0$ ; since  $v$  is an overlap,  $w_{(j+v)p+1} = w_{(j+v+l')p+1}$  for all  $v \in \{1, \dots, l'\}$ . But we have also  $w_{jp+k} = w_{(j+l')p+k}$ , and since a subblock is uniquely determined by its  $k$ th symbol,  $w_{jp+1} = w_{(j+l')p+1}$ . So, the word  $w_{jp+1}w_{(j+1)p+1} \dots w_{(j+2l')p+1}$  is  $l'$ -periodic, and it is the needed overlap in  $w'$ .  $\square$

As it follows from Claims 3 and 4, we have found a fixed point of a morphism of  $S(m)$  containing an overlap of length  $l' = l/p$ . But if  $p > 1$ , this contradicts to the minimality of our counter-example. On the other hand, if  $p = 1$ , then it follows from (3) that

$$\varphi(0) = 0\bar{c} \overline{2c} \dots \overline{(m-1)c}.$$

But a fixed point of such a morphism cannot be a counter-example according to Lemma 1. A contradiction. Theorem 2 is proved.  $\square$

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## References

- [1] J.-P. Allouche and J. Shallit. The ubiquitous Prouhet-Thue-Morse sequence. In C. Ding, T. Helleseht and H. Niederreiter, eds., *Sequences and Their Applications, Proceedings of SETA'98*, Springer-Verlag, 1999, 1–16.
- [2] J.-P. Allouche and J. Shallit. Sums of digits, overlaps, and palindromes. *Discr. Math. Theoret. Comput. Sci.* **4** (2000), 1–10.
- [3] J. Berstel. Axel Thue's work on repetitions in words. In P. Leroux and C. Reutenauer, eds., *Séries Formelles et Combinatoire Algébrique*, no. 11 in Publications du LACIM, Université du Québec à Montréal, 1992, 65–80.
- [4] J. Berstel and L. Boasson. Partial words and a theorem of Fine and Wilf. *Theoret. Comput. Sci.* **218** (1999), no. 1, 135–141.

- [5] J. Cassaigne. Unavoidable patterns. In M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, to appear.
- [6] C. Choffrut and J. Karhumäki. Combinatorics on words. In G. Rozenberg and A. Salomaa, eds., *Handbook of Formal Languages*, v. 1, chapter 6. Springer-Verlag, 1997.
- [7] F. Dejean. Sur un théorème de Thue. *J. Combin. Theory. Ser. A* **13** (1972), 25–36.
- [8] E. D. Fife. Binary sequences which contain no *BBb*. *Trans. Amer. Math. Soc.* **261** (1980), 115–136.
- [9] N. J. Fine and H. S. Wilf. Uniqueness theorem for periodic functions. *Proc. Amer. Math. Soc.* **16** (1965), 109–114.
- [10] A. Frid. Applying a uniform marked morphism to a word. *Discr. Math. Theoret. Comput. Sci.* **3** (1999), 125–140.
- [11] V. Keränen. Abelian squares are avoidable on 4 letters. In *Automata, Languages and Programming, Proceedings of ICALP'92 Lecture Notes in Comput. Sci.*; V. 700, Berlin: Springer, 1992, 41–52.
- [12] G. Lallement. *Semigroups and Combinatorial Applications*, Wiley, 1979.
- [13] P. Séébold. Sequences generated by infinitely iterated morphisms. *Discrete Appl. Math.* **11** (1985), 255–264.
- [14] J. Shallit, private communication.
- [15] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Norske vid. Selsk. Skr. Mat. Nat. Kl.* **1** (1912), 1–67. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell, ed., Universitetsforlaget, Oslo, 1977, 413–478.