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The b -chromatic number of some power graphs

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Let G be a graph on vertices v_1, v_2, \dots, v_n . The b -chromatic number of G is defined as the maximum number k of colors that can be used to color the vertices of G , such that we obtain a proper coloring and each color i , with $1 \leq i \leq k$, has at least one representant x_i adjacent to a vertex of every color j , $1 \leq j \neq i \leq k$. In this paper, we give the exact value for the b -chromatic number of power graphs of a path and we determine bounds for the b -chromatic number of power graphs of a cycle.

Keywords: b -chromatic number, coloring, cycle, path, power graphs

1 Introduction

We consider graphs without loops or multiple edges. Let G be a graph with a vertex set V and an edge set E . We denote by $d(x)$ the degree of the vertex x in G , and by $dist_G(x, y)$ the distance between two vertices x and y in G . The p -th power graph G^p is a graph obtained from G by adding an edge between every pair of vertices at distance p or less, with $p \geq 1$. It is easy to see that $G^1 = G$. In the literature, power graphs of several classes have been investigated [2, 3, 8]. In this note we study a vertex coloring of power graphs. The power graph of a path and the power graph of a cycle can be also considered as respectively subclasses of *distance graphs* and *circulant graphs*. The *distance graph* $G(D)$ with distance set $D = \{d_1, d_2, \dots\}$ has the set Z of integers as vertex set, with two vertices $i, j \in Z$ adjacent if and only if $|i - j| \in D$. The *circulant graph* can be defined as follows. Let n be a natural number and let $S = \{k_1, k_2, \dots, k_r\}$ with $k_1 < k_2 < \dots < k_r \leq n/2$. Then the vertex set of the circulant graph $G(n, S)$ is $\{0, 1, \dots, n-1\}$ and the set of neighbors of the vertex i is $\{(i \pm k_j) \bmod n \mid j = 1, 2, \dots, r\}$.

The study of distance graphs was initiated by Eggleton and al. [4]. Recently, the problem of coloring of this class of graphs has attracted considerable attention, see e.g. [12, 13]. Circulant graphs have been extensively studied and have a vast number of applications to multicomputer networks and distributed computation (see [1, 10]). The special cases we consider are the distance graph $G(D)$ with finite distance set $D = \{1, 2, \dots, p\}$ which is isomorphic to the p -th power graph of a path and the circulant graph $G(n, S)$ with $S = \{1, 2, \dots, p\}$ which is isomorphic to the p -th power graph of a cycle.

A k -coloring of G is defined as a function c on $V(G) = \{v_1, v_2, \dots, v_n\}$ into a set of colors $C = \{1, 2, \dots, k\}$ such that for each vertex v_i , with $1 \leq i \leq n$, we have $c_{v_i} \in C$. A *proper k -coloring* is a k -coloring satisfying the condition $c_x \neq c_y$ for each pair of adjacent vertices $x, y \in V(G)$. A *dominating proper k -coloring* is a proper k -coloring satisfying the following property P : for each i , $1 \leq i \leq k$, there exists a vertex x_i of color i such that, for each j , with $1 \leq j \neq i \leq k$, there exists a vertex y_j of color j adjacent to x_i . A set of vertices satisfying the property P is called a *dominating system*. Each vertex of a dominating system is called a *dominating vertex*. The *b-chromatic number* $\phi(G)$ of a graph G is defined as the maximum k such that G admits a dominating proper k -coloring.

The b-chromatic number was introduced in [7]. The motivation, similarly as for the previously studied *achromatic number* (cf. e.g. [5, 6]), comes from algorithmic graph theory. The achromatic number $\psi(G)$ of a graph G is the largest number of colors which can be assigned to the vertices of G such that the coloring is proper and every pair of distinct colors appears on an edge. A proper coloring of a graph G using $k > \chi(G)$ colors could be improved if the vertices of two color classes could be recolored by a single color so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by the achromatic number. A more versatile form of recoloring strategy would be to allow the vertices of a single color class to be redistributed among the colors of the remaining classes, so as to obtain a proper coloring. The largest number of colors for which such a recoloring strategy is not possible is given by $\phi(G)$ (these recolorings are discussed in [7] and [11]). Thus $\phi(G) \leq \psi(G)$ (also given in [7]). From this point of view, both complexity results and tight bounds for the b-chromatic number are interesting. The following bounds of b-chromatic number are already presented in [7].

Proposition 1 *Assume that the vertices x_1, x_2, \dots, x_n of G are ordered such that $d(x_1) \geq d(x_2) \geq \dots \geq d(x_n)$. Then $\phi(G) \leq m(G) \leq \Delta(G) + 1$, where $m(G) = \max\{1 \leq i \leq n : d(x_i) \geq i - 1\}$ and $\Delta(G)$ is the maximum degree of G .*

R. W. Irving and D. F. Manlove [7] proved that finding the b-chromatic number of any graph is a NP-hard problem, and they gave a polynomial-time algorithm for finding the b-chromatic number of trees. Kouider and Mahéo [9] gave some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. They gave, in particular, a lower bound for the b-chromatic number of the cartesian product of two graphs where each one has a stable dominating system. More recently in [11], the authors characterized bipartite graphs for which the lower bound on the b-chromatic number is attained and proved the NP-completeness of the problem to decide whether there is a dominating proper k -coloring even for connected bipartite graphs and $k = \Delta(G) + 1$. They also determine the asymptotic behavior for the b-chromatic number of random graphs.

In this paper, we present several exact values and determine bounds for the b-chromatic number of power graphs of paths and cycles.

Let $Diam(G)$ be the diameter of a graph G , defined as the maximum distance between any pair of vertices of G . Let us begin with the following observation.

Fact 2 *For any graph G of order n , if $Diam(G) \leq p$, then $\phi(G^p) = n$, with $p \geq 2$.*

Proof. If $Diam(G) \leq p$, it is trivial to see that G^p is a complete graph. So $\phi(G^p) = n$. □

Let G be a path or a cycle on vertices x_1, x_2, \dots, x_n . We fix an orientation of G (left to right if G is a path and clockwise if G is a cycle). For each $1 \leq i \leq n$, we denote by x_i^+ (resp. x_i^-) the successor (resp.

predecessor) of x_i in G (if any). For $1 \leq i \neq j \leq n$, we define $[x_i, x_j]_G$, $[x_i, x_j)_G$ and $(x_i, x_j)_G$ as the set of consecutive vertices on G from respectively x_i to x_j , x_i to x_j^- and x_i^+ to x_j^- , following the fixed orientation of G . If there is no ambiguity, we denote $[x_i, x_j]_G$, $[x_i, x_j)_G$ and $(x_i, x_j)_G$ by respectively $[x_i, x_j]$, $[x_i, x_j)$ and (x_i, x_j) .

In all figures, the graph G is represented with solid edges. Edges added in a p -th power graph G^p are represented with dashed edges. In some figures, vertices are surrounded and represent a dominating system of the coloring. In any coloring of a graph G , we will say that a vertex x of G is *adjacent* to a color i if there exists a neighbor of x which is colored by i .

2 Power Graph of a Path

In this section, we determine the b-chromatic number of a p -th power graph of a path, with $p \geq 1$. First we give a lemma used in the proof of Theorem 4. Then the b-chromatic number of a p -th power graph of a path is computed.

Lemma 3 *For any $p \geq 1$, and for any $n \geq p + 1$, let P_n be the path on vertices x_1, x_2, \dots, x_n . For each integer k , with $p + 1 \leq k \leq \min(2p + 1, n)$, there exists a proper k -coloring on P_n^p . Moreover each vertex x , such that $x \in \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$, is adjacent to each color j , with $1 \leq j \neq c_x \leq k$.*

Proof. As $k \geq p + 1$, it is easy to see that if we put the set of colors $\{1, 2, \dots, k\}$ cyclically on $V(P_n)$, then two adjacent vertices will not have the same color. The coloring is thus a proper k -coloring.

Let $S = \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$. First we show that each vertex of S is adjacent to at least $k - 1$ vertices. Observe that the vertex x_{k-p} is adjacent to $(k - p - 1) + p = k - 1$ vertices. And the vertex $x_{n-k+p+1}$ is adjacent to $p + n - (n - k + p + 1) = k - 1$ vertices. Since each vertex x_i , with $k - p + 1 \leq i \leq n - k + p$, has a degree $d(x_i) \geq d(x_{k-p})$, then each vertex of S is adjacent to at least $k - 1$ other vertices.

Next, we can see by the construction that all the colors $\{1, 2, \dots, k\} \setminus \{c_{x_i}\}$ appear between the first and the last neighbor of x_i . Therefore each vertex x_i of S is adjacent to each color j , with $1 \leq j \neq c_{x_i} \leq k$ and $k - p \leq i \leq n - k + p + 1$. \square

The b-chromatic number of a p -th power graph of a path is given by:

Theorem 4 *Let P_n be a path on vertices x_1, x_2, \dots, x_n . The b-chromatic number of P_n^p , with $p \geq 1$, is given by:*

$$\varphi(P_n^p) = \begin{cases} n & \text{if } n \leq p + 1, & (1) \\ p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor & \text{if } p + 2 \leq n \leq 4p + 1, & (2) \\ 2p + 1 & \text{if } n \geq 4p + 2 & (3) \end{cases}$$

Proof.

1. If $n \leq p + 1$, then $\text{Diam}(P_n) \leq p$. So, by Fact 2, $\varphi(P_n^p) = n$.
2. We prove first that $\varphi(P_n^p) \geq p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$ for $p + 2 \leq n \leq 4p + 1$. Let $k = p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$. By Lemma 3, we give a proper k -coloring of P_n^p . For example, Figure 1 shows a dominating proper 5-coloring of P_8^3 .

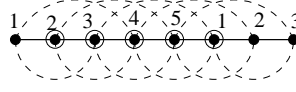


Fig. 1: Coloring of P_8^3

Let S' be the set of vertices $\{x_{k-p}, x_{k-p+1}, \dots, x_{2k-p-1}\}$. Since $2k-p-1 \leq n-k+p+1$, then $S' \subseteq \{x_{k-p}, x_{k-p+1}, \dots, x_{n-k+p+1}\}$. By Lemma 3, S' is a dominating system. As the coloring is proper and has a dominating system, we obtain a dominating proper k -coloring. So, $\varphi(P_n^p) \geq p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

Next we prove that $\varphi(P_n^p) \leq p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$ for $p+2 \leq n \leq 4p+1$. The proof is by contradiction. Suppose that there exists a dominating proper k' -coloring such that

$$k' > p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor. \quad (1)$$

Let $W = \{w_1, w_2, \dots, w_{k'}\}$ be a dominating system of the coloring on P_n^p (following the orientation of P_n , we meet $w_1, w_2, \dots, w_{k'}$). The vertices w_1 and $w_{k'}$ are adjacent to, at most, p different colors in $[w_1, w_{k'}]$. As w_1 (respectively $w_{k'}$) is a dominating vertex, it must be adjacent to at least $k' - 1$ different colors. Then, there are at least $k' - p - 1$ vertices on $[x_1, w_1]$ (respectively $(w_{k'}, x_n)$). Therefore, $n - k' \geq n - |[w_1, w_{k'}]| \geq 2(k' - p - 1)$.

On the other hand by hypothesis $k' \geq p+2 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$, so that $n - k' \leq n - p - 2 - \left\lfloor \frac{n-p-1}{3} \right\rfloor$. These two results give the following inequality,

$$2(k' - p - 1) \leq n - k' \leq n - p - 2 - \left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

$$k' \leq \frac{1}{2}(n+p - \left\lfloor \frac{n-p-1}{3} \right\rfloor). \quad (2)$$

By (1) and (2), we obtain,

$$\frac{1}{2}(n+p - \left\lfloor \frac{n-p-1}{3} \right\rfloor) \geq k' \geq p+2 + \left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

$$n-p-4 \geq 3 \left\lfloor \frac{n-p-1}{3} \right\rfloor,$$

which is a contradiction. Hence such a coloring does not exist. Therefore, $\varphi(P_n^p) \leq p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

We deduce from these two parts that $\varphi(P_n^p) = p+1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor$.

3. $\Delta(P_n^p) = 2p$, so by Proposition 1, $\varphi(P_n^p) \leq 2p + 1$. Lemma 3 gives a proper $(2p + 1)$ -coloring and shows that each vertex x of the set $\{x_{p+1}, x_{p+2}, \dots, x_{3p+1}\}$ is adjacent to each color j with $1 \leq j \neq c_x \leq k$. So this set is a dominating system and $\varphi(P_n^p) \geq 2p + 1$. Therefore $\varphi(P_n^p) = 2p + 1$. For example, Figure 2 gives a dominating proper 7-coloring of P_{15}^3 .

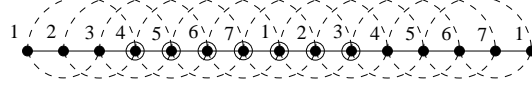


Fig. 2: Coloring of P_{15}^3

3 Power Graph of a Cycle

In this section, we study the b -chromatic number of a p -th power graph of a cycle, with $p \geq 1$. First we give two lemmas used in the proof of Theorem 7. Then we bound the b -chromatic number of a p -th power graph of a cycle.

Lemma 5 *Let C_n^p be a p -th power graph of a cycle C_n , with $p \geq 2$. For any $2p + 3 \leq n \leq 4p$, let $k \geq \min(n - p - 1, p + 1 + \lfloor \frac{n-p-1}{3} \rfloor)$. Then $n \leq 2k$.*

Proof. The proof is by contradiction. Suppose $n \geq 2k + 1$. We consider two cases. Firstly, $k \geq n - p - 1$. So,

$$\begin{aligned} n &\geq 2k + 1 \geq 2(n - p - 1) + 1, \\ n &\leq 2p + 1, \end{aligned}$$

which is a contradiction. Secondly, $k \geq p + 1 + \lfloor \frac{n-p-1}{3} \rfloor$. So,

$$\begin{aligned} n &\geq 2k + 1 \geq 2\left(p + 1 + \left\lfloor \frac{n-p-1}{3} \right\rfloor\right) + 1, \\ n - 2p - 3 &\geq 2 \left\lfloor \frac{n-p-1}{3} \right\rfloor, \end{aligned}$$

which is a contradiction too. □

Lemma 6 *For any $p \geq 2$, and for any $2p + 3 \leq n \leq 4p$, let C_n be the cycle on vertices x_1, x_2, \dots, x_n . Let $k = \min(n - p - 1, p + 1 + \lfloor \frac{n-p-1}{3} \rfloor)$. So there exists a proper k -coloring on C_n^p . Moreover each vertex x , such that $x \in \{x_{k-p}, x_{k-p+1}, \dots, x_{2k-p-1}\}$, is adjacent to each color j , with $1 \leq j \neq c_x \leq k$.*

Proof. We put the set of colors $\{1, 2, \dots, k\}$ cyclically on $V(C_n)$. As $k \leq p + 1 + \lfloor \frac{n-p-1}{3} \rfloor$ and $n \leq 4p$, then $k \leq 2p + 1$. Moreover, by Lemma 5 we deduce that $2k \geq n \geq 2p + 3 \geq k + 2$. So, the full set of colors $\{1, 2, \dots, k\}$ appears consecutively at least once, and at most twice, in the cyclic coloring of C_n^p . As $2k \geq n \geq 2p + 3$, we have $k \geq p + 1$. Furthermore, by definition of k we have $n - k \geq p + 1$. Thus, as $k \geq p + 1$ and $n - k \geq p + 1$, the coloring is proper.

Let P_n be the subpath of C_n induced by x_1, x_2, \dots, x_n . Let $S = \{x_{k-p}, x_{k-p+1}, \dots, x_{k+(k-p-1)}\}$. As $p+1 \leq k \leq 2p+1$ and $2k-p-1 \leq n-k+p+1$, then by Lemma 3 each vertex x_i of S , with $k-p \leq i \leq 2k-p-1$, is adjacent to each color q , with $1 \leq q \neq c_{x_i} \leq k$, on P_n^p . Therefore each vertex x_i of S is adjacent to each color q , with $1 \leq q \neq c_{x_i} \leq k$, on C_n^p . \square

Theorem 7 *Let C_n be a cycle on vertices x_1, x_2, \dots, x_n . The b -chromatic number of C_n^p , with $p \geq 1$, is*

$$\varphi(C_n^p) = \begin{cases} n & \text{if } n \leq 2p+1, & (1) \\ p+1 & \text{if } n = 2p+2, & (2) \\ (\geq) \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor) & \text{if } 2p+3 \leq n \leq 3p & (3) \\ p+1 + \lfloor \frac{n-p-1}{3} \rfloor & \text{if } 3p+1 \leq n \leq 4p & (4) \\ 2p+1 & \text{if } n \geq 4p+1 & (5) \end{cases}$$

Proof.

1. If $n \leq 2p+1$, then $\text{Diam}(C_n) \leq p$. So, by Fact 2, $\varphi(C_n^p) = n$.
2. To color the graph, we put the set of colors $\{1, 2, \dots, p+1\}$ cyclically twice. One can easily see that this coloring is a proper $(p+1)$ -coloring. Let S be the set of vertices $\{x_1, x_2, \dots, x_{p+1}\}$. Each vertex x_i , with $1 \leq i \leq p+1$, is adjacent to $n-2$ vertices. Since $n-2 \geq p+1$, then each vertex x_i ($1 \leq i \leq p+1$) of S is adjacent to all colors other than c_{x_i} . So the set S is a dominating system. We now show that, in any dominating proper coloring, vertices x_i and x_{i+p+1} must have the same color. For the subgraph induced by vertices x_1, x_2, \dots, x_{p+1} , we have a clique and we can assume without loss of generality that these vertices are colored by $1, 2, \dots, p+1$ respectively. If there exists a dominating vertex of color j , for some $j > p+1$, then this vertex is x_{p+1+i} for some i ($1 \leq i \leq p+1$). Vertex x_{p+1+i} is not adjacent to x_i , but every other vertex is adjacent to x_i , so that x_{p+1+i} cannot be a dominating vertex, a contradiction. Therefore $\varphi(C_n^p) = p+1$ for $n = 2p+2$.
3. Let $k = \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$.
By Lemma 6 there exists a dominating proper k -coloring for $2p+3 \leq n \leq 3p$. Therefore $\varphi(C_n^p) \geq \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$. For example, in Figure 3, we give a dominating proper 6-coloring of C_{11}^4 .
4. Let $k = p+1 + \lfloor \frac{n-p-1}{3} \rfloor$.
For $3p+1 \leq n \leq 4p$, Lemma 6 gives a dominating proper k -coloring. This proves that $\varphi(C_n^p) \geq \min(n-p-1, p+1 + \lfloor \frac{n-p-1}{3} \rfloor)$. For example, Figure 4 shows a dominating proper 6-coloring of C_{11}^3 .
Next, we prove that $\varphi(C_n^p) \leq k$. Suppose there exists a dominating proper k' -coloring for C_n^p , with $k' \geq p+2 + \lfloor \frac{n-p-1}{3} \rfloor$, for the sake of contradiction. Let $W = \{w_1, w_2, \dots, w_{k'}\}$ be a set of dominating vertices on C_n (following the orientation of C_n , we meet $w_1, w_2, \dots, w_{k'}$). We distinguish two cases.

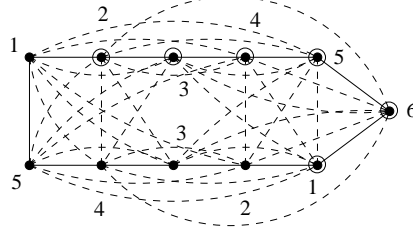


Fig. 3: Coloring of C_{11}^4 ($n - p - 1 = 6$, $p + 1 + \lfloor \frac{n-p-1}{3} \rfloor = 7$ and $\phi(C_{11}^4) \geq 6$)

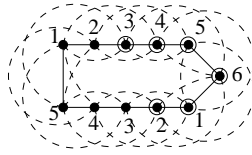


Fig. 4: Coloring of C_{11}^3

Case 1: for each i , with $1 \leq i \leq k'$, $|(w_i, w_{i+1})| \leq p - 1$.

As $k' \geq p + 2 + \lfloor \frac{n-p-1}{3} \rfloor$, by a straightforward modification of the proof of Lemma 5, we have $n < 2k'$. So, there exists at least one color c not repeated in C_n^p (i.e. there are not two distinct vertices with the same color c). Without loss of generality, suppose that c appears on the vertex x , with $x \in V(C_n)$. Therefore x is a dominating vertex and each other dominating vertex is adjacent to x . Then, $|(w_1, x)| \leq p$ and $|(x, w_{k'})| \leq p$. As for each i , with $1 \leq i \leq k'$, we have $|(w_i, w_{i+1})| \leq p - 1$ and since on the cycle the next dominating vertex from $w_{k'}$ is w_1 , then

$$|(w_{k'}, w_1)| \leq p - 1,$$

where

$$|(w_{k'}, w_1)| = n - |[w_1, x]| - |[x, w_{k'}]| - 1.$$

Therefore, we have

$$n - |[w_1, x]| - |[x, w_{k'}]| - 1 \leq p - 1,$$

$$n - 2p - 1 \leq p - 1,$$

$$n \leq 3p,$$

which is a contradiction.

Case 2: There exists r , with $1 \leq r \leq k'$ and r is taken modulo k' , such that $|(w_r, w_{r+1})| \geq p$.

Let X be the set of vertices of $[w_{r+1}, w_r]$ (see Figure 5). Let X_C be the set of colors appearing in X . Let $\Gamma_X(x_i)$ be the set of neighbors of x_i in X and $\Gamma_X^c(x_i)$ the set of colors appearing in $\Gamma_X(x_i)$, with $1 \leq i \leq n$. Let $A = X_C \setminus (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$. Let $B = X_C \setminus (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$. We discuss two subcases.

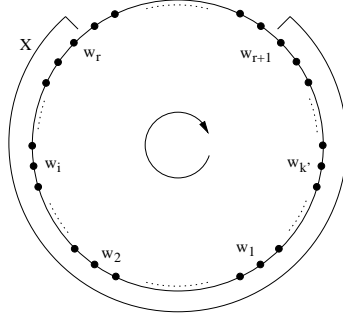


Fig. 5: A dominating system on C_n^p and the set X

Subcase 1: $|X| \leq 2p + 2$. Since all dominating vertices belong to X , we have $|X| \geq k'$. Then, $|(w_r, w_{r+1})| \leq n - k'$ and $|X_C| = k'$. As the vertices of $\Gamma_X(w_r)$ form a clique, then $|\Gamma_X^c(w_r)| = |\Gamma_X(w_r)| = p$. So we have $|A| = |X_C| - |\Gamma_X^c(w_r)| - 1 = k' - p - 1$. In the same way, we deduce that $|B| = k' - p - 1$. As $|X| \leq 2p + 2$, we have $X \subseteq (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.a). So, $A \subseteq (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$ and $B \subseteq (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$. Let $q \in \{1, 2, \dots, k'\}$. If $q \in A$ (resp. $q \in B$) then $q \notin (\Gamma_X^c(w_r) \cup \{c_{w_r}\})$ (resp. $q \notin (\Gamma_X^c(w_{r+1}) \cup \{c_{w_{r+1}}\})$) and so $q \notin B$ (resp. $q \notin A$). Therefore, $A \cap B = \emptyset$. As w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}) . Therefore,

$$\begin{aligned} |A| + |B| &\leq |(w_r, w_{r+1})|, \\ 2(k' - p - 1) &\leq n - k', \\ 3k' &\leq n + 2p + 2, \\ 3 \left\lfloor \frac{n - p - 1}{3} \right\rfloor &\leq n - p - 4, \end{aligned}$$

which is a contradiction.

Subcase 2: $|X| \geq 2p + 3$. As in Subcase 1, we have $|A| = k' - p - 1$ and $|B| = k' - p - 1$. Let $X' = X \setminus (\Gamma_X(w_r) \cup \Gamma_X(w_{r+1}) \cup \{w_r, w_{r+1}\})$ (see Figure 6.b). So $|X'| \geq |A \cap B|$. Since w_r (resp. w_{r+1}) is a dominating vertex and $|(w_r, w_{r+1})| \geq p$, the colors of A (resp. B) must be repeated in (w_r, w_{r+1}) . Then,

$$\begin{aligned} |A| + |B| - |A \cap B| &\leq |(w_r, w_{r+1})| \leq n - 2p - 2 - |X'|, \\ 2(k' - p - 1) - |A \cap B| &\leq n - 2p - 2 - |A \cap B|, \\ 2(p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor) &\leq n, \end{aligned}$$

which is a contradiction. Therefore there does not exist a dominating proper k' -coloring, with $k' \geq p + 2 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.

This completes the proof of $\phi(C_n^p) = p + 1 + \left\lfloor \frac{n - p - 1}{3} \right\rfloor$.

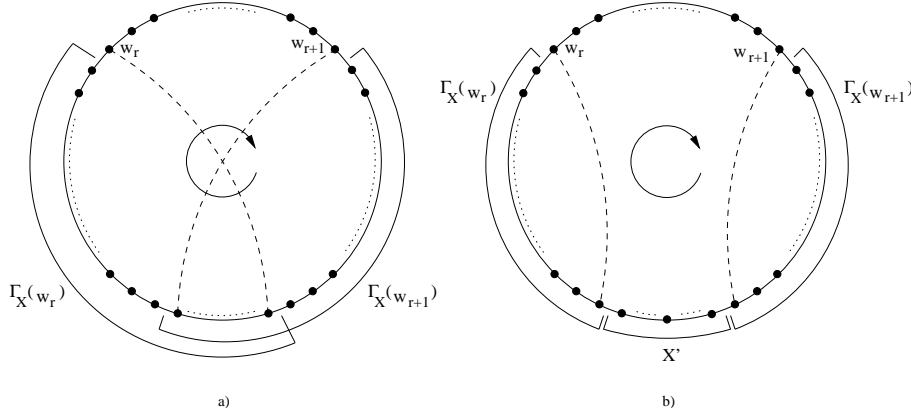


Fig. 6: Neighborhoods of w_r and w_{r+1} on X when a) $|X| \leq 2p + 2$ and b) $|X| \geq 2p + 3$

5. As $\Delta = 2p$, by Proposition 1, $\varphi(C_n^p) \leq 2p + 1$.

We then give a proper $(2p + 1)$ -coloring. It is constructed in two steps. First, we put $(2p + 1)$ different colors on the $(2p + 1)$ first vertices ($c_{x_i} := i$ for $1 \leq i \leq 2p + 1$). In the second step, we have two cases. If $n = 4p + 1$, we color the remaining vertices as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p + 2 \leq i \leq n$. If $n \geq 4p + 2$, then the remaining vertices are colored as follows: $c_{x_i} := c_{x_{i-2p-1}}$ for $2p + 2 \leq i \leq 4p + 2$, and $c_{x_i} := c_{x_{i-p-1}}$ for $4p + 3 \leq i \leq n$. Then the distance between two vertices colored by the same color c is at least $p + 1$. So the coloring is proper. By an analogue proof of Lemma 3, one can prove that each vertex x_i , with $p + 1 \leq i \leq 3p + 1$, is a dominating vertex. So this coloring is a dominating proper $(2p + 1)$ -coloring. This construction shows that $\varphi(C_n^p) \geq 2p + 1$. Therefore we have proved that $\varphi(C_n^p) = 2p + 1$. For example, Figure 7 gives a dominating proper 7-coloring C_{16}^3 . \square

4 Open Problem

In section 3, we have obtained the exact values of $\varphi(C_n^p)$, except in case $2p + 3 \leq n \leq 3p$ where we give a lower bound. We believe that $\min(n - p - 1, p + 1 + \lfloor \frac{n-p-1}{3} \rfloor)$ is the exact value of $\varphi(C_n^p)$ for $2p + 3 \leq n \leq 3p$.

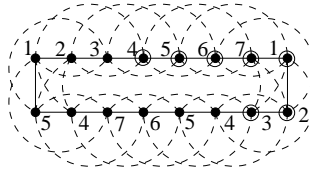


Fig. 7: Coloring of C_{16}^3

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References

- [1] J.-C. BERMOND, F. CORMELLAS, D. F. HSU, Distributed loop computer networks: a survey, *Journal of Parallel and Distributed Computing* 24 (1995) 2-10.
- [2] A. BRANDSTÄDT, V. D. CHEPOI, F. F. DRAGAN, Perfect elimination orderings of chordal powers of graphs, *Discrete Mathematics* 158 (1996) 273-278.
- [3] E. DAHLHAUS AND P. DUCHET, On strongly chordal graphs, *Ars Combinatoria* 24B (1987) 23-30.
- [4] R.B. EGGLETON, P. ERDŐS AND D.K. SKILTON, Coloring the real line, *Journal of Combinatorial Theory* B39 (1985) 86-100.
- [5] F. HARARY AND S. HEDETNIEMI, The achromatic number of a graph, *Journal of Combinatorial Theory* 8 (1970) 154-161.
- [6] F. HUGHES AND G. MACGILLIVRAY, The achromatic number of graphs: A survey and some new results, *Bull. Inst. Comb. Appl.* 19 (1997) 27-56.
- [7] R. W. IRVING AND D. F. MANLOVE, The b-chromatic number of a graph, *Discrete Applied Mathematics* 91 (1999) 127-141.
- [8] H.KHEDDOUCI, J.F.SACLÉ AND M.WOŹNIAK, Packing of two copies of a tree into its fourth power, *Discrete Mathematics* 213 (1-3) (2000) 169-178.
- [9] M.KOUIDER AND M.MAHEO, Some bounds for the b-chromatic number of a graph, *Discrete Mathematics* 256, Issues 1-2, (2002) 267-277.
- [10] D.E. KNUTH, The Art of Computer Programming, Vol. 3 *Addison-Wesley, Reading, MA.* 1975.
- [11] J. KRATOCHVÍL, Z. TUZA AND M. VOIGT, On the b-chromatic number of graphs, *Proceedings WG'02 - 28th International Workshop on Graph-Theoretic Concepts in Computer Science, Cesky Krumlov, Czech Republic, volume 2573 of Lecture Notes in Computer Science.* Springer Verlag 2002.
- [12] I.Z. RUZSA, Z. TUZA AND M. VOIGT, Distance Graphs with Finite Chromatic Number, *Journal of Combinatorial Theory* B85 (2002) 181-187.
- [13] X. ZHU, Pattern periodic coloring of distance graphs, *Journal of Combinatorial Theory* B73 (1998) 195-206.