

Fountains, histograms, and q-identities

Peter Paule, Helmut Prodinger

► **To cite this version:**

Peter Paule, Helmut Prodinger. Fountains, histograms, and q-identities. *Discrete Mathematics and Theoretical Computer Science*, DMTCS, 2003, 6 (1), pp.101-106. hal-00958993

HAL Id: hal-00958993

<https://hal.inria.fr/hal-00958993>

Submitted on 13 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fountains, histograms, and q -identities

Peter Paule^{1†} and Helmut Prodinger^{2‡}

¹Research Institute for Symbolic Computation, Johannes Kepler University Linz, A-4040 Linz, Austria

e-mail: Peter.Paule@risc.uni-linz.ac.at

Url: <http://www.risc.uni-linz.ac.at/research/combinat/>

²The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa

e-mail: helmut@maths.wits.ac.za

Url: <http://www.wits.ac.za/helmut/>

received Jan 14, 2003, accepted Aug 15, 2003.

We solve the recursion $S_n = S_{n-1} - q^n S_{n-p}$, both, explicitly, and in the limit for $n \rightarrow \infty$, proving in this way a formula due to Merlini and Sprugnoli. It is also discussed how computer algebra could be applied.

Keywords: q -identities, fountains, histograms, Schur polynomials

1 Fountains and histograms

Merlini and Sprugnoli [6] discuss *fountains* and *histograms*; for the reader's convenience, we review a few key issues here.

A *fountain with n coins* is an arrangement of n coins in rows such that each coin in a higher row touches exactly two coins in the next lower row.

A *p -histogram* is a sequence of columns in which the height of the $(j+1)$ st column is at most $k+p$, if k is the height of column j ; the first column has height r , with $1 \leq r \leq p$.

It can be shown that the enumeration of coins in a fountain is equivalent with the enumeration of 1-histograms. The paper [6] addresses the enumeration of p -histograms with respect to area (=number of cells). Let $f_n^{[p]}$ be the number p -histograms with area n and $F^{[p]}(q)$ the corresponding generating function $F^{[p]}(q) = \sum_n f_n^{[p]} q^n$. The authors of [6] use two different approaches: one produces the answer in the form

$$F^{[p]}(q) = \lim_{m \rightarrow \infty} \frac{D_m}{E_m},$$

[†]Partially supported by SFB grant F1305 of the Austrian FWF.

[‡]The research was started when this author visited the RISC in 2002. Partially supported by NRF grant 2053748.

with some polynomials D_m, E_m defined in the next section, and the other gives it as

$$F^{[p]}(q) = \sum_{k \geq 0} \frac{(-1)^k q^{p \binom{k+1}{2}}}{(1-q) \dots (1-q^k)} \bigg/ \sum_{k \geq 0} \frac{(-1)^k q^{k+p \binom{k}{2}}}{(1-q) \dots (1-q^k)}.$$

According to [5], it would be nice to have a direct argument that these two answers coincide. This is the subject of the present note.

2 Generalized Schur polynomials

The polynomials mentioned in the introduction are for fixed $p \geq 1$ defined as follows:

$$\begin{aligned} E_n &= E_{n-1} - q^n E_{n-p}, \quad n \geq p, & E_0 &= \dots = E_{p-1} = 1, \\ D_n &= D_{n-1} - q^n D_{n-p}, \quad n \geq p, & D_i &= 1 - \sum_{j=1}^i q^j, \quad i = 0, \dots, p-1. \end{aligned}$$

They can be compared with the classical Schur polynomials [8], which occur for $p = 2$ and $q = -1$. Then Merlini and Sprugnoli want a direct proof of the formulæ

$$\begin{aligned} E_\infty &:= \lim_{n \rightarrow \infty} E_n = \sum_{k \geq 0} \frac{(-1)^k q^{p \binom{k+1}{2}}}{(1-q) \dots (1-q^k)}, \\ D_\infty &:= \lim_{n \rightarrow \infty} D_n = \sum_{k \geq 0} \frac{(-1)^k q^{k+p \binom{k}{2}}}{(1-q) \dots (1-q^k)}. \end{aligned}$$

We will not only achieve that but actually derive *explicit* expressions for these polynomials!

It should be mentioned that Cigler [4] developed independently a combinatorial method to deal with recursions as ours, but also more general ones.

Let us study the generic recursion

$$S_n = S_{n-1} + tq^{n-p} S_{n-p},$$

with unspecified initial values S_0, \dots, S_{p-1} . For $p = 2$, these polynomials were studied by Andrews (and others) in the context of *Schur polynomials*, see [2].

We will use standard notation from q -calculus, see [1]:

$$(x)_n = (1-x)(1-xq) \dots (1-xq^{n-1}), \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

It will be convenient to define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < 0$ or $k > n$.

Now we will proceed as in [1] and consider noncommutative variables x, η , such that $x\eta = q\eta x$; all other variables commute.

Lemma 1.

$$(x + x^p \eta)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} \eta^{n-k}.$$

Proof. We write

$$(x + x^p \eta)^n = \sum_{k=0}^n a_{n,k} x^{k+p(n-k)} \eta^{n-k},$$

and $(x + x^p \eta)^{n+1} = (x + x^p \eta)^n (x + x^p \eta)$ resp. as $(x + x^p \eta)^{n+1} = (x + x^p \eta)(x + x^p \eta)^n$, compare coefficients, and get the recursions

$$\begin{aligned} a_{n+1,k} &= a_{n,k-1} + a_{n,k} q^{k+p(n-k)}, \\ a_{n+1,k} &= a_{n,k-1} q^{n+1-k} + a_{n,k} q^{p(n-k)}. \end{aligned}$$

From this we derive, taking differences,

$$a_{n,k} = \frac{1 - q^{n+1-k}}{1 - q^k} q^{-p(n-k)} a_{n,k-1}.$$

The result follows from iteration by noting that $a_{n,0} = q^{p \binom{n}{2}}$. □

Of course we also have

$$(x + tx^p \eta)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \eta^{n-k}.$$

Now we derive the generating function for

$$F(x) = \sum_{n \geq 0} S_n x^n;$$

the following procedure is inspired by [2]. Note that we can alternatively view η as an operator, defined by $\eta f(x) = f(qx)$. Cigler worked also much with this technique [3, 4]. We find

$$\sum_{n \geq p} S_n x^n = \sum_{n \geq p} S_{n-1} x^n + \sum_{n \geq p} t q^{n-p} S_{n-p} x^n = x \sum_{n \geq p-1} S_n x^n + t x^p \sum_{n \geq 0} \eta S_n x^n$$

or

$$F(x) - \sum_{n < p} S_n x^n = x F(x) - x \sum_{n < p-1} S_n x^n + t x^p \eta F(x),$$

and

$$F(x) = \frac{1}{1 - x - t x^p \eta} \left(\sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1} \right).$$

Now we can apply our lemma and write

$$\begin{aligned} F(x) &= \sum_{n \geq 0} (x + tx^p \eta)^n \left(\sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1} \right) \\ &= \sum_{n \geq 0} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \eta^{n-k} \left(\sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1} \right) \\ &= \sum_{n \geq 0} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \left(\sum_{i < p} S_i q^{i(n-k)} x^i - \sum_{i < p-1} S_i q^{(i+1)(n-k)} x^{i+1} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} q^{p \binom{k}{2}} x^{n-k+pk} t^k \left(\sum_{i < p} S_i q^{ik} x^i - \sum_{i < p-1} S_i q^{(i+1)k} x^{i+1} \right) \\
&= \sum_{k, n \geq 0} \binom{n+k}{k} q^{p \binom{k}{2}} x^{n+pk} t^k \left(\sum_{i < p} S_i q^{ik} x^i - \sum_{i < p-1} S_i q^{(i+1)k} x^{i+1} \right) \\
&= \sum_{k \geq 0} q^{p \binom{k}{2}} x^{pk} t^k \frac{1}{(x)_{k+1}} \left(\sum_{i < p} S_i q^{ik} x^i - \sum_{i < p-1} S_i q^{(i+1)k} x^{i+1} \right).
\end{aligned}$$

From this we find an explicit formula for S_n (the quantity S_{-1} has to be interpreted as 0):

$$S_n = \sum_{0 \leq i < p} (S_i - S_{i-1}) \sum_{k \geq 0} \binom{n - (p-1)k - i}{k} q^{p \binom{k}{2} + ik} t^k.$$

Now we specialize this to our instance. Here, $t = -q^p$, and thus

$$S_n = \sum_{0 \leq i < p} (S_i - S_{i-1}) \sum_{k \geq 0} \binom{n - (p-1)k - i}{k} q^{p \binom{k+1}{2} + ik} (-1)^k.$$

Therefore

$$E_n = \sum_{k \geq 0} \binom{n - (p-1)k}{k} q^{p \binom{k+1}{2}} (-1)^k.$$

From this, the limit of E_n is immediate. For D_n we eventually get the following form

$$D_n = \sum_{k \geq 0} \binom{n - (p-1)(k-1)}{k} q^{k+p \binom{k}{2}} (-1)^k,$$

from which the formula for D_∞ is immediate. To prove it, we need a simple lemma whose proof is just a routine calculation.

Lemma 2.

$$\binom{m-i}{k} q^{i(k+1)} = g(i) - g(i-1) \quad \text{where} \quad g(i) = -\binom{m-i}{k+1} q^{(i+1)(k+1)}.$$

□

Now we can plug into the general formula above and compute

$$\begin{aligned}
D_n &= E_n - \sum_{i=1}^{p-1} \sum_{k \geq 0} \binom{n - (p-1)k - i}{k} q^{p \binom{k+1}{2} + i(k+1)} (-1)^k \\
&= E_n - \sum_{k \geq 0} (-1)^k q^{p \binom{k+1}{2}} \sum_{i=1}^{p-1} \binom{n - (p-1)k - i}{k} q^{i(k+1)} \\
&= E_n - \sum_{k \geq 0} (-1)^k q^{p \binom{k+1}{2}} \left\{ q^{k+1} \binom{n - (p-1)k}{k+1} - q^{p(k+1)} \binom{n - (p-1)(k+1)}{k+1} \right\} \\
&= 1 - \sum_{k \geq 0} (-1)^k q^{p \binom{k+1}{2}} q^{k+1} \binom{n - (p-1)k}{k+1},
\end{aligned}$$

which is the announced formula after a simple change of variable. Note that in the penultimate step the telescoping property of the lemma has been used.

3 Computer algebra proofs

The polynomial families (E_n) and (D_n) give rise to the following study with respect to possible computer proofs. Let us take as input our sum representations of E_n and D_n :

$$\begin{aligned} E_n &= \sum_{k \geq 0} \binom{n - (p-1)k}{k} q^{p \binom{k+1}{2}} (-1)^k, \\ D_n &= \sum_{k \geq 0} \binom{n - (p-1)(k-1)}{k} q^{k+p \binom{k}{2}} (-1)^k. \end{aligned} \quad (3.1)$$

Then, if p is chosen as a specific positive integer, Riese's package `qZeil` [7] returns the recurrences $S_n = S_{n-1} - q^n S_{n-p}$ ($n \geq p$) together with a certificate function `Cert` for independent verification. Despite the fact that for general "generic" integer parameter p there is no algorithm available, a general pattern can be easily guessed from running the algorithm for $p = 1$, $p = 2$, and $p = 3$, say.

For example, let $F(n, k)$ be the k th summand in our sum representation (3.1) of E_n , then the recurrence for E_n can be refined to the following statement.

Theorem 3.1. *For $n \geq p$ and $\delta_k f(n, k) = f(n, k) - f(n, k-1)$, we have*

$$F(n, k) - F(n-1, k) + q^n F(n-p, k) = \delta_k \text{Cert}(n, k) F(n, k), \quad (3.2)$$

where

$$\text{Cert}(n, k) = q^n \frac{(q^{n-p(k+1)+1})_p}{(q^{n-(p-1)(k+1)})_p}.$$

Proof. After dividing both sides of (3.2) by $F(n, k)$ the proof reduces to checking equality of rational functions. Namely, note that

$$\begin{aligned} \frac{F(n-1, k)}{F(n, k)} &= \frac{1 - q^{n-pk}}{1 - q^{n-(p-1)k}}, \\ \frac{F(n, k-1)}{F(n, k)} &= -\frac{q^{pk}}{1 - q^k} \frac{(q^{n-pk+1})_p}{(q^{n-(p-1)k+1})_{p-1}}, \end{aligned}$$

and

$$\frac{F(n-p, k)}{F(n, k)} = q^{-n} \text{Cert}(n, k).$$

□

Analogously, there is a refined version of the recurrence for D_n . The certificate in this case is

$$\text{Cert}(n, k) = q^n \frac{(q^{n-pk})_p}{(q^{n-(p-1)k})_p}.$$

Summarizing, with the sum representation for E_n and D_n in hand, the corresponding recurrences follow immediately by summing both sides of the computer recurrences (3.2) over all $k \geq 0$.

References

- [1] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, 1999.
- [2] G. E. Andrews, *Fibonacci numbers and Rogers–Ramanujan identities*, The Fibonacci Quarterly, to appear (2003), 15 pp.
- [3] J. Cigler, *Elementare q -Identitäten*, Séminaire Lotharingien de Combinatoire **B05a** (1981), 29 pp.
- [4] J. Cigler, *Some algebraic aspects of Morse code sequences*, Discrete Mathematics and Theoretical Computer Science **6** (2003), 55–68.
- [5] D. Merlini, *Private communication*, (2002).
- [6] D. Merlini and R. Sprugnoli, *Fountains and histograms*, J. Algorithms **44** (2002), no. 1, 159–176.
- [7] P. Paule and A. Riese, *A Mathematica q -analogue of Zeilberger’s algorithm based on an algebraically motivated approach to q -hypergeometric telescoping*, Special functions, q -series and related topics (Toronto, ON, 1995), Fields Inst. Commun., vol. 14, Amer. Math. Soc., Providence, RI, 1997, pp. 179–210.
- [8] I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl., 1917, 302–321, reprinted in I. Schur, *Gesammelte Abhandlungen*, vol. 2, pp. 117–136, Springer, 1973.