

# On a Unimodality Conjecture in Matroid Theory

W. M. B. Dukes

► **To cite this version:**

W. M. B. Dukes. On a Unimodality Conjecture in Matroid Theory. *Discrete Mathematics and Theoretical Computer Science, DMTCS*, 2002, 5, pp.181-190. <hal-00958981>

**HAL Id: hal-00958981**

**<https://hal.inria.fr/hal-00958981>**

Submitted on 13 Mar 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On a Unimodality Conjecture in Matroid Theory

W.M.B. Dukes

School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

received Aug 6, 2001, accepted Jul 28, 2002.

---

A certain unimodal conjecture in matroid theory states the number of rank- $r$  matroids on a set of size  $n$  is unimodal in  $r$  and attains its maximum at  $r = \lfloor n/2 \rfloor$ . We show that this conjecture holds up to  $r = 3$  by constructing a map from a class of rank-2 matroids into the class of loopless rank-3 matroids. Similar inequalities are proven for the number of non-isomorphic loopless matroids, loopless matroids and matroids.

**Keywords:** Matroid Theory, Unimodal Conjecture, Rank-2 matroids, Rank-3 matroids

---

## 1 Introduction

Let us first recall some elementary definitions (further details may be found in Oxley (1), Welsh (2) and the excellent appendix of White (3)). Let  $S_n$  be a finite set of size  $n$ . A matroid  $M$  on the ground set  $S_n$  is a collection of subsets  $I(M)$  of  $S_n$  satisfying

- $I(M) \neq \emptyset$ ,
- if  $X \in I(M)$  and  $Y \subseteq X$  then  $Y \in I(M)$ ,
- if  $X, Y \in I(M)$  with  $|X| = |Y| + 1$ , then there exists  $x \in X \setminus Y$  such that  $Y \cup \{x\} \in I(M)$ .

Sets in  $I(M)$  are called *independent sets*. The *rank* of a set  $X \subseteq S_n$ , denoted  $r(X)$ , is the size of the largest independent set which it contains. The rank of the matroid  $r(M) := r(S_n)$ . A set  $X$  is *closed* (or termed a *flat*) if  $r(X \cup \{x\}) = r(X) + 1$  for all  $x \in S_n \setminus X$ . We denote by  $\mathcal{F}(M)$  the closed sets of  $M$ . The *loops* of  $M$  are the elements of the rank-0 flat. Also note that  $x \in S_n$  is a loop if it is not contained in any of the independent sets of  $M$ .

A certain unimodal conjecture in matroid theory states that the sequence of the number of non-isomorphic rank- $r$  matroids on  $S_n$ ,  $\{f_r(n) : 1 \leq r \leq n\}$ , is unimodal in  $r$  and attains its maximum at  $r = \lfloor n/2 \rfloor$  (see Oxley (1) or Welsh (4) p.300). It is easily seen that  $f_1(n) \leq f_2(n)$  holds since  $f_1(n) = n$  and  $f_2(n) = p(1) + \dots + p(n) - n$ , where  $p(n)$  is the number of integer partitions of  $n$ . The step between rank-2 and rank-3 is not as clear since the exact value of  $f_3(n)$  remains unknown. We show, through construction of a map between a class of rank-2 matroids and loopless rank-3 matroids and known values of these numbers from the On-line Encyclopedia of Integer Sequences, that this unimodal conjecture holds for

these rank-2 versus rank-3 matroids. Furthermore, we show the corresponding inequalities hold for the number of rank-2, 3 non-isomorphic loopless matroids,  $g_2(n) \leq g_3(n)$ , loopless matroids,  $c_2(n) \leq c_3(n)$ , and matroids,  $m_2(n) < m_3(n)$ .

Let  $b_i(n)$  be the number of partitions of the set  $S_n$  into  $i$  parts and  $b(n)$  be the  $n^{\text{th}}$  Bell number. Let  $p_i(n)$  be the number of partitions of the integer  $n$  into  $i$  parts. The number of rank-2 matroids can be enumerated through considering the points and lines of the associated geometry. We have  $c_2(n) = b(n) - 1$ ,  $g_2(n) = p(n) - 1$  and  $m_2(n) = b(n+1) - 2^n$  (for proofs see Dukes (5)). The main results of this paper are given in Theorems 2.5, 2.6, 2.11 and 2.12.

## 2 Mapping rank-2 to rank-3 matroids

Let  $\mathcal{M}_r(n)$  be the collection of rank- $r$  matroids on  $S_n$ . Let  $\mathcal{A}_r(n)$  be the collection of rank- $r$  matroids on  $S_n$  with at least one loop and  $\mathcal{B}_r(n) := \mathcal{M}_r(n) \setminus \mathcal{A}_r(n)$ . We define the map  $\sigma : \mathcal{A}_2(n) \rightarrow \mathcal{B}_3(n)$  as follows: given  $M \in \mathcal{A}_2(n)$  with loops  $F_0$  and rank-1 flats  $\mathcal{F}_1(M) = \{F_0 \cup F_1, \dots, F_0 \cup F_m\}$  define  $M' = \sigma(M)$  as:

$$\begin{aligned} \mathcal{F}_0(M') &:= \{\emptyset\} \\ \mathcal{F}_1(M') &:= \{F_0, F_1, \dots, F_m\} \\ \mathcal{F}_2(M') &:= \{F_0 \cup F_i \mid 1 \leq i \leq m\} \cup \{F_1 \cup \dots \cup F_m\} \\ \mathcal{F}_3(M') &:= \{S_n\}. \end{aligned}$$

It is easily checked that these collections of flats satisfy the axioms for a loopless rank-3 matroid. For  $M \in \mathcal{M}_r(n)$ , let us write  $d(M)$  for the number of rank-1 flats of  $M$  (which we will refer to as the *degree* of  $M$ ). Let us mention that for any loopless matroid  $M$ , the rank-1 flats of  $M$  partition the ground set. Similarly, for any matroid, the rank-1 flats partition the ground set less the set of loops. Also note that in the collection  $\mathcal{F}_2(M')$ , there are precisely  $d(M)$  sets containing  $F_0$ , 2 sets containing  $F_i$  (for any  $1 \leq i \leq d(M)$ ) and one set containing  $F_i \cup F_j$  (for all  $0 \leq i \neq j \leq d(M)$ ).

The following lemma shows that to each rank-2 matroid with at least one loop, there corresponds a rank-3 loopless matroid (although not necessarily unique). The following lemma classifies those matroids which map to a unique loopless matroid in  $\mathcal{B}_3(n)$  and those which do not.

**Lemma 2.1** *Let  $M_1, M_2 \in \mathcal{A}_2(n)$  be such that  $\mathcal{F}_0(M_1) = \{F_0^{(1)}\}$ ,  $\mathcal{F}_0(M_2) = \{F_0^{(2)}\}$ ,  $\mathcal{F}_1(M_1) = \{F_0^{(1)} \cup F_1^{(1)}, \dots, F_0^{(1)} \cup F_{d(M_1)}^{(1)}\}$  and  $\mathcal{F}_1(M_2) = \{F_0^{(2)} \cup F_1^{(2)}, \dots, F_0^{(2)} \cup F_{d(M_2)}^{(2)}\}$ . Then  $\sigma(M_1) = \sigma(M_2)$  if and only if  $d(M_1) = d(M_2) = 2$  and*

$$\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\}.$$

**Proof:** ONLY IF: Let  $M_1, M_2 \in \mathcal{A}_2(n)$  be such that  $M_1 \neq M_2$  and  $\sigma(M_1) = \sigma(M_2)$ . Let  $M'_1 := \sigma(M_1)$  and  $M'_2 := \sigma(M_2)$ . Then we must have  $\mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2)$  and  $\mathcal{F}_2(M'_1) = \mathcal{F}_2(M'_2)$ . Now  $\mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2) \Rightarrow d(M_1) = d(M_2)$  and  $\{F_i^{(1)}\}_{i=0}^{d(M_1)} = \{F_i^{(2)}\}_{i=0}^{d(M_2)}$ . If  $d(M_1) > 2$  then we must have  $F_0^{(1)} = F_0^{(2)}$  which would imply  $M_1 = M_2$ . Hence  $d(M_1) = 2 = d(M_2)$ . This gives  $\mathcal{F}_2(M'_1) = \{F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}\} = \mathcal{F}_2(M'_2)$  only if  $\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\}$ .

IF: This is trivial as  $\{F_0^{(1)}, F_1^{(1)}, F_2^{(1)}\} = \{F_0^{(2)}, F_1^{(2)}, F_2^{(2)}\}$  gives  $\mathcal{F}_1(M'_1) = \mathcal{F}_1(M'_2)$  and

$$\begin{aligned} \mathcal{F}_2(M'_1) &:= \left\{ F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)}, F_0^{(1)} \cup F_1^{(1)} \right\} \\ &= \left\{ F_0^{(2)} \cup F_1^{(2)}, F_0^{(2)} \cup F_1^{(2)}, F_0^{(2)} \cup F_1^{(2)} \right\} =: \mathcal{F}_2(M'_2). \end{aligned}$$

□

Thus it is seen for each matroid  $M \in \sigma(\mathcal{A}_2(n))$  such that  $d(M) = 2$ , there are precisely three different matroids  $M_1, M_2, M_3 \in \mathcal{A}_2(n)$  such that  $\sigma(M_1) = \sigma(M_2) = \sigma(M_3) = M$ .

**Lemma 2.2** For all  $n \geq 3$ ,  $c_3(n) \geq b(n+1) - b(n) - 3^{n-1}$ .

**Proof:** We show that the number of unique matroids in the image of  $\mathcal{A}_2(n)$  under  $\sigma$  is given by  $b(n+1) - b(n) - 3^{n-1}$ , thereby lower-bounding  $c_3(n)$ . In the enumeration below, we divide the matroids to be counted in the image into two classes, those matroids  $M$  with  $d(M) = 2$  and those with  $d(M) > 2$ . The former class projects different matroids to the same matroid in  $\mathcal{B}_3(n)$  and through the use of the previous lemma we take care of this over-counting, hence

$$\begin{aligned} &\#\{\sigma(M) | M \in \mathcal{A}_2(n)\} \\ &= \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} + \sum_{i=3}^n \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} \\ &= \frac{1}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} + \sum_{i=3}^n \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} \\ &= \sum_{i=2}^n \#\{\sigma(M) | M \in \mathcal{A}_2(n) \text{ and } d(M) = i\} - \frac{2}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} \\ &= \#\mathcal{A}_2(n) - \frac{2}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} \\ &= b(n+1) - 2^n - (b(n) - 1) - \frac{2}{3} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\}. \end{aligned}$$

Note that  $\#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} = \sum_{l=2}^{n-1} \binom{n}{l} b_2(l)$  and  $b_2(l) = \frac{1}{2} \sum_{j=1}^{l-1} \binom{l}{j} = 2^{l-1} - 1$ , giving:

$$\begin{aligned} \#\{M | M \in \mathcal{A}_2(n) \text{ and } d(M) = 2\} &= \sum_{l=2}^{n-1} \binom{n}{l} (2^{l-1} - 1) \\ &= \frac{1}{2} (3^n - 2^n - 2n - 1) - (2^n - n - 2) \\ &= \frac{3}{2} (3^{n-1} - 2^n + 1). \end{aligned}$$

Thus

$$\begin{aligned} \#\{\sigma(M) | M \in \mathcal{A}_2(n)\} &= b(n+1) - 2^n - b(n) + 1 - \frac{2}{3} \frac{3}{2} (3^{n-1} - 2^n + 1) \\ &= b(n+1) - b(n) - 3^{n-1}. \end{aligned}$$

□

The corresponding inequality for the number of non-isomorphic loopless matroids is proved in Lemma 2.3. We do this in a similar manner as before, by showing that each rank-2 matroid (which is not a loopless matroid) of degree greater than 3 corresponds uniquely to a rank-3 loopless matroid.

**Lemma 2.3** For all  $n \geq 4$ ,  $g_3(n) \geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$ .

**Proof:** We show the number of non-isomorphic matroids in the image of  $\mathcal{A}_2(n)$  under  $\sigma$  is given by  $\sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$  which lower bounds  $g_3(n)$ .

Let us identify  $\mathcal{A}_2^*(n) \subseteq \mathcal{A}_2(n)$  by placing an ordering on the elements of  $S_n = \{x_1, \dots, x_n\}$ . Given  $M \in \mathcal{A}_2(n)$  with  $d(M) = m$ , loops  $F_0$  and rank-1 flats  $\{F_0 \cup F_1, \dots, F_0 \cup F_m\}$ , let  $M \in \mathcal{A}_2^*(n)$  if and only if  $F_0 = \{x_1, \dots, x_{|F_0|}\}$ ,  $F_1$  contains the next  $|F_1|$  elements of  $S_n$ , i.e.  $\{x_{|F_0|+1}, \dots, x_{|F_0|+|F_1|}\}$  and so forth. Define

$$\mathcal{T}(n) := \{M \in \mathcal{A}_2^*(n) \mid d(M) = 2 \text{ and } |F_0| \leq |F_1| \leq |F_2|\}$$

and for  $3 \leq i \leq n-1$ ,  $3 \leq j \leq i$ , define

$$\Omega_{i,j}(n) := \{M \in \mathcal{A}_2^*(n) \mid d(M) = j, |F_0| = n-i \text{ and } |F_1| \leq \dots \leq |F_j|\}.$$

Let us now write

$$\mathcal{A}_2^{**}(n) := \mathcal{T}(n) \cup \bigcup_{i=3}^{n-1} \bigcup_{j=3}^i \Omega_{i,j}(n) \subseteq \mathcal{A}_2^*(n).$$

It is obvious that no two matroids in  $\mathcal{T}(n)$  are isomorphic to one-another. Similarly with  $\Omega_{k,l}(n)$ . We have simply reduced our class of matroids from  $\mathcal{A}_2(n)$  to  $\mathcal{A}_2^{**}(n)$  in the same manner as one moves from the set of partitions of a finite set of size  $n$  to the set of integer partitions of  $n$ .

The unions in the definition of  $\mathcal{A}_2^{**}(n)$  are strictly disjoint and no isomorphisms may occur between matroids in different classes or matroids in the same class. The same is true of the image of  $\mathcal{A}_2^{**}(n)$  under the map  $\sigma$ . We may directly enumerate the number of non-isomorphic matroids in  $\mathcal{B}_3(n)$  in the image of  $\mathcal{A}_2^{**}(n)$  under  $\sigma$  as

$$p_3(n) + \sum_{i=3}^{n-1} \sum_{j=3}^i p_j(i).$$

The rightmost term is bounded below;

$$\begin{aligned}
 \sum_{i=3}^{n-1} \sum_{j=3}^i p_j(i) &= \sum_{i=2}^{n-1} \{p(i) - p_1(i) - p_2(i)\} \\
 &= \sum_{i=3}^{n-1} \{p(i) - 1 - \lfloor i/2 \rfloor\} \\
 &= -(n-3) + \sum_{i=3}^{n-1} p(i) - \sum_{i=3}^{n-1} \lfloor i/2 \rfloor \\
 &= \begin{cases} \sum_{i=1}^{n-1} p(i) - \frac{n(n+2)}{4} + 1, & n \text{ even,} \\ \sum_{i=1}^{n-1} p(i) - \frac{(n+1)^2}{4} + 1, & n \text{ odd,} \end{cases} \\
 &\geq \sum_{i=1}^{n-1} p(i) - \frac{(n+1)^2}{4},
 \end{aligned}$$

for all  $n \geq 2$ . As for  $p_3(n)$ , from Hall (6) [p.32], we have

$$\begin{aligned}
 p_3(n) &= \begin{cases} \lfloor n^2/12 \rfloor, & \text{for } n \not\equiv 3 \pmod{6}, \\ \lceil n^2/12 \rceil, & \text{for } n \equiv 3 \pmod{6}, \end{cases} \\
 &\geq \frac{n^2}{12} - 1,
 \end{aligned}$$

and so

$$\begin{aligned}
 g_3(n) &\geq \sum_{i=1}^{n-1} p(i) + \frac{n^2}{12} - \frac{(n+1)^2}{4} - 1 \\
 &= \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1.
 \end{aligned}$$

□

## 2.1 Matroids

The following lemma is needed in order to support the theorem which follows it.

**Lemma 2.4** For all  $n \geq 1$ ,  $b(n+1) - 2^n \geq 2^n - (1+n)$ .

**Proof:** We have that  $b(i) \geq 2$  for all  $i \geq 2$ . Since  $n \geq 2$ , it follows that

$$\begin{aligned}
 b(n+1) - 2^n &= \sum_{i=0}^n \binom{n}{i} (b(i) - 1) \\
 &\geq \sum_{i=2}^n \binom{n}{i} 1 \\
 &= 2^n - (1+n).
 \end{aligned}$$

□

**Theorem 2.5** For all  $n > 4$ ,  $c_3(n) \geq c_2(n)$ .

**Proof:** From Lemma 2.2 we have that  $c_3(n) \geq b(n+1) - b(n) - 3^{n-1}$ . We know  $c_2(n) = b(n) - 1$ . It suffices to show that

$$b(n+1) - b(n) - 3^{n-1} \geq b(n) - 1$$

for all  $n > 4$ . Let us look at the value  $b(n+1) - 2b(n) - 3^{n-1} + 2^{n-1}$ :

$$\begin{aligned} b(n+1) - 2b(n) - 3^{n-1} + 2^{n-1} &= \sum_{i=0}^{n-1} \binom{n}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} b(i) - \sum_{i=0}^{n-1} \binom{n-1}{i} 2^i + \sum_{i=0}^{n-1} \binom{n-1}{i} \\ &= \sum_{i=0}^{n-2} \binom{n-1}{i} (b(i+1) - 2^i) \end{aligned}$$

and using Lemma 2.4,

$$\begin{aligned} &\geq \sum_{i=1}^{n-2} \binom{n-1}{i} (2^i - (1+i)) \\ &= 3^{n-1} - 2^n + 1 - \sum_{i=1}^{n-2} \binom{n-1}{i} i \\ &= 3^{n-1} - (n+3)2^{n-2} + n. \end{aligned}$$

The problem has been reduced to showing  $3^{n-1} - (n+3)2^{n-2} + n \geq 2^{n-1} - 1$  for all  $n \geq 5$ , which is easily shown by induction.  $\square$

We now show the number of rank-3 matroids dominates the number of rank-2 matroids by using two things: the first is the result proved previously, that the number of rank-3 loopless matroids is at least as large as the number of rank-2 loopless matroids; the second is the first few known values of the numbers  $c_2(n)$  and  $c_3(n)$ . The latter knowledge makes the inequality strict.

**Theorem 2.6** For all  $n \geq 5$ ,  $m_3(n) \geq m_2(n)$ .

**Proof:** The number of rank- $r$  matroids on  $S_n$  is related to the number of loopless matroids on  $S_n$  by

$$m_r(n) = \sum_{i=r}^n \binom{n}{i} c_r(i).$$

In Theorem 2.5 we showed that  $c_3(n) \geq c_2(n)$  for all  $n \geq 5$ . Replacing  $r = 3$  in the above expression and using the first few values of  $c_3(n)$  (taken from row 3, table A058710, of Sloane (7)),

$$\begin{aligned} m_3(n) &= \sum_{i=3}^n \binom{n}{i} c_3(i) \\ &= 1 \binom{n}{3} + 11 \binom{n}{4} + 106 \binom{n}{5} + 1232 \binom{n}{6} + \sum_{i=7}^n \binom{n}{i} c_3(i) \\ &\geq 1 \binom{n}{3} + 11 \binom{n}{4} + 106 \binom{n}{5} + 1232 \binom{n}{6} + \sum_{i=7}^n \binom{n}{i} c_2(i) \\ &= 830 \binom{n}{6} + 75 \binom{n}{5} - 3 \binom{n}{4} - 3 \binom{n}{3} - \binom{n}{2} + \sum_{i=2}^n \binom{n}{i} c_2(i) \end{aligned}$$

A simple check shows that  $830\binom{n}{6} + 75\binom{n}{5} - 3\binom{n}{4} - 3\binom{n}{3} - \binom{n}{2}$  is greater than zero and increasing for all  $n \geq 7$ . From Table 1 (see Appendix), the result is also seen to hold for  $n = 5, 6$ . Equality holds only for  $n = 5$ , for all other values of  $n$  the inequality is strict.  $\square$

## 2.2 Non-isomorphic matroids

Proving the corresponding inequalities for the non-isomorphic numbers is more difficult. We first prove several lemmas related to the numbers  $p(n)$  which we will need in the proofs of the two remaining theorems.

**Lemma 2.7** For all  $n \geq 1$ ,  $p(n+1) \geq p(n) + \lfloor \frac{n+1}{2} \rfloor$ .

**Proof:** The number of partitions of the integer  $n+1$  whose first part contains the integer 1 is precisely  $p(n)$ . The number beginning with  $i$ , for any  $2 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$  is at least 1 since we can have the partition  $n+1 = i + (n+1-i)$ . Also, the number  $n+1$  is a partition by itself, hence,

$$\begin{aligned} p(n+1) &\geq p(n) + \left( \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \right) + 1 \\ &= p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor. \end{aligned}$$

$\square$

**Lemma 2.8** For all  $n \geq 1$ ,  $p(n) \geq 1 + \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil \geq \frac{n^2+3}{4}$ .

**Proof:** From Lemma 2.7 we have

$$p(n+1) \geq p(n) + \left\lfloor \frac{n+1}{2} \right\rfloor$$

for all  $n \geq 1$ . Applying this lemma recursively gives

$$\begin{aligned} p(n) &\geq p(n-1) + \left\lfloor \frac{n}{2} \right\rfloor \\ &\geq p(n-2) + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ &\vdots \\ &\geq p(1) + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ &\geq 1 + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned} \tag{1}$$

Now we wish to evaluate the sum  $\sum_{i=2}^n \lfloor \frac{i}{2} \rfloor$ . Let  $n = 2m+1$  for some  $m \geq 1$ , then

$$\begin{aligned} \sum_{i=2}^n \left\lfloor \frac{i}{2} \right\rfloor &= \sum_{i=2}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor \\ &= \sum_{i=1}^m \left\lfloor \frac{2i}{2} \right\rfloor + \left\lfloor \frac{2i+1}{2} \right\rfloor \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \end{aligned}$$



For the  $n = 2m$  case with  $m \geq 1$ , we simply remove the last term in the previous expression, thus

$$\begin{aligned} \sum_{i=2}^n \left\lfloor \frac{i}{2} \right\rfloor &= \sum_{i=2}^{2m+1} \left\lfloor \frac{i}{2} \right\rfloor - \left\lfloor \frac{2m+1}{2} \right\rfloor \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil. \end{aligned}$$

Continuing to the inequality in Equation 1 above,

$$\begin{aligned} p(n) &\geq 1 + \left\lfloor \frac{1+1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ &= 1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil, \end{aligned}$$

for all  $n \geq 1$ . If  $n$  is even, then  $\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \frac{n^2}{4}$ . If  $n$  is odd, then  $\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil = \frac{n-1}{2} \frac{n+1}{2}$ . In either case,  $1 + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \geq 1 + \frac{n^2-1}{4}$ .  $\square$

**Lemma 2.9** For all  $n \geq 5$ ,  $p(n+1) < 2p(n) - \frac{n+2}{3}$ .

**Proof:** Let  $x_1 + x_2 + \cdots + x_k = n+1$  be a partition of  $n+1$  with  $1 \leq x_1 < \cdots < x_k$ . There are precisely  $p(n)$  partitions with  $x_1 = 1$ , since  $x_2 + \cdots + x_k = n+1-1$ .

For all those partitions with  $x_1 \geq 2$ , we see that reducing  $x_1$  by 1 will yield a partition of  $n$ . Thus an upper bound on the number beginning with  $x_2 \geq 2$  is  $p(n)$ . For all partitions starting with  $x_1 = 2$ , we see that  $x_2 \neq 1$ , thus we may remove all those sequences with  $x_2 = 1 \leq x_3 \leq \cdots \leq x_k$  such that  $2 + 1 + x_3 + \cdots + x_k = n+1$ . Reformulated, this means all those partitions with  $x_3 + \cdots + x_k = n-2$  and  $1 \leq x_3 \leq \cdots \leq x_k$  of which there are  $p(n-2)$ .

Thus we see that  $p(n+1) < p(n) + p(n) - p(n-2) = 2p(n) - p(n-2)$ . From lemma 2.8 we know that for  $n \geq 3$ ,

$$p(n-2) \geq \frac{(n-2)^2 + 3}{4} = \frac{n^2 - 4n + 7}{4}.$$

Now, we see that the simple inequality  $(3n-13)(n-1) \geq 0$  holds for all  $n \geq \frac{13}{3}$ , i.e.  $\frac{(n^2-4n+7)}{4} \geq \frac{(n+2)}{3}$ . From above, this gives

$$\begin{aligned} p(n+1) &< 2p(n) - p(n-2) \\ &\leq 2p(n) - \frac{(n^2 - 4n + 7)}{4} \\ &\leq 2p(n) - \frac{(n+2)}{3}, \end{aligned}$$

for all  $n \geq 5$  and we are done. A check of the first few values of  $p(n)$  shows the stated inequality to hold for all  $n \geq 2$ .  $\square$

**Lemma 2.10** For all  $n \geq 7$ ,  $\sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2 + 6n + 3)$ .

**Proof:** By simple induction. The result is true for  $n = 7$  since  $p(1) + p(2) + \dots + p(6) = 30$  and  $p(7) + \frac{1}{12}(2(7)^2 + 6(7) + 3) < 27$ . Suppose it to be true for some  $n = m \geq 7$ , then:

$$\begin{aligned} \sum_{i=1}^m p(i) &= p(m) + \sum_{i=1}^{m-1} p(i) \\ &\geq p(m) + p(m) + \frac{1}{12}(2m^2 + 6m + 3) \\ &= 2p(m) + \frac{1}{12}(2m^2 + 6m + 3) \end{aligned}$$

and using Lemma 2.9,

$$\begin{aligned} &> p(m+1) + \frac{m+2}{3} + \frac{1}{12}(2m^2 + 6m + 3) \\ &= p(m+1) + \frac{1}{12}(2(m+1)^2 + 6(m+1) + 3). \end{aligned}$$

□

**Theorem 2.11** For all  $n \geq 5$ ,  $g_3(n) \geq g_2(n)$ .

**Proof:** We have that  $g_2(n) = p(n) - 1$ . Also, we know from Theorem 2.6 that  $g_3(n) \geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1$ . From Lemma 2.10, we have  $\sum_{i=1}^{n-1} p(i) > p(n) + \frac{1}{12}(2n^2 + 6n + 3)$  for all  $n \geq 7$ . Combining these facts gives

$$\begin{aligned} g_3(n) &\geq \sum_{i=1}^{n-1} p(i) - \frac{1}{12}(2n^2 + 6n + 3) - 1 \\ &\geq p(n) - 1 \end{aligned}$$

which is  $g_2(n)$ . From Table 1, the result is also seen to hold for  $n = 5, 6$ .

□

**Theorem 2.12** For all  $n \geq 5$ ,  $f_3(n) \geq f_2(n)$ .

**Proof:** The number of non-isomorphic rank-3 matroids on  $S_n$  in terms of loopless non-isomorphic rank-3 matroids is given through the relation  $f_3(n) = \sum_{i=3}^n g_3(i)$  for all  $n \geq 3$ . The value  $f_2(n) = p(1) + p(2) + \dots + p(n) - n$  for all  $n \geq 2$ . From Theorem 2.11 we have  $g_3(n) \geq g_2(n)$  for all  $n \geq 7$ . Applying the above expression for  $f_3(n)$ , using the known value for  $g_3(n)$  (from Sloane (7), row 3 of A058716) and assuming  $n \geq 7$ ,

$$\begin{aligned} f_3(n) &= 38 + \sum_{i=7}^n g_3(i) \\ &> 23 + \sum_{i=7}^n g_2(i) \\ &= \sum_{i=2}^n g_2(i) \end{aligned}$$

which is precisely  $f_2(n)$ . From Table 1, the result is seen to hold for  $n = 5, 6$ . Note that the above inequality is strict for  $n \geq 6$  and equality holds only for  $n = 5$ .

□

Note that, by duality, an immediate Corollary of Theorems 2.6 and 2.12 is the following.

**Corollary 2.13** For all  $n \geq 6$ ,

$$\begin{aligned} f_n(n) &\leq f_{n-1}(n) \leq f_{n-2}(n) \leq f_{n-3}(n) \\ m_n(n) &\leq m_{n-1}(n) \leq m_{n-2}(n) \leq m_{n-3}(n). \end{aligned}$$

## Acknowledgements

The author would like to thank the anonymous referees for their helpful comments.

## References

- [1] J.G. Oxley. *Matroid Theory*. Oxford University Press, first edition, 1992.
- [2] D.J.A. Welsh. *Matroid Theory*. Academic Press, 1<sup>st</sup> edition, 1976.
- [3] Neil White. *Theory of Matroids*. Cambridge University Press, 1<sup>st</sup> edition, 1986.
- [4] D.J.A. Welsh. *Combinatorial Mathematics and its Applications*. Academic Press, 1<sup>st</sup> edition, 1971.
- [5] W.M.B. Dukes. Enumerating Low Rank Matroids and Their Asymptotic Probability of Occurrence. Technical Report DIAS-STP-01-10, Dublin Institute for Advanced Studies, 2001.
- [6] M. Hall Jr. *Combinatorial Theory*. Wiley, 2<sup>nd</sup> edition, 1986.
- [7] N.J.A. Sloane. The On-line Encyclopedia of Integer Sequences, published electronically at <http://www.research.att.com/~njas/sequences>, 2001.

## Appendix

$n$	2	3	4	5	6	7	8	OLEIS Number	Row
$g_2(n)$	1	2	4	6	10	14	21	A058716	2
$g_3(n)$		1	3	9	25	70	217	A058716	3
$f_2(n)$	1	3	7	13	23	37	58	A053534	2
$f_3(n)$		1	4	13	38	108	325	A053534	3
$c_2(n)$	1	4	14	31	202	876	4139	A058710	2
$c_3(n)$		1	11	106	1232	22172	803583	A058710	3
$m_2(n)$	1	7	36	171	813	4012	20891	A058669	2
$m_3(n)$		1	15	171	2053	33442	1022217	A058669	3

Table 1: Known values for the number of rank-2 and rank-3 matroids taken from Sloane (7).