The b-chromatic number of power graphs

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Discrete Mathematics and Theoretical Computer Science **6**, 2003, 0045–054

The b-chromatic number of some power graphs

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received Jul 19, 2002, revised Oct 29, 2002, accepted Apr 22, 2003.

Let G be a graph on vertices $v_1, v_2, \dots, v_n$. The b-chromatic number of G is defined as the maximum number k of colors that can be used to color the vertices of G, such that we obtain a proper coloring and each color i, with $1 \leq i \leq k$, has at least one representant x_i adjacent to a vertex of every color j, $1 \leq j \neq i \leq k$. In this paper, we give the exact value for the b-chromatic number of power graphs of a path and we determine bounds for the b-chromatic number of power graphs of a cycle.

Keywords: b-chromatic number, coloring, cycle, path, power graphs

1 Introduction

We consider graphs without loops or multiple edges. Let G be a graph with a vertex set V and an edge set E. We denote by $d(x)$ the degree of the vertex x in G, and by $dist_G(x,y)$ the distance between two vertices x and y in G. The *p-th power graph* G^p is a graph obtained from G by adding an edge between every pair of vertices at distance p or less, with $p \geq 1$. It is easy to see that $G^1 = G$. In the literature, power graphs of several classes have been investigated [2, 3, 8]. In this note we study a vertex coloring of power graphs. The power graph of a path and the power graph of a cycle can be also considered as respectively subclasses of *distance graphs* and *circulant graphs*. The *distance graph* $G(D)$ with distance set $D = \{d_1, d_2, \dots\}$ has the set Z of integers as vertex set, with two vertices $i, j \in Z$ adjacent if and only if $|i-j| \in D$. The *circulant graph* can be defined as follows. Let n be a natural number and let $S = \{k_1, k_2, \dots, k_r\}$ with $k_1 < k_2 < \dots < k_r \leq n/2$. Then the vertex set of the circulant graph $G(n,S)$ is $\{0, 1, \dots, n-1\}$ and the set of neighbors of the vertex i is $\{(i \pm k_j) mod\ n | j = 1, 2 \dots, r\}$.

The study of distance graphs was initiated by Eggleton and al. [4]. Recently, the problem of coloring of this class of graphs has attracted considerable attention, see e.g. [12, 13]. Circulant graphs have been extensively studied and have a vast number of applications to multicomputer networks and distributed computation (see [1, 10]). The special cases we consider are the distance graph $G(D)$ with finite distance set $D = \{1, 2, \dots, p\}$ which is isomorphic to the *p-th* power graph of a path and the circulant graph $G(n,S)$ with $S = \{1, 2, \dots, p\}$ which is isomorphic to the *p-th* power graph of a cycle.

A *k-coloring* of G is defined as a function c on $V(G) = \{v_1, v_2, \dots, v_n\}$ into a set of colors $C = \{1, 2, \dots, k\}$ such that for each vertex v_i, with $1 \le i \le n$, we have $c_{v_i} \in C$. A *proper k-coloring* is a k-coloring satisfying the condition $c_x \neq c_y$ for each pair of adjacent vertices $x, y \in V(G)$. A *dominating proper k-coloring* is a proper k-coloring satisfying the following property P: for each i, $1 \le i \le k$, there exists a vertex x_i of color i such that, for each j, with $1 \le j \neq i \le k$, there exists a vertex y_j of color j adjacent to x_i. A set of vertices satisfying the property P is called a *dominating system*. Each vertex of a dominating system is called a *dominating vertex*. The *b-chromatic number* $\varphi(G)$ of a graph G is defined as the maximum k such that G admits a dominating proper k-coloring.

The b-chromatic number was introduced in [7]. The motivation, similarly as for the previously studied *achromatic number* (cf. e.g. [5, 6]), comes from algorithmic graph theory. The achromatic number $\psi(G)$ of a graph G is the largest number of colors which can be assigned to the vertices of G such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph G using $k > \chi(G)$ colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by $\varphi(G)$ (these recolorings are discussed in [7] and [11]). Thus $\varphi(G) \le \psi(G)$ (also given in [7]). From this point of view, both complexity results and tight bounds for the b-chromatic number are interesting. The following bounds of b-chromatic number are already presented in [7].

Proposition 1 *Assume that the vertices $x_1, x_2, \dots, x_n$ of G are ordered such that $d(x_1) \ge d(x_2) \ge \dots \ge d(x_n)$. Then $\varphi(G) \le m(G) \le \Delta(G) + 1$, where $m(G) = max\{1 \le i \le n : d(x_i) \ge i - 1\}$ and $\Delta(G)$ is the maximum degree of G.*

R. W. Irving and D. F. Manlove [7] proved that finding the b-chromatic number of any graph is a NP-hard problem, and they gave a polynomial-time algorithm for finding the b-chromatic number of trees. Kouider and Mahéo [9] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. They gave, in particular, a lower bound for the b-chromatic number of the cartesian product of two graphs where each one has a stable dominating system. More recently in [11], the authors characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper k-coloring even for connected bipartite graphs and $k = \Delta(G) + 1$. They also determine the asymptotic behavior for the b-chromatic number of random graphs.

In this paper, we present several exact values and determine bounds for the b-chromatic number of power graphs of paths and cycles.

Let $Diam(G)$ be the diameter of a graph G, defined as the maximum distance between any pair of vertices of G. Let us begin with the following observation.

Fact 2 *For any graph G of order n, if $Diam(G) \le p$, then $\varphi(G^p) = n$, with $p \ge 2$.*

Proof. If $Diam(G) \le p$, it is trivial to see that G^p is a complete graph. So $\varphi(G^p) = n$. □

Let G be a path or a cycle on vertices $x_1, x_2, \dots, x_n$. We fix an orientation of G (left to right if G is a path and clockwise if G is a cycle). For each $1 \le i \le n$, we denote by x_i^+ (resp. x_i^-) the successor (resp.

predecessor) of x_i in G (if any). For $1 \le i \ne j \le n$, we define $[x_i,x_j]_G$, $[x_i,x_j)_G$ and $(x_i,x_j)_G$ as the set of consecutive vertices on G from respectively x_i to x_j, x_i to x_j^- and x_i^+ to x_j^-, following the fixed orientation of G. If there is no ambiguity, we denote $[x_i,x_j]_G$, $[x_i,x_j)_G$ and $(x_i,x_j)_G$ by respectively $[x_i,x_j]$, $[x_i,x_j)$ and (x_i,x_j).

In all figures, the graph G is represented with solid edges. Edges added in a *p-th* power graph G^p are represented with dashed edges. In some figures, vertices are surrounded and represent a dominating system of the coloring. In any coloring of a graph G, we will say that a vertex x of G is *adjacent* to a color i if there exists a neighbor of x which is colored by i.

2 Power Graph of a Path

In this section, we determine the b-chromatic number of a *p-th* power graph of a path, with $p \ge 1$. First we give a lemma used in the proof of Theorem 4. Then the b-chromatic number of a *p-th* power graph of a path is computed.

Lemma 3 *For any $p \ge 1$, and for any $n \ge p+1$, let P_n be the path on vertices $x_1,x_2,\dots,x_n$. For each integer k, with $p+1 \le k \le \min(2p+1,n)$, there exists a proper k-coloring on P_n^p. Moreover each vertex x, such that $x \in \{x_{k-p},x_{k-p+1},\dots,x_{n-k+p+1}\}$, is adjacent to each color j, with $1 \le j \ne c_x \le k$.*

Proof. As $k \ge p+1$, it is easy to see that if we put the set of colors $\{1,2,\dots,k\}$ cyclically on $V(P_n)$, then two adjacent vertices will not have the same color. The coloring is thus a proper k-coloring.
Let $S = \{x_{k-p},x_{k-p+1},\dots,x_{n-k+p+1}\}$. First we show that each vertex of S is adjacent to at least $k-1$ vertices. Observe that the vertex x_{k-p} is adjacent to $(k-p-1)+p = k-1$ vertices. And the vertex $x_{n-k+p+1}$ is adjacent to $p+n-(n-k+p+1) = k-1$ vertices. Since each vertex x_i, with $k-p+1 \le i \le n-k+p$, has a degree $d(x_i) \ge d(x_{k-p})$, then each vertex of S is adjacent to at least $k-1$ other vertices.
Next, we can see by the construction that all the colors $\{1,2,\dots,k\} \setminus \{c_{x_i}\}$ appear between the first and the last neighbor of x_i. Therefore each vertex x_i of S is adjacent to each color j, with $1 \le j \ne c_{x_i} \le k$ and $k-p \le i \le n-k+p+1$. □

The b-chromatic number of a *p-th* power graph of a path is given by:

Theorem 4 *Let P_n be a path on vertices $x_1,x_2,\dots,x_n$. The b-chromatic number of P_n^p, with $p \ge 1$, is given by:*

$$\varphi(P_n^p) = \begin{cases} n \text{ if } n \le p+1, & (1) \\ p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor \text{ if } p+2 \le n \le 4p+1, & (2) \\ 2p+1 \text{ if } n \ge 4p+2 & (3) \end{cases}$$

Proof.

1. If $n \le p+1$, then $Diam(P_n) \le p$. So, by Fact 2, $\varphi(P_n^p) = n$.

2. We prove first that $\varphi(P_n^p) \ge p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$ for $p+2 \le n \le 4p+1$. Let $k = p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$. By Lemma 3, we give a proper k-coloring of P_n^p. For example, Figure 1 shows a dominating proper 5-coloring of P_8^3.

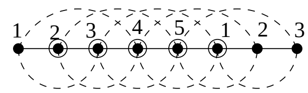

Fig. 1: Coloring of P_8^3

Let S' be the set of vertices $\{x_{k-p}, x_{k-p+1}, \ldots, x_{2k-p-1}\}$. Since $2k-p-1 \leq n-k+p+1$, then $S' \subseteq \{x_{k-p}, x_{k-p+1}, \ldots, x_{n-k+p+1}\}$. By Lemma 3, S' is a dominating system. As the coloring is proper and has a dominating system, we obtain a dominating proper k-coloring. So, $\varphi(P_n^p) \geq p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$.

Next we prove that $\varphi(P_n^p) \leq p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$ for $p+2 \leq n \leq 4p+1$. The proof is by contradiction. Suppose that there exists a dominating proper k'-coloring such that

$$k' > p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor. \tag{1}$$

Let $W = \{w_1, w_2, \ldots, w_{k'}\}$ be a dominating system of the coloring on P_n^p (following the orientation of P_n, we meet $w_1, w_2, \ldots, w_{k'}$). The vertices w_1 and $w_{k'}$ are adjacent to, at most, p different colors in $[w_1, w_{k'}]$. As w_1 (respectively $w_{k'}$) is a dominating vertex, it must be adjacent to at least $k'-1$ different colors. Then, there are at least $k'-p-1$ vertices on $[x_1, w_1)$ (respectively $(w_{k'}, x_n]$). Therefore, $n-k' \geq n-|[w_1, w_{k'}]| \geq 2(k'-p-1)$.
On the other hand by hypothesis $k' \geq p+2+\left\lfloor \frac{n-p-1}{3} \right\rfloor$, so that $n-k' \leq n-p-2-\left\lfloor \frac{n-p-1}{3} \right\rfloor$.
These two results give the following inequality,

$$2(k'-p-1) \leq n-k' \leq n-p-2-\left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

$$k' \leq \frac{1}{2}\left(n+p-\left\lfloor \frac{n-p-1}{3} \right\rfloor\right). \tag{2}$$

By (1) and (2), we obtain,

$$\frac{1}{2}\left(n+p-\left\lfloor \frac{n-p-1}{3} \right\rfloor\right) \geq k' \geq p+2+\left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

$$n-p-4 \geq 3\left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

which is a contradiction. Hence such a coloring does not exist. Therefore, $\varphi(P_n^p) \leq p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$.
We deduce from these two parts that $\varphi(P_n^p) = p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$.

3. $\Delta(P_n^p) = 2p$, so by Proposition 1, $\varphi(P_n^p) \le 2p+1$. Lemma 3 gives a proper $(2p+1)$-coloring and shows that each vertex x of the set $\{x_{p+1}, x_{p+2}, \dots, x_{3p+1}\}$ is adjacent to each color j with $1 \le j \ne c_x \le k$. So this set is a dominating system and $\varphi(P_n^p) \ge 2p+1$. Therefore $\varphi(P_n^p) = 2p+1$. For example, Figure 2 gives a dominating proper 7-coloring of P_{15}^3.

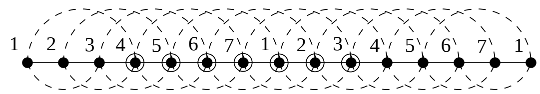

Fig. 2: Coloring of P_{15}^3

3 Power Graph of a Cycle

In this section, we study the b-chromatic number of a *p-th* power graph of a cycle, with $p \ge 1$. First we give two lemmas used in the proof of Theorem 7. Then we bound the b-chromatic number of a *p-th* power graph of a cycle.

Lemma 5 *Let C_n^p be a p-th power graph of a cycle C_n, with $p \ge 2$. For any $2p+3 \le n \le 4p$, let $k \ge \min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor)$. Then $n \le 2k$.*

Proof. The proof is by contradiction. Suppose $n \ge 2k+1$. We consider two cases. Firstly, $k \ge n-p-1$. So,

$$n \ge 2k+1 \ge 2(n-p-1)+1,$$

$$n \le 2p+1,$$

which is a contradiction. Secondly, $k \ge p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$. So,

$$n \ge 2k+1 \ge 2(p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor)+1,$$

$$n-2p-3 \ge 2\left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

which is a contradiction too. □

Lemma 6 *For any $p \ge 2$, and for any $2p+3 \le n \le 4p$, let C_n be the cycle on vertices $x_1, x_2, \dots, x_n$. Let $k = \min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor)$. So there exists a proper k-coloring on C_n^p. Moreover each vertex x, such that $x \in \{x_{k-p}, x_{k-p+1}, \dots, x_{2k-p-1}\}$, is adjacent to each color j, with $1 \le j \ne c_x \le k$.*

Proof. We put the set of colors $\{1, 2, \dots, k\}$ cyclically on $V(C_n)$. As $k \le p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$ and $n \le 4p$, then $k \le 2p+1$. Moreover, by Lemma 5 we deduce that $2k \ge n \ge 2p+3 \ge k+2$. So, the full set of colors $\{1, 2, \dots, k\}$ appears consecutively at least once, and at most twice, in the cyclic coloring of C_n^p. As $2k \ge n \ge 2p+3$, we have $k \ge p+1$. Furthermore, by definition of k we have $n-k \ge p+1$. Thus, as $k \ge p+1$ and $n-k \ge p+1$, the coloring is proper.

Let P_n be the subpath of C_n induced by $x_1, x_2, \ldots, x_n$. Let $S = \{x_{k-p}, x_{k-p+1}, \ldots, x_{k+(k-p-1)}\}$. As $p+1 \leq k \leq 2p+1$ and $2k-p-1 \leq n-k+p+1$, then by Lemma 3 each vertex x_i of S, with $k-p \leq i \leq 2k-p-1$, is adjacent to each color q, with $1 \leq q \neq c_{x_i} \leq k$, on P_n^p. Therefore each vertex x_i of S is adjacent to each color q, with $1 \leq q \neq c_{x_i} \leq k$, on C_n^p. □

Theorem 7 *Let C_n be a cycle on vertices $x_1, x_2, \ldots, x_n$. The b-chromatic number of C_n^p, with $p \geq 1$, is*

$$\varphi(C_n^p) = \begin{cases} n \text{ if } n \leq 2p+1, & (1) \\ p+1 \text{ if } n = 2p+2, & (2) \\ (\geq) \min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor) \text{ if } 2p+3 \leq n \leq 3p & (3) \\ p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor \text{ if } 3p+1 \leq n \leq 4p & (4) \\ 2p+1 \text{ if } n \geq 4p+1 & (5) \end{cases}$$

Proof.

1. If $n \leq 2p+1$, then $Diam(C_n) \leq p$. So, by Fact 2, $\varphi(C_n^p) = n$.

2. To color the graph, we put the set of colors $\{1, 2, \ldots, p+1\}$ cyclically twice. One can easily see that this coloring is a proper $(p+1)$-coloring. Let S be the set of vertices $\{x_1, x_2, \ldots, x_{p+1}\}$. Each vertex x_i, with $1 \leq i \leq p+1$, is adjacent to $n-2$ vertices. Since $n-2 \geq p+1$, then each vertex x_i $(1 \leq i \leq p+1)$ of S is adjacent to all colors other than c_{x_i}. So the set S is a dominating system. We now show that, in any dominating proper coloring, vertices x_i and x_{i+p+1} must have the same color. For the subgraph induced by vertices $x_1, x_2, \ldots, x_{p+1}$, we have a clique and we can assume without loss of generality that these vertices are colored by $1, 2, \ldots, p+1$ respectively. If there exists a dominating vertex of color j, for some $j > p+1$, then this vertex is x_{p+1+i} for some i $(1 \leq i \leq p+1)$. Vertex x_{p+1+i} is not adjacent to x_i, but every other vertex is adjacent to x_i, so that x_{p+1+i} cannot be a dominating vertex, a contradiction. Therefore $\varphi(C_n^p) = p+1$ for $n = 2p+2$.

3. Let $k = \min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor)$.

   By Lemma 6 there exists a dominating proper k-coloring for $2p+3 \leq n \leq 3p$. Therefore $\varphi(C_n^p) \geq \min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor)$. For example, in Figure 3, we give a dominating proper 6-coloring of C_{11}^4.

4. Let $k = p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$.

   For $3p+1 \leq n \leq 4p$, Lemma 6 gives a dominating proper k-coloring. This proves that $\varphi(C_n^p) \geq \min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor)$. For example, Figure 4 shows a dominating proper 6-coloring of C_{11}^3.

   Next, we prove that $\varphi(C_n^p) \leq k$. Suppose there exists a dominating proper k'-coloring for C_n^p, with $k' \geq p+2+\left\lfloor \frac{n-p-1}{3} \right\rfloor$, for the sake of contradiction. Let $W = \{w_1, w_2, \ldots, w_{k'}\}$ be a set of dominating vertices on C_n (following the orientation of C_n, we meet $w_1, w_2, \ldots, w_{k'}$). We distinguish two cases.

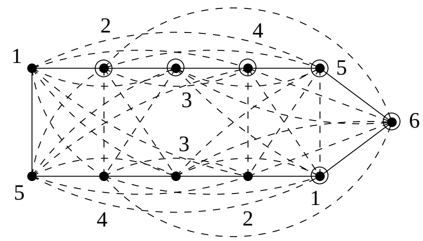

Fig. 3: Coloring of C_{11}^4 ($n-p-1=6$, $p+1+\left\lfloor\frac{n-p-1}{3}\right\rfloor=7$ and $\varphi(C_{11}^4)\geq 6$)

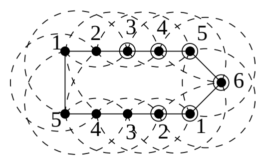

Fig. 4: Coloring of C_{11}^3

Case 1: for each i, with $1\leq i\leq k'$, $|(w_i,w_{i+1})|\leq p-1$.
As $k'\geq p+2+\left\lfloor\frac{n-p-1}{3}\right\rfloor$, by a straightforward modification of the proof of Lemma 5, we have $n<2k'$. So, there exists at least one color c not repeated in C_n^p (i.e. there are not two distinct vertices with the same color c). Without loss of generality, suppose that c appears on the vertex x, with $x\in V(C_n)$. Therefore x is a dominating vertex and each other dominating vertex is adjacent to x. Then, $|[w_1,x)|\leq p$ and $|[x,w_{k'})|\leq p$. As for each i, with $1\leq i\leq k'$, we have $|(w_i,w_{i+1})|\leq p-1$ and since on the cycle the next dominating vertex from $w_{k'}$ is w_1, then

$$|(w_{k'},w_1)|\leq p-1,$$

where

$$|(w_{k'},w_1)|=n-|[w_1,x)|-|[x,w_{k'})|-1.$$

Therefore, we have

$$n-|[w_1,x)|-|[x,w_{k'})|-1\leq p-1,$$
$$n-2p-1\leq p-1,$$
$$n\leq 3p,$$

which is a contradiction.

Case 2: There exists r, with $1\leq r\leq k'$ and r is taken modulo k', such that $|(w_r,w_{r+1})|\geq p$.
Let X be the set of vertices of $[w_{r+1},w_r]$ (see Figure 5). Let X_C be the set of colors appearing in X. Let $\Gamma_X(x_i)$ be the set of neighbors of x_i in X and $\Gamma_X^c(x_i)$ the set of colors appearing in $\Gamma_X(x_i)$, with $1\leq i\leq n$. Let $A=X_C\backslash(\Gamma_X^c(w_r)\cup\{c_{w_r}\})$. Let $B=X_C\backslash(\Gamma_X^c(w_{r+1})\cup\{c_{w_{r+1}}\})$. We discuss two subcases.

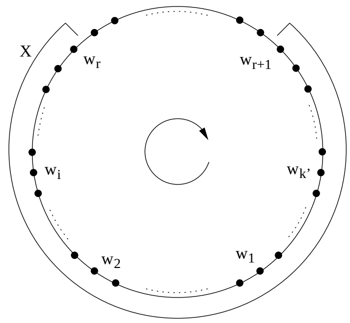

Fig. 5: A dominating system on C_n^p and the set X

Subcase 1: $|X| \leq 2p+2$. Since all dominating vertices belong to X, we have $|X| \geq k'$. Then, $|(w_r, w_{r+1})| \leq n-k'$ and $|X_c| = k'$. As the vertices of $\Gamma_X(w_r)$ form a clique, then $|\Gamma_X^c(w_r)| = |\Gamma_X(w_r)| = p$. So we have $|A| = |X_C| - |\Gamma_X^c(w_r)| - 1 = k'-p-1$. In the same way, we deduce that $|B| = k'-p-1$. As $|X| \leq 2p+2$, we have $X \subseteq (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.a). So, $A \subseteq (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$ and $B \subseteq (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$. Let $q \in \{1,2,\dots,k'\}$. If $q \in A$ (resp. $q \in B$) then $q \notin (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$ (resp. $q \notin (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$) and so $q \notin B$ (resp. $q \notin A$). Therefore, $A \cap B = \emptyset$. As w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}). Therefore,

$$|A| + |B| \leq |(w_r, w_{r+1})|,$$

$$2(k'-p-1) \leq n-k',$$

$$3k' \leq n+2p+2,$$

$$3\left\lfloor \frac{n-p-1}{3} \right\rfloor \leq n-p-4,$$

which is a contradiction.

Subcase 2: $|X| \geq 2p+3$. As in Subcase 1, we have $|A| = k'-p-1$ and $|B| = k'-p-1$. Let $X' = X \backslash (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.b). So $|X'| \geq |A \cap B|$. Since w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}). Then,

$$|A| + |B| - |A \cap B| \leq |(w_r, w_{r+1})| \leq n-2p-2-|X'|,$$

$$2(k'-p-1) - |A \cap B| \leq n-2p-2-|A \cap B|,$$

$$2(p+2+\left\lfloor \frac{n-p-1}{3} \right\rfloor) \leq n,$$

which is a contradiction. Therefore there does not exist a dominating proper k'-coloring, with $k' \geq p+2+\left\lfloor \frac{n-p-1}{3} \right\rfloor$.

This completes the proof of $\varphi(C_n^p) = p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor$.

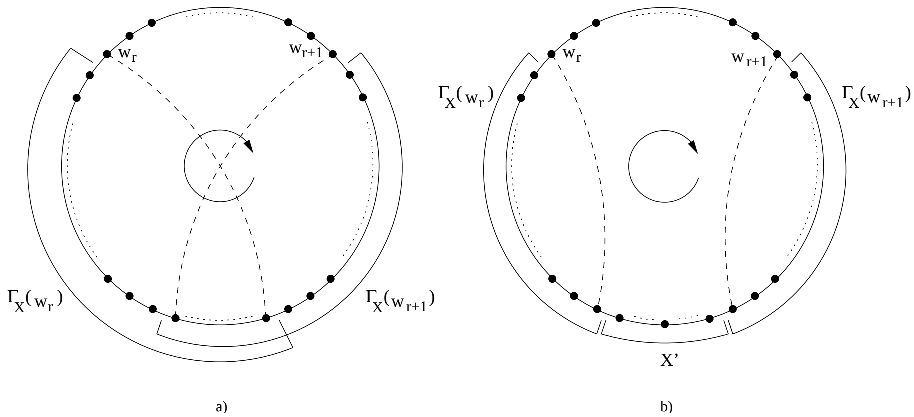

Fig. 6: Neighborhoods of w_r and w_{r+1} on X when a) $|X| \leq 2p+2$ and b) $|X| \geq 2p+3$

5. As $\Delta = 2p$, by Proposition 1, $\varphi(C_n^p) \leq 2p+1$.
We then give a proper $(2p+1)$-coloring. It is constructed in two steps. First, we put $(2p+1)$ different colors on the $(2p+1)$ first vertices ($c_{x_i} := i$ for $1 \leq i \leq 2p+1$). In the second step, we have two cases. If $n = 4p+1$, we color the remaining vertices as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p+2 \leq i \leq n$. If $n \geq 4p+2$, then the remaining vertices are colored as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p+2 \leq i \leq 4p+2$, and $c_{x_i} := c_{x_{i-p-1}}$ for $4p+3 \leq i \leq n$. Then the distance between two vertices colored by the same color c is at least $p+1$. So the coloring is proper. By an analogue proof of Lemma 3, one can prove that each vertex x_i, with $p+1 \leq i \leq 3p+1$, is a dominating vertex. So this coloring is a dominating proper $(2p+1)$-coloring. This construction shows that $\varphi(C_n^p) \geq 2p+1$. Therefore we have proved that $\varphi(C_n^p) = 2p+1$. For example, Figure 7 gives a dominating proper 7-coloring C_{16}^3. $\square$

4 Open Problem

In section 3, we have obtained the exact values of $\varphi(C_n^p)$, except in case $2p+3 \leq n \leq 3p$ where we give a lower bound. We believe that $\min(n-p-1, p+1+\left\lfloor \frac{n-p-1}{3} \right\rfloor)$ is the exact value of $\varphi(C_n^p)$ for $2p+3 \leq n \leq 3p$.

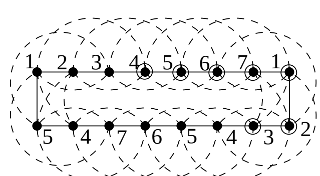

Fig. 7: Coloring of C_{16}^3

Acknowledgments

The authors thank the referees for useful suggestions that led to this improved version.

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