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The b-chromatic number of some power graphs

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Let $G$ be a graph on vertices $v_1, v_2, \ldots, v_n$. The b-chromatic number of $G$ is defined as the maximum number $k$ of colors that can be used to color the vertices of $G$, such that we obtain a proper coloring and each color $i$, with $1 \leq i \leq k$, has at least one representant $x_i$ adjacent to a vertex of every color $j$, $1 \leq j \neq i \leq k$. In this paper, we give the exact value for the b-chromatic number of power graphs of a path and we determine bounds for the b-chromatic number of power graphs of a cycle.

Keywords: b-chromatic number, coloring, cycle, path, power graphs

1 Introduction

We consider graphs without loops or multiple edges. Let $G$ be a graph with a vertex set $V$ and an edge set $E$. We denote by $d(x)$ the degree of the vertex $x$ in $G$, and by $dist_G(x, y)$ the distance between two vertices $x$ and $y$ in $G$. The $p$-th power graph $G^p$ is a graph obtained from $G$ by adding an edge between every pair of vertices at distance $p$ or less, with $p \geq 1$. It is easy to see that $G^1 = G$. In the literature, power graphs of several classes have been investigated [2, 3, 8]. In this note we study a vertex coloring of power graphs. The power graph of a path and the power graph of a cycle can be also considered as respectively subclasses of distance graphs and circulant graphs. The distance graph $G(D)$ with distance set $D = \{d_1, d_2, \ldots\}$ has the set $Z$ of integers as vertex set, with two vertices $i, j \in Z$ adjacent if and only if $|i - j| \in D$. The circulant graph can be defined as follows. Let $n$ be a natural number and let $S = \{k_1, k_2, \ldots, k_r\}$ with $k_1 < k_2 < \ldots < k_r \leq n/2$. Then the vertex set of the circulant graph $G(n, S)$ is $\{0, 1, \ldots, n-1\}$ and the set of neighbors of the vertex $i$ is $\{(i \pm k_j) \mod n | j = 1, 2, \ldots, r\}$.

The study of distance graphs was initiated by Eggleton and al. [4]. Recently, the problem of coloring of this class of graphs has attracted considerable attention, see e.g. [12, 13]. Circulant graphs have been extensively studied and have a vast number of applications to multicomputer networks and distributed computation (see [10, 11]). The special cases we consider are the distance graph $G(D)$ with finite distance set $D = \{1, 2, \ldots, p\}$ which is isomorphic to the $p$-th power graph of a path and the circulant graph $G(n, S)$ with $S = \{1, 2, \ldots, p\}$ which is isomorphic to the $p$-th power graph of a cycle.

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A $k$-coloring of $G$ is defined as a function $c$ on $V(G) = \{v_1, v_2, \ldots, v_n\}$ into a set of colors $C = \{1, 2, \ldots, k\}$ such that for each vertex $v_i$, with $1 \leq i \leq n$, we have $c(v_i) \in C$. A proper $k$-coloring is a $k$-coloring satisfying the condition $c_x \neq c_y$ for each pair of adjacent vertices $x, y \in V(G)$. A dominating proper $k$-coloring is a proper $k$-coloring satisfying the following property $P$: for each $i$, $1 \leq i \leq k$, there exists a vertex $x_i$ of color $i$ such that, for each $j$, with $1 \leq j \neq i \leq k$, there exists a vertex $y_j$ of color $j$ adjacent to $x_i$. A set of vertices satisfying the property $P$ is called a dominating system. Each vertex of a dominating system is called a dominating vertex. The b-chromatic number $\varphi(G)$ of a graph $G$ is defined as the maximum $k$ such that $G$ admits a dominating proper $k$-coloring.

The b-chromatic number was introduced in [7]. The motivation, similarly as for the previously studied achromatic number (cf. e.g. [5, 6]), comes from algorithmic graph theory. The achromatic number $\psi(G)$ of a graph $G$ is the largest number of colors which can be assigned to the vertices of $G$ such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph $G$ using $k > \chi(G)$ colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by $\varphi(G)$ (these recolorings are discussed in [7] and [11]). Thus $\varphi(G) \leq \psi(G)$ (also given in [2]). From this point of view, both complexity results and tight bounds for the b-chromatic number are interesting. The following bounds of b-chromatic number are already presented in [7].

**Proposition 1** Assume that the vertices $x_1, x_2, \ldots, x_n$ of $G$ are ordered such that $d(x_1) \geq d(x_2) \geq \ldots \geq d(x_n)$. Then $\varphi(G) \leq m(G) \leq \Delta(G) + 1$, where $m(G) = \max\{1 \leq i \leq n : d(x_i) \geq i - 1\}$ and $\Delta(G)$ is the maximum degree of $G$.

R. W. Irving and D. F. Manlove [7] proved that finding the b-chromatic number of any graph is an NP-hard problem, and they gave a polynomial-time algorithm for finding the b-chromatic number of trees. Kouider and Mahéo [9] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. They gave, in particular, a lower bound for the b-chromatic number of the cartesian product of two graphs where each one has a stable dominating system. More recently in [11], the authors characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper $k$-coloring even for connected bipartite graphs and $k = \Delta(G) + 1$. They also determine the asymptotic behavior for the b-chromatic number of random graphs.

In this paper, we present several exact values and determine bounds for the b-chromatic number of power graphs of paths and cycles.

Let $Diam(G)$ be the diameter of a graph $G$, defined as the maximum distance between any pair of vertices of $G$. Let us begin with the following observation.

**Fact 2** For any graph $G$ of order $n$, if $Diam(G) \leq p$, then $\varphi(G^p) = n$, with $p \geq 2$.

**Proof.** If $Diam(G) \leq p$, it is trivial to see that $G^p$ is a complete graph. So $\varphi(G^p) = n$. $\square$

Let $G$ be a path or a cycle on vertices $x_1, x_2, \ldots, x_n$. We fix an orientation of $G$ (left to right if $G$ is a path and clockwise if $G$ is a cycle). For each $1 \leq i \leq n$, we denote by $x_i^-$ (resp. $x_i^+$) the successor (resp. predecessor).
The predecessor) of \(x_i\) in \(G\) (if any). For \(1 \leq i \neq j \leq n\), we define \([x_i,x_j])_G, [x_i,x_j]_G\) and \((x_i,x_j)_G\) as the set of consecutive vertices on \(G\) from respectively \(x_i\) to \(x_j\), \(x_i\) to \(x_j\) and \(x_i\) to \(x_j\), following the fixed orientation of \(G\). If there is no ambiguity, we denote \([x_i,x_j])_G, [x_i,x_j]_G\) and \((x_i,x_j)_G\) by respectively \([x_i,x_j], [x_i,x_j]\) and \((x_i,x_j)\).

In all figures, the graph \(G\) is represented with solid edges. Edges added in a \(p\)-th power graph \(G^p\) are represented with dashed edges. In some figures, vertices are surrounded and represent a dominating system of the coloring. In any coloring of a graph \(G\), we will say that a vertex \(x\) of \(G\) is adjacent to a color \(i\) if there exists a neighbor of \(x\) which is colored by \(i\).

## 2 Power Graph of a Path

In this section, we determine the \(b\)-chromatic number of a \(p\)-th power graph of a path, with \(p \geq 1\). First we give a lemma used in the proof of Theorem 4. Then the \(b\)-chromatic number of a \(p\)-th power graph of a path is computed.

**Lemma 3** For any \(p \geq 1\), and for any \(n \geq p + 1\), let \(P_n\) be the path on vertices \(x_1,x_2,\ldots,x_n\). For each integer \(k\), with \(p + 1 \leq k \leq \min(2p + 1, n)\), there exists a proper \(k\)-coloring on \(P_n^p\). Moreover each vertex \(x\), such that \(x \in \{x_{k-p},x_{k-p+1},\ldots,x_{n-k+p+1}\}\), is adjacent to each color \(j\), with \(1 \leq j \neq c_i \leq k\).

**Proof.** As \(k \geq p + 1\), it is easy to see that if we put the set of colors \(\{1,2,\ldots,k\}\) cyclically on \(V(P_n)\), then two adjacent vertices will not have the same color. The coloring is thus a proper \(k\)-coloring.

Let \(S = \{x_{k-p},x_{k-p+1},\ldots,x_{n-k+p+1}\}\). First we show that each vertex of \(S\) is adjacent to at least \(k - 1\) vertices. Observe that the vertex \(x_{k-p}\) is adjacent to \((k - p - 1) + p = k - 1\) vertices. And the vertex \(x_{n-k+p+1}\) is adjacent to \(p + n - (n - k + p + 1) = k - 1\) vertices. Since each vertex \(x_i\), with \(k - p + 1 \leq i \leq n - k + p\), has a degree \(d(x_i) \geq d(x_{k-p})\), then each vertex of \(S\) is adjacent to at least \(k - 1\) other vertices.

Next, we can see by the construction that all the colors \(\{1,2,\ldots,k\}\) \(\backslash c_i\) appear between the first and the last neighbor of \(x_i\). Therefore each vertex \(x_i\) of \(S\) is adjacent to each color \(j\), with \(1 \leq j \neq c_i \leq k\) and \(k - p \leq i \leq n - k + p + 1\).

The \(b\)-chromatic number of a \(p\)-th power graph of a path is given by:

**Theorem 4** Let \(P_n\) be a path on vertices \(x_1,x_2,\ldots,x_n\). The \(b\)-chromatic number of \(P_n^p\), with \(p \geq 1\), is given by:

\[
\varphi(P_n^p) = \begin{cases} 
  n & \text{if } n \leq p + 1, \\
  p + 1 + \left\lfloor \frac{n-p-1}{2} \right\rfloor & \text{if } p + 2 \leq n \leq 4p + 1, \\
  2p + 1 & \text{if } n \geq 4p + 2
\end{cases}
\]

**Proof.**

1. If \(n \leq p + 1\), then \(\text{Diam}(P_n) \leq p\). So, by Fact 2, \(\varphi(P_n^p) = n\).

2. We prove first that \(\varphi(P_n^p) \geq p + 1 + \left\lfloor \frac{n-p-1}{2} \right\rfloor\) for \(p + 2 \leq n \leq 4p + 1\). Let \(k = p + 1 + \left\lfloor \frac{n-p-1}{2} \right\rfloor\).

   By Lemma 3, we give a proper \(k\)-coloring of \(P_n^p\). For example, Figure 1 shows a dominating proper 5-coloring of \(P_n^5\).
Let \( S' \) be the set of vertices \( \{x_{k-p}, x_{k-p+1}, \ldots, x_{2k-p-1}\} \). Since \( 2k - p - 1 \leq n - k + p + 1 \), then \( S' \subseteq \{x_{k-p}, x_{k-p+1}, \ldots, x_{n-k+p+1}\} \). By Lemma 3, \( S' \) is a dominating system. As the coloring is proper and has a dominating system, we obtain a dominating proper \( k \)-coloring. So, \( \phi(P^p_n) \geq p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil \).

Next we prove that \( \phi(P^p_n) \leq p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil \) for \( p + 2 \leq n \leq 4p + 1 \). The proof is by contradiction. Suppose that there exists a dominating proper \( k' \)-coloring such that

\[
k' > p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil .
\]  

Let \( W = \{w_1, w_2, \ldots, w_{k'}\} \) be a dominating system of the coloring on \( P^p_n \) (following the orientation of \( P_n \), we meet \( w_1, w_2, \ldots, w_{k'} \)). The vertices \( w_1 \) and \( w_{k'} \) are adjacent to, at most, \( p \) different colors in \([w_1, w_{k'}]\). As \( w_1 \) (respectively \( w_{k'} \)) is a dominating vertex, it must be adjacent to at least \( k' - 1 \) different colors. Then, there are at least \( k' - p - 1 \) vertices on \([x_1, w_1]\) (respectively \((w_{k'}, x_n]\)). Therefore, \( n - k' \geq n - |[w_1, w_{k'}]| \geq 2(k' - p - 1) \).

On the other hand by hypothesis \( k' \geq p + 2 + \left\lceil \frac{n-p-1}{3} \right\rceil \), so that \( n - k' \leq n - p - 2 - \left\lceil \frac{n-p-1}{3} \right\rceil \).

These two results give the following inequality,

\[
2(k' - p - 1) \leq n - k' \leq n - p - 2 - \left\lfloor \frac{n-p-1}{3} \right\rfloor ,
\]

\[
k' \leq \frac{1}{2}(n + p - \left\lfloor \frac{n-p-1}{3} \right\rfloor) .
\]  

By (1) and (2), we obtain,

\[
\frac{1}{2}(n + p - \left\lfloor \frac{n-p-1}{3} \right\rfloor) \geq k' \geq p + 2 + \left\lceil \frac{n-p-1}{3} \right\rceil ,
\]

\[
n - p - 4 \geq 3 \left\lfloor \frac{n-p-1}{3} \right\rfloor ,
\]

which is a contradiction. Hence such a coloring does not exist. Therefore, \( \phi(P^p_n) \leq p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil \).

We deduce from these two parts that \( \phi(P^p_n) = p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil \).
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3. \( \Delta(P_n^p) = 2p \), so by Proposition \(^5\), \( \varphi(P_n^p) \leq 2p + 1 \). Lemma \(^3\) gives a proper \((2p + 1)\)-coloring and shows that each vertex \( x \) of the set \( \{x_{p+1}, x_{p+2}, \ldots, x_{3p+1}\} \) is adjacent to each color \( j \) with \( 1 \leq j \neq c_x \leq k \). So this set is a dominating system and \( \varphi(P_n^p) \geq 2p + 1 \). Therefore \( \varphi(P_n^p) = 2p + 1 \). For example, Figure \(^2\) gives a dominating proper 7-coloring of \( P_{15}^3 \).

3. **Power Graph of a Cycle**

In this section, we study the b-chromatic number of a \( p \)-th power graph of a cycle, with \( p \geq 1 \). First we give two lemmas used in the proof of Theorem \(^4\). Then we bound the b-chromatic number of a \( p \)-th power graph of a cycle.

**Lemma 5** Let \( C_n^p \) be a \( p \)-th power graph of a cycle \( C_n \), with \( p \geq 2 \). For any \( 2p + 3 \leq n \leq 4p \), let \( k \geq \min(n - p - 1, p + 1 + \lfloor \frac{n - p - 1}{3} \rfloor) \). Then \( n \leq 2k \).

**Proof.** The proof is by contradiction. Suppose \( n \geq 2k + 1 \). We consider two cases. Firstly, \( k \geq n - p - 1 \). So,

\[
n \geq 2k + 1 \geq 2(n - p - 1) + 1, \]

\[
n \leq 2p + 1,
\]

which is a contradiction. Secondly, \( k \geq p + 1 + \lfloor \frac{n - p - 1}{3} \rfloor \). So,

\[
n \geq 2k + 1 \geq 2(p + 1 + \lfloor \frac{n - p - 1}{3} \rfloor) + 1,
\]

\[
n - 2p - 3 \geq 2 \left\lfloor \frac{n - p - 1}{3} \right\rfloor,
\]

which is a contradiction too. \( \square \)

**Lemma 6** For any \( p \geq 2 \), and for any \( 2p + 3 \leq n \leq 4p \), let \( C_n \) be the cycle on vertices \( x_1, x_2, \ldots, x_n \). Let 

\[
 k = \min(n - p - 1, p + 1 + \lfloor \frac{n - p - 1}{3} \rfloor).
\]

So there exists a proper \( k \)-coloring on \( C_n^p \). Moreover each vertex \( x \), such that \( x \in \{x_{k-p}, x_{k-p+1}, \ldots, x_{2k-p-1}\} \), is adjacent to each color \( j \), with \( 1 \leq j \neq c_x \leq k \).

**Proof.** We put the set of colors \( \{1, 2, \ldots, k\} \) cyclically on \( V(C_n) \). As \( k \leq p + 1 + \lfloor \frac{n - p - 1}{3} \rfloor \) and \( n \leq 4p \), then \( k \leq 2p + 1 \). Moreover, by Lemma \(^3\) we deduce that \( 2k \geq n \geq 2p + 3 \geq k + 2 \). So, the full set of colors \( \{1, 2, \ldots, k\} \) appears consecutively at least once, and at most twice, in the cyclic coloring of \( C_n^p \). As \( 2k \geq n \geq 2p + 3 \), we have \( k \geq p + 1 \). Furthermore, by definition of \( k \) we have \( n - k \geq p + 1 \). Thus, as \( k \geq p + 1 \) and \( n - k \geq p + 1 \), the coloring is proper.

![Coloring of P^3_{15}](image-url)
Let $P_n$ be the subpath of $C_n$ induced by $x_1, x_2, \ldots, x_n$. Let $S = \{x_{k-p}, x_{k-p+1}, \ldots, x_{k+(k-p-1)}\}$. As $p + 1 \leq k \leq 2p + 1$ and $2k - p - 1 \leq n - k + p + 1$, then by Lemma 7 each vertex $x_i$ of $S$, with $k - p \leq i \leq 2k - p - 1$, is adjacent to each color $q$, with $1 \leq q \neq c_{x_i} \leq k$, on $P_n$. Therefore each vertex $x_i$ of $S$ is adjacent to each color $q$, with $1 \leq q \neq c_{x_i} \leq k$, on $C_n$. \hfill $\Box$

**Theorem 7** Let $C_n$ be a cycle on vertices $x_1, x_2, \ldots, x_n$. The $b$-chromatic number of $C_n$, with $p \geq 1$, is

\[
\varphi(C_n^b) = \begin{cases} 
  n & \text{if } n \leq 2p + 1, \\
  p + 1 & \text{if } n = 2p + 2, \\
  \lceil \min(n - p - 1, p + 1 + \frac{n-p-1}{3}) \rceil & \text{if } 2p + 3 \leq n \leq 3p \\
  p + 1 + \frac{n-p-1}{3} & \text{if } 3p + 1 \leq n \leq 4p \\
  2p + 1 & \text{if } n \geq 4p + 1
\end{cases}
\]

**Proof.**

1. If $n \leq 2p + 1$, then $Diam(C_n) \leq p$. So, by Fact 2, $\varphi(C_n^b) = n$.

2. To color the graph, we put the set of colors $\{1, 2, \ldots, p + 1\}$ cyclically twice. One can easily see that this coloring is a proper $(p + 1)$-coloring. Let $S$ be the set of vertices $\{x_1, x_2, \ldots, x_{p+1}\}$. Each vertex $x_i$, with $1 \leq i \leq p + 1$, is adjacent to $n - 2$ vertices. Since $n - 2 \geq p + 1$, then each vertex $x_i$ is adjacent to all colors other than $c_{x_i}$. So the set $S$ is a dominating system. We now show that, in any dominating proper coloring, vertices $x_i$ and $x_{i+p+1}$ must have the same color. For the subgraph induced by vertices $x_1, x_2, \ldots, x_{p+1}$, we have a clique and we can assume without loss of generality that these vertices are colored by $1, 2, \ldots, p + 1$ respectively. If there exists a dominating vertex of color $j$, for some $j > p + 1$, then this vertex is $x_{p+1+i}$ for some $i$ ($1 \leq i \leq p + 1$). Vertex $x_{p+1+i}$ is not adjacent to $x_i$, but every other vertex is adjacent to $x_i$, so that $x_{p+1+i}$ cannot be a dominating vertex, a contradiction. Therefore $\varphi(C_n^b) = p + 1$ for $n = 2p + 2$.

3. Let $k = \min(n - p - 1, p + 1 + \frac{n-p-1}{3})$.

By Lemma 8 there exists a dominating proper $k$-coloring for $2p + 3 \leq n \leq 3p$. Therefore $\varphi(C_n^b) \geq \min(n - p - 1, p + 1 + \frac{n-p-1}{3})$. For example, in Figure 8 we give a dominating proper 6-coloring of $C_{11}$.

4. Let $k = p + 1 + \frac{n-p-1}{3}$.

For $3p + 1 \leq n \leq 4p$, Lemma 8 gives a dominating proper $k$-coloring. This proves that $\varphi(C_n^b) \geq \min(n - p - 1, p + 1 + \frac{n-p-1}{3})$. For example, Figure 8 shows a dominating proper 6-coloring of $C_{11}$.

Next, we prove that $\varphi(C_n^b) \leq k$. Suppose there exists a dominating proper $k'$-coloring for $C_n^b$, with $k' \geq p + 2 + \frac{n-p-1}{3}$, for the sake of contradiction. Let $W = \{w_1, w_2, \ldots, w_{k'}\}$ be a set of dominating vertices on $C_n$ (following the orientation of $C_n$, we meet $w_1, w_2, \ldots, w_{k'}$). We distinguish two cases.
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Fig. 3: Coloring of $C_{11}^4$ ($n - p - 1 = 6$, $p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil = 7$ and $\varphi(C_{11}^4) \geq 6$)

Fig. 4: Coloring of $C_{11}^3$

Case 1: for each $i$, with $1 \leq i \leq k'$, $|(w_i, w_{i+1})| \leq p - 1$.

As $k' \geq p + 2 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$, by a straightforward modification of the proof of Lemma 5, we have $n < 2k'$. So, there exists at least one color $c$ not repeated in $C_p^n$ (i.e. there are not two distinct vertices with the same color $c$). Without loss of generality, suppose that $c$ appears on the vertex $x$, with $x \in V(C_n)$. Therefore $x$ is a dominating vertex and each other dominating vertex is adjacent to $x$. Then, $|w_1, x| \leq p$ and $|x, w_{k'}| \leq p$. As for each $i$, with $1 \leq i \leq k'$, we have $|(w_i, w_{i+1})| \leq p - 1$ and since on the cycle the next dominating vertex from $w_{k'}$ is $w_1$, then

$$|(w_{k'}, w_1)| \leq p - 1,$$

where

$$|(w_{k'}, w_1)| = n - |w_1, x| - |x, w_{k'}| - 1.$$

Therefore, we have

$$n - |w_1, x| - |x, w_{k'}| - 1 \leq p - 1,$$

$$n - 2p - 1 \leq p - 1,$$

$$n \leq 3p,$$

which is a contradiction.

Case 2: There exists $r$, with $1 \leq r \leq k'$ and $r$ is taken modulo $k'$, such that $|(w_r, w_{r+1})| \geq p$.

Let $X$ be the set of vertices of $[w_{r+1}, w_r]$ (see Figure 5). Let $X_c$ be the set of colors appearing in $X$. Let $\Gamma_X(x_i)$ be the set of neighbors of $x_i$ in $X$ and $\Gamma_X(x_i)$ the set of colors appearing in $\Gamma_X(x_i)$, with $1 \leq i \leq n$. Let $A = X_c \setminus (\Gamma_X(w_r) \cup \{c_{w_r}\})$. Let $B = X_c \setminus (\Gamma_X(w_{r+1}) \cup \{c_{w_{r+1}}\})$. We discuss two subcases.
Subcase 1: $|X| \leq 2p + 2$. Since all dominating vertices belong to $X$, we have $|X| \geq k'$. Then, $|(w_r, w_{r+1})| \leq n - k'$ and $|X| = k'$. As the vertices of $\Gamma_X(w_r)$ form a clique, then $|\Gamma_X(w_r)| = |\Gamma_X(w_r)| = p$. So we have $|A| = |X| = |\Gamma_X(w_r)| - 1 = k' - p - 1$. In the same way, we deduce that $|B| = k' - p - 1$. As $|X| \leq 2p + 2$, we have $X \subseteq (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 5). So, $A \subseteq (\Gamma_X(w_r) \cup \{c_{w_{r+1}}\})$ and $B \subseteq (\Gamma_X(w_r) \cup \{c_{w_r}\})$. Let $q \in \{1, 2, \ldots, k'\}$. If $q \in A$ (resp. $q \in B$) then $q \in (\Gamma_X(w_r) \cup \{c_{w_r}\})$ (resp. $q \in (\Gamma_X(w_{r+1}) \cup \{c_{w_{r+1}}\})$) and so $q \notin B$ (resp. $q \notin A$). Therefore, $A \cap B = \emptyset$. As $w_r$ (resp. $w_{r+1}$) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of $A$ (resp. $B$) must be repeated in $(w_r, w_{r+1})$. Therefore,

$$
|A| + |B| \leq |(w_r, w_{r+1})|,
$$

$$
2(k' - p - 1) \leq n - k',
$$

$$
3k' \leq n + 2p + 2,
$$

$$
3 \left\lfloor \frac{n - p - 1}{3} \right\rfloor \leq n - p - 4,
$$

which is a contradiction.

Subcase 2: $|X| \geq 2p + 3$. As in Subcase 1, we have $|A| = k' - p - 1$ and $|B| = k' - p - 1$. Let $X' = X \setminus (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 5). So $|X'| \geq |A \cap B|$. Since $w_r$ (resp. $w_{r+1}$) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of $A$ (resp. $B$) must be repeated in $(w_r, w_{r+1})$. Then,

$$
|A| + |B| - |A \cap B| \leq |(w_r, w_{r+1})| - n - 2p - 2 - |X'|,
$$

$$
2(k' - p - 1) - |A \cap B| \leq n - 2p - 2 - |A \cap B|,
$$

$$
2(p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor) \leq n,
$$

which is a contradiction. Therefore there does not exist a dominating proper $k'$-coloring, with $k' \geq p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.

This completes the proof of $\Phi(C^p_n) = p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.
5. As $\Delta = 2p$, by Proposition 1, $\phi(C_p^n) \leq 2p + 1$.

We then give a proper $(2p + 1)$-coloring. It is constructed in two steps. First, we put $(2p + 1)$ different colors on the $(2p + 1)$ first vertices ($c_{x_i} := i$ for $1 \leq i \leq 2p + 1$). In the second step, we have two cases. If $n = 4p + 1$, we color the remaining vertices as follows: $c_{x_i} := c_{x_i-2p-1}$ for $2p + 2 \leq i \leq n$. If $n \geq 4p + 2$, then the remaining vertices are colored as follows: $c_{x_i} := c_{x_i-2p-1}$ for $2p + 2 \leq i \leq 4p + 2$, and $c_{x_i} := c_{x_i-p-1}$ for $4p + 3 \leq i \leq n$. Then the distance between two vertices colored by the same color $c$ is at least $p + 1$. So the coloring is proper. By an analogue proof of Lemma 3, one can prove that each vertex $x_i$, with $p + 1 \leq i \leq 3p + 1$, is a dominating vertex. So this coloring is a dominating proper $(2p + 1)$-coloring. This construction shows that $\phi(C_p^n) \geq 2p + 1$.

Therefore we have proved that $\phi(C_p^n) = 2p + 1$. For example, Figure 7 gives a dominating proper 7-coloring $C^3_{16}$.

4 Open Problem

In section 3, we have obtained the exact values of $\phi(C_p^n)$, except in case $2p + 3 \leq n \leq 3p$ where we give a lower bound. We believe that $\min(n - p - 1, p + 1 + \left\lceil \frac{n-p-1}{3} \right\rceil)$ is the exact value of $\phi(C_p^n)$ for $2p + 3 \leq n \leq 3p$.
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References


