

**3x+1 Minus the +**  
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► **To cite this version:**

Kenneth G. Monks. 3x+1 Minus the +. Discrete Mathematics and Theoretical Computer Science, DMTCS, 2002, 5, pp.47-54. <hal-00958971>

**HAL Id: hal-00958971**

**<https://hal.inria.fr/hal-00958971>**

Submitted on 13 Mar 2014

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# $3x + 1$ Minus the +

Kenneth G. Monks

Department of Mathematics, University of Scranton, Scranton, PA, 18510 USA  
 email: monks@scranton.edu

received Aug 10, 2001, accepted April 9, 2002.

We use Conway's *Fractran* language to derive a function  $R : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  of the form

$$R(n) = r_i n \text{ if } n \equiv i \pmod{d}$$

where  $d$  is a positive integer,  $0 \leq i < d$  and  $r_0, r_1, \dots, r_{d-1}$  are rational numbers, such that the famous  $3x + 1$  conjecture holds if and only if the  $R$ -orbit of  $2^n$  contains 2 for all positive integers  $n$ . We then show that the  $R$ -orbit of an arbitrary positive integer is a constant multiple of an orbit that contains a power of 2. Finally we apply our main result to show that any cycle  $\{x_0, \dots, x_{m-1}\}$  of positive integers for the  $3x + 1$  function must satisfy

$$\sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in \mathcal{O}} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$

where  $\mathcal{O} = \{i : x_i \text{ is odd}\}$ ,  $\mathcal{E} = \{i : x_i \text{ is even}\}$ , and  $k = |\mathcal{O}|$ . The method used illustrates a general mechanism for deriving mathematical results about the iterative dynamics of arbitrary integer functions from *Fractran* algorithms.

**Keywords:** Collatz conjecture,  $3x + 1$  problem, *Fractran*, discrete dynamical systems

## 1 Introduction and Main Results

The famous  $3x + 1$  conjecture (cf. [3],[4]) states that for every  $n \in \mathbb{Z}^+$  there exists  $k \in \mathbb{Z}^+$  such that  $T^k(n) = 1$  where

$$T(n) = \begin{cases} \frac{1}{2}n & \text{if } n \text{ is even} \\ \frac{3}{2}n + \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

and  $T^k = \underbrace{T \circ T \circ \dots \circ T}_k$  denotes the  $k$ -fold composition of  $T$  with itself. If we let  $T_0(x) = \frac{x}{2}$  and  $T_1(x) = \frac{3}{2}x + \frac{1}{2}$ , then for any  $n$  and  $k$ ,  $T^k(n) = T_{v_{k-1}} \circ T_{v_{k-2}} \circ \dots \circ T_{v_0}(n)$  for some  $v_0, \dots, v_{k-1} \in \{0, 1\}$  and  $v_i \equiv T^i(n) \pmod{2}$ . Several authors (cf. [3]) have given explicit formulas for this composition, e.g.

$$T_{v_{k-1}} \circ T_{v_{k-2}} \circ \dots \circ T_{v_0}(n) = \frac{3^m}{2^k} n + \sum_{i=0}^{k-1} v_i \frac{3^{v_{i+1} + \dots + v_{k-1}}}{2^{k-i}} \text{ where } m = \sum_{i=0}^{k-1} v_i.$$

Compare this somewhat unwieldy expression with the much simpler one

$$R_{v_{k-1}} \circ R_{v_{k-2}} \circ \cdots \circ R_{v_0}(n) = \frac{3^m}{2^k} n$$

when  $R_0(n) = \frac{1}{2}n$  and  $R_1(n) = \frac{3}{2}n$ . With this example in mind, it is natural to ask if there is some function of the form

$$R(n) = \begin{cases} r_0 n & \text{if } n \equiv 0 \pmod{d} \\ r_1 n & \text{if } n \equiv 1 \pmod{d} \\ \vdots & \vdots \\ r_{d-1} n & \text{if } n \equiv d-1 \pmod{d} \end{cases} \quad (1.1)$$

where  $r_1, \dots, r_{d-1}$  are rational numbers and  $d \geq 2$  such that knowledge of certain  $R$ -orbits would settle the  $3x+1$  problem, i.e. is there an addition-free variant of the  $3x+1$  function whose dynamics encode the conjecture? We answer this question in the affirmative with the following result

**Theorem 1** *There are infinitely many functions  $R$  of the form (1.1) having the property that the  $3x+1$  conjecture is true if and only if for all positive integers  $n$  the  $R$ -orbit of  $2^n$  contains 2. In particular,*

$$R(n) = \begin{cases} \frac{1}{11}n & \text{if } 11 \mid n \\ \frac{136}{15}n & \text{if } 15 \mid n \text{ and NOTA} \\ \frac{5}{17}n & \text{if } 17 \mid n \text{ and NOTA} \\ \frac{4}{5}n & \text{if } 5 \mid n \text{ and NOTA} \\ \frac{26}{21}n & \text{if } 21 \mid n \text{ and NOTA} \\ \frac{7}{13}n & \text{if } 13 \mid n \text{ and NOTA} \\ \frac{1}{7}n & \text{if } 7 \mid n \text{ and NOTA} \\ \frac{33}{4}n & \text{if } 4 \mid n \text{ and NOTA} \\ \frac{5}{2}n & \text{if } 2 \mid n \text{ and NOTA} \\ 7n & \text{otherwise} \end{cases} \quad (1.2)$$

(where NOTA means “None of the Above” conditions hold) is one such function. Furthermore, for any nonnegative integer  $n$  the  $R$ -orbit of  $2^n$  contains the subsequence

$$2^n, 2^{T(n)}, 2^{T^2(n)}, 2^{T^3(n)} \dots$$

and these are the only powers of two that occur.

Note that the function  $R$  given in the theorem is of the form (1.1) if we take

$$d = \text{lcm}(11, 15, 17, 5, 21, 13, 7, 4, 2) = 1021020$$

since the first condition satisfied by  $n$  will also be the first condition satisfied by  $n + dj$  for any  $j$ .

*Proof:* The proof is a straightforward application of Conway’s *Fractran* language and its mathematical consequences. We refer the reader to [2] for details. A *Fractran program* consists of a finite list of positive rational numbers,  $[r_1, \dots, r_t]$ . The state of a *Fractran machine* consists of a single positive integer

$S$ . The exponents of the primes in the prime factorization of  $S$  are used as registers for storing nonnegative integers. The program is executed by multiplying  $S$  by the first rational number in the list for which the product is a nonnegative integer (and halts if no such integer exists). Thus, each *FRACTRAN* program corresponds to a function of the form (1.1) where execution of the program corresponds to iteration of the function.

The *FRACTRAN* program

$$\left[ \frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7 \right] \tag{1.3}$$

when started with  $S = 2^n$ , will produce  $S = 2^{T(n)}$  as the next  $S$  power of 2 in the orbit. To see this, consider the flowchart for this program indicated in Figure 1. (In what follows we will only be concerned with an initial state that is a power of 2, as required.)

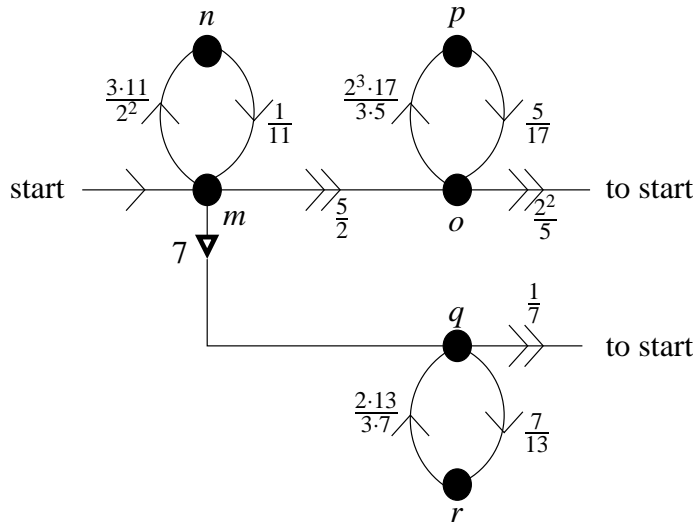


Fig. 1: A Fractran program for  $T$

The edges of the flowchart are labeled in order of decreasing priority using a single arrow, double arrow, and triangle respectively. At a given node, the current state  $S$  is multiplied by the fraction labeling the edge of highest priority for which the product is a positive integer. The powers of the primes 5, 7, 11, 13, 17 in  $S$  correspond to the nodes  $o, q, n, r, p$  respectively, a positive exponent of one of the primes indicating the program is at that node (and it is at node  $m$  if it is at no other node). The exponents of 2 and 3 in  $S$  are used as registers to compute  $T$ . We will refer to these exponents as  $\alpha$  and  $\beta$  respectively.

When the program is started with  $S = 2^n$  at node  $m$ , it will execute the loop between nodes  $m$  and  $n$  exactly  $q = \lfloor \frac{n}{2} \rfloor$  times, each time decreasing  $\alpha$  by 2 and incrementing  $\beta$ . This results in  $S = 2^{n \bmod 2} 3^q$ .

If  $n$  is odd then  $n = 2q + 1$  for some positive integer  $q$  and execution proceeds to node  $o$  where the state becomes  $S = 3^q 5$ . The loop between nodes  $o$  and  $p$  then produces  $S = 2^{3q} 5$  which is then multiplied by

$\frac{2^2}{5}$  to produce

$$S = 2^{3q+2} = 2^{(6q+4)/2} = 2^{(6q+3+1)/2} = 2^{(3(2q+1)+1)/2} = 2^{(3n+1)/2} = 2^{T(n)}$$

as required.

If  $n$  is even, then upon completion of the  $mn$  loop  $S$  is multiplied by 7 moving execution to node  $q$ . The loop between nodes  $q$  and  $r$  produces  $S = 2^q 7$  which is then multiplied by  $1/7$  to produce

$$S = 2^q = 2^{n/2} = 2^{T(n)}$$

as required.

Iteration of the function  $R$  given in the theorem starting with seed  $2^n$  corresponds exactly to execution of this *Fractran* program (the sequence of states being the  $R$ -orbit of  $2^n$ ). Since the choice of primes and algorithm used in this program was arbitrary, there are infinitely many such programs, and thus infinitely many such functions. This completes the proof.  $\square$

Theorem 1 shows the relationship between the  $R$ -orbits of two powers and the  $3x + 1$  problem. One might ask for its own sake<sup>†</sup> how the iterates of  $R$  behave for arbitrary positive integer inputs. We answer this question with the following result.

**Theorem 2** *Let  $R$  be defined as in (1.2). Then for all  $a, b, c, d, e, f, g, h \in \mathbb{N}$*

1. *for all  $m \in \mathbb{Z}^+$  with  $\gcd(m, 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17) = 1$ ,*

$$R\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g m\right) = m \cdot R\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g\right)$$

and

2. *there exists  $k \in \mathbb{N}$  such that  $R^k\left(2^a 3^b 5^c 7^d 11^e 13^f 17^g\right) = 2^j$  for some  $j$ .*

Thus if we iterate  $R$  starting with an arbitrary positive integer  $n$ , the prime factors of  $n$  that are greater than 17 are left unchanged, and the iterates of the remaining factor eventually reach a two power (after which the behavior proceeds as indicated in Theorem 1).

*Proof:* The proof of part (1) follows immediately from the definition of  $R$ , since prime factors greater than 17 are not affected when a positive integer is multiplied by any of the rational numbers listed in (1.3).

To prove part (2), let  $S$  be the set of positive integers that are not divisible by a prime greater than 17. Since no prime greater than 17 is a factor of the numerator of any fraction in (1.3),  $R$  maps elements of  $S$  to elements of  $S$ .

Let  $S'$  be the subset of  $S$  consisting of integers of the form  $2^a 3^b$  for some  $a, b \in \mathbb{N}$ . Let  $a, b \in \mathbb{N}$ . By the definition of  $R$ ,  $R^2\left(2^{a+2} 3^b\right) = 2^a 3^{b+1}$  so that  $R^{2b}\left(2^{a+2b}\right) = 2^a 3^b$ . Thus any element of  $S'$  is in the  $R$ -orbit of a power of two. Since the  $R$ -orbit of  $2^{a+2b}$  contains infinitely many terms that are powers of two by Theorem 1, so does the  $R$ -orbit of  $2^a 3^b$  for any  $a, b \in \mathbb{N}$ . Thus it suffices to show that the  $R$ -orbit of any element of  $S$  contains an element of  $S'$ .

Define  $\alpha : S \rightarrow \mathbb{N}$  by  $\alpha\left(2^{e_1} 3^{e_2} 5^{e_3} 7^{e_4} 11^{e_5} 13^{e_6} 17^{e_7}\right) = \sum_{i=2}^7 e_i$ . We argue by contradiction, and suppose that we have an element  $n$  of  $S$  so that all iterates  $R^k(n) \notin S'$ . Then all terms in the  $R$ -orbit of  $n$  are divisible

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<sup>†</sup> Thanks to the anonymous referee of an earlier draft of this paper for suggesting this line of inquiry.

by some prime in  $\{5, 7, 11, 13, 17\}$ . Thus by the definition of  $R$ , for all  $k \geq 1$ ,  $R^k(n) = r_k R^{k-1}(n)$  for some  $r_k \in \{\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}\}$ . For any  $k \in \mathbb{N}$ , if  $r_{k+1} \in \{\frac{1}{11}, \frac{136}{15}, \frac{4}{5}, \frac{26}{21}, \frac{1}{7}\}$  then

$$\alpha(R^{k+1}(n)) = \alpha(r_{k+1}R^k(n)) < \alpha(R^k(n))$$

and if  $r_{k+1} \in \{\frac{5}{17}, \frac{7}{13}\}$  then

$$\alpha(R^{k+1}(n)) = \alpha(r_{k+1}R^k(n)) = \alpha(R^k(n)).$$

So the  $R$ -orbit of  $n$  has nonincreasing values of  $\alpha$ , i.e. the sequence

$$\alpha(n), \alpha(R(n)), \alpha(R^2(n)), \dots \tag{1.4}$$

is a nonincreasing. Since none of the terms are a two power (by our assumption), (1.4) is a nonincreasing sequence of positive integers whose terms are all less than or equal to  $\alpha(n)$ . Thus there must be some  $h \geq 0$  such that  $\alpha(R^k(n)) = \alpha(R^h(n))$  for all  $k \geq h$ . So  $r_k \in \{\frac{5}{17}, \frac{7}{13}\}$  for all  $k \geq h$ . But multiplication by these values of  $r_k$  decreases the exponent of either 13 or 17 in the prime factorization of an integer, so that repeated multiplication by these fractions eventually produces a non-integer value. This contradicts our assumption and completes the proof.  $\square$

Conway [1] used an argument similar to the proof of Theorem 1 to show that there exist functions of the form (1.1) for which the fate of the orbit of an arbitrary positive integer is algorithmically undecidable. In Theorem 1 we turn this method around to obtain a positive result, and now illustrate how this result can be used to obtain mathematical results about the conjecture itself.

## 2 An Application

Let  $x_0, \dots, x_{n-1}$  be positive integers such that  $x_i = T(x_{i-1})$  for  $0 < i < n$  and  $x_0 = T(x_{n-1})$ . In this situation we say  $\{x_0, \dots, x_{n-1}\}$  is a  $T$ -cycle. If the  $3x+1$  conjecture is true, then the only  $T$ -cycle of positive integers is  $\{1, 2\}$  (the existence of any other positive integer in a  $T$ -cycle being a counterexample). Thus it is of interest to study the properties of positive integer  $T$ -cycles.

Suppose  $\{x_0, \dots, x_{n-1}\}$  is a  $T$ -cycle of positive integers with  $x_i = T(x_{i-1})$  for  $0 < i < n$  and  $x_0 = T(x_{n-1})$ . Then by Theorem 1 the  $R$ -orbit of  $2^{x_0}$  is also cyclic and contains  $\{2^{x_0}, \dots, 2^{x_{n-1}}\}$  as a subset. Thus there exists some positive integer  $t$  such that  $R^t(x_0) = x_0$ . But each application of  $R$  is simply multiplication by one of the rational numbers in  $\{\frac{1}{11}, \frac{136}{15}, \frac{5}{17}, \frac{4}{5}, \frac{26}{21}, \frac{7}{13}, \frac{1}{7}, \frac{33}{4}, \frac{5}{2}, 7\}$  so that we must have

$$x_0 = R^t(x_0) = \left(\frac{1}{11}\right)^a \left(\frac{136}{15}\right)^b \left(\frac{5}{17}\right)^c \left(\frac{4}{5}\right)^d \left(\frac{26}{21}\right)^e \left(\frac{7}{13}\right)^f \left(\frac{1}{7}\right)^g \left(\frac{33}{4}\right)^h \left(\frac{5}{2}\right)^i 7^j x_0$$

for some nonnegative integers  $a, b, c, d, e, f, g, h, i, j$  with  $a+b+c+d+e+f+g+h+i+j=t$ . Collecting prime factors on the right hand side and dividing by  $x_0$  gives us

$$2^{3b+2d+e-2h-i} 3^{-b-e+h} 5^{-b+c-d+i} 7^{-e+f-g+j} 11^{-a+h} 13^{e-f} 17^{b-c} = 1.$$

This yields the system of linear equations

$$\begin{aligned}
 3b + 2d + e - 2h - i &= 0 \\
 -b - e + h &= 0 \\
 -b + c - d + i &= 0 \\
 -e + f - g + j &= 0 \\
 -a + h &= 0 \\
 e - f &= 0 \\
 b - c &= 0
 \end{aligned}$$

which is equivalent to the system

$$\begin{aligned}
 a &= 2c + i & (2.1) \\
 b &= c \\
 d &= i \\
 e &= c + i \\
 f &= c + i \\
 g &= j \\
 h &= 2c + i.
 \end{aligned}$$

Now define  $O = \{i : x_i \text{ is odd}\}$  and  $\mathcal{E} = \{i : x_i \text{ is even}\}$  and let  $k = |O|$  so that  $|\mathcal{E}| = n - k$ . Then as explained in the proof of Theorem 1 we see that

$$\begin{aligned}
 i &= k & (2.2) \\
 j &= n - k \\
 c &= \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor \\
 a &= \sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor
 \end{aligned}$$

Substituting (2.2) into  $a = 2c + i$  from (2.1) we obtain

$$\sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor = 2 \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor + k. \quad (2.3)$$

But  $\sum_{i=0}^{n-1} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor + \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor$ . Substituting this into (2.3) and simplifying proves

**Corollary 1** *If  $\{x_0, \dots, x_{n-1}\}$  is a  $T$ -cycle of positive integers and*

$$O = \{i : x_i \text{ is odd}\} \text{ and } \mathcal{E} = \{i : x_i \text{ is even}\}$$

*then*

$$\sum_{i \in \mathcal{E}} \left\lfloor \frac{x_i}{2} \right\rfloor = \sum_{i \in O} \left\lfloor \frac{x_i}{2} \right\rfloor + k.$$

It should be noted that this formula can be proven directly from the known relationship

$$\sum_{i \in \mathcal{E}} x_i = \sum_{i \in \mathcal{O}} x_i + k \quad (2.4)$$

(obtained by noticing that  $\{x_0, \dots, x_{n-1}\} = \{T(x_0), \dots, T(x_{n-1})\}$  so that  $\sum x_i = \sum T(x_i)$  and thus  $\sum_{i \in \mathcal{E}} x_i + \sum_{i \in \mathcal{O}} x_i = \sum_{i \in \mathcal{O}} \frac{3x_i+1}{2} + \sum_{i \in \mathcal{E}} \frac{x_i}{2}$  which can be solved to obtain (2.4)). However, the method used here reveals the results of the Corollary without specifically searching for those results. Thus this method provides a general approach for discovering new mathematical results by simply coding different algorithms for computing  $T$  (or any other computable integer function) and solving a simple linear system.

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