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# Fountains, histograms, and $q$ -identities

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We solve the recursion  $S_n = S_{n-1} - q^n S_{n-p}$ , both, explicitly, and in the limit for  $n \rightarrow \infty$ , proving in this way a formula due to Merlini and Sprugnoli. It is also discussed how computer algebra could be applied.

**Keywords:**  $q$ -identities, fountains, histograms, Schur polynomials

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## 1 Fountains and histograms

Merlini and Sprugnoli [6] discuss *fountains* and *histograms*; for the reader's convenience, we review a few key issues here.

A *fountain with  $n$  coins* is an arrangement of  $n$  coins in rows such that each coin in a higher row touches exactly two coins in the next lower row.

A  *$p$ -histogram* is a sequence of columns in which the height of the  $(j+1)$ st column is at most  $k+p$ , if  $k$  is the height of column  $j$ ; the first column has height  $r$ , with  $1 \leq r \leq p$ .

It can be shown that the enumeration of coins in a fountain is equivalent with the enumeration of 1-histograms. The paper [6] addresses the enumeration of  $p$ -histograms with respect to area (=number of cells). Let  $f_n^{[p]}$  be the number  $p$ -histograms with area  $n$  and  $F^{[p]}(q)$  the corresponding generating function  $F^{[p]}(q) = \sum_n f_n^{[p]} q^n$ . The authors of [6] use two different approaches: one produces the answer in the form

$$F^{[p]}(q) = \lim_{m \rightarrow \infty} \frac{D_m}{E_m},$$

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with some polynomials  $D_m, E_m$  defined in the next section, and the other gives it as

$$F^{[p]}(q) = \sum_{k \geq 0} \frac{(-1)^k q^{p \binom{k+1}{2}}}{(1-q) \dots (1-q^k)} \bigg/ \sum_{k \geq 0} \frac{(-1)^k q^{k+p \binom{k}{2}}}{(1-q) \dots (1-q^k)}.$$

According to [5], it would be nice to have a direct argument that these two answers coincide. This is the subject of the present note.

## 2 Generalized Schur polynomials

The polynomials mentioned in the introduction are for fixed  $p \geq 1$  defined as follows:

$$\begin{aligned} E_n &= E_{n-1} - q^n E_{n-p}, \quad n \geq p, & E_0 &= \dots = E_{p-1} = 1, \\ D_n &= D_{n-1} - q^n D_{n-p}, \quad n \geq p, & D_i &= 1 - \sum_{j=1}^i q^j, \quad i = 0, \dots, p-1. \end{aligned}$$

They can be compared with the classical Schur polynomials [8], which occur for  $p = 2$  and  $q = -1$ . Then Merlini and Sprugnoli want a direct proof of the formulæ

$$\begin{aligned} E_\infty &:= \lim_{n \rightarrow \infty} E_n = \sum_{k \geq 0} \frac{(-1)^k q^{p \binom{k+1}{2}}}{(1-q) \dots (1-q^k)}, \\ D_\infty &:= \lim_{n \rightarrow \infty} D_n = \sum_{k \geq 0} \frac{(-1)^k q^{k+p \binom{k}{2}}}{(1-q) \dots (1-q^k)}. \end{aligned}$$

We will not only achieve that but actually derive *explicit* expressions for these polynomials!

It should be mentioned that Cigler [4] developed independently a combinatorial method to deal with recursions as ours, but also more general ones.

Let us study the generic recursion

$$S_n = S_{n-1} + tq^{n-p} S_{n-p},$$

with unspecified initial values  $S_0, \dots, S_{p-1}$ . For  $p = 2$ , these polynomials were studied by Andrews (and others) in the context of *Schur polynomials*, see [2].

We will use standard notation from  $q$ -calculus, see [1]:

$$(x)_n = (1-x)(1-xq) \dots (1-xq^{n-1}), \quad \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

It will be convenient to define  $\begin{bmatrix} n \\ k \end{bmatrix} = 0$  for  $n < 0$  or  $k > n$ .

Now we will proceed as in [1] and consider noncommutative variables  $x, \eta$ , such that  $x\eta = q\eta x$ ; all other variables commute.

*Lemma 1.*

$$(x + x^p \eta)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} \eta^{n-k}.$$

**Proof.** We write

$$(x + x^p \eta)^n = \sum_{k=0}^n a_{n,k} x^{k+p(n-k)} \eta^{n-k},$$

and  $(x + x^p \eta)^{n+1} = (x + x^p \eta)^n (x + x^p \eta)$  resp. as  $(x + x^p \eta)^{n+1} = (x + x^p \eta)(x + x^p \eta)^n$ , compare coefficients, and get the recursions

$$\begin{aligned} a_{n+1,k} &= a_{n,k-1} + a_{n,k} q^{k+p(n-k)}, \\ a_{n+1,k} &= a_{n,k-1} q^{n+1-k} + a_{n,k} q^{p(n-k)}. \end{aligned}$$

From this we derive, taking differences,

$$a_{n,k} = \frac{1 - q^{n+1-k}}{1 - q^k} q^{-p(n-k)} a_{n,k-1}.$$

The result follows from iteration by noting that  $a_{n,0} = q^{p \binom{n}{2}}$ . □

Of course we also have

$$(x + tx^p \eta)^n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \eta^{n-k}.$$

Now we derive the generating function for

$$F(x) = \sum_{n \geq 0} S_n x^n;$$

the following procedure is inspired by [2]. Note that we can alternatively view  $\eta$  as an operator, defined by  $\eta f(x) = f(qx)$ . Cigler worked also much with this technique [3, 4]. We find

$$\sum_{n \geq p} S_n x^n = \sum_{n \geq p} S_{n-1} x^n + \sum_{n \geq p} t q^{n-p} S_{n-p} x^n = x \sum_{n \geq p-1} S_n x^n + t x^p \sum_{n \geq 0} \eta S_n x^n$$

or

$$F(x) - \sum_{n < p} S_n x^n = x F(x) - x \sum_{n < p-1} S_n x^n + t x^p \eta F(x),$$

and

$$F(x) = \frac{1}{1 - x - t x^p \eta} \left( \sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1} \right).$$

Now we can apply our lemma and write

$$\begin{aligned} F(x) &= \sum_{n \geq 0} (x + t x^p \eta)^n \left( \sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1} \right) \\ &= \sum_{n \geq 0} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \eta^{n-k} \left( \sum_{i < p} S_i x^i - \sum_{i < p-1} S_i x^{i+1} \right) \\ &= \sum_{n \geq 0} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{p \binom{n}{2} - pnk + p \binom{k+1}{2}} x^{k+p(n-k)} t^{n-k} \left( \sum_{i < p} S_i q^{i(n-k)} x^i - \sum_{i < p-1} S_i q^{(i+1)(n-k)} x^{i+1} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} q^{p \binom{k}{2}} x^{n-k+pk} t^k \left( \sum_{i < p} S_i q^{ik} x^i - \sum_{i < p-1} S_i q^{(i+1)k} x^{i+1} \right) \\
&= \sum_{k, n \geq 0} \binom{n+k}{k} q^{p \binom{k}{2}} x^{n+pk} t^k \left( \sum_{i < p} S_i q^{ik} x^i - \sum_{i < p-1} S_i q^{(i+1)k} x^{i+1} \right) \\
&= \sum_{k \geq 0} q^{p \binom{k}{2}} x^{pk} t^k \frac{1}{(x)_{k+1}} \left( \sum_{i < p} S_i q^{ik} x^i - \sum_{i < p-1} S_i q^{(i+1)k} x^{i+1} \right).
\end{aligned}$$

From this we find an explicit formula for  $S_n$  (the quantity  $S_{-1}$  has to be interpreted as 0):

$$S_n = \sum_{0 \leq i < p} (S_i - S_{i-1}) \sum_{k \geq 0} \binom{n - (p-1)k - i}{k} q^{p \binom{k}{2} + ik} t^k.$$

Now we specialize this to our instance. Here,  $t = -q^p$ , and thus

$$S_n = \sum_{0 \leq i < p} (S_i - S_{i-1}) \sum_{k \geq 0} \binom{n - (p-1)k - i}{k} q^{p \binom{k+1}{2} + ik} (-1)^k.$$

Therefore

$$E_n = \sum_{k \geq 0} \binom{n - (p-1)k}{k} q^{p \binom{k+1}{2}} (-1)^k.$$

From this, the limit of  $E_n$  is immediate. For  $D_n$  we eventually get the following form

$$D_n = \sum_{k \geq 0} \binom{n - (p-1)(k-1)}{k} q^{k+p \binom{k}{2}} (-1)^k,$$

from which the formula for  $D_\infty$  is immediate. To prove it, we need a simple lemma whose proof is just a routine calculation.

*Lemma 2.*

$$\binom{m-i}{k} q^{i(k+1)} = g(i) - g(i-1) \quad \text{where} \quad g(i) = -\binom{m-i}{k+1} q^{(i+1)(k+1)}.$$

□

Now we can plug into the general formula above and compute

$$\begin{aligned}
D_n &= E_n - \sum_{i=1}^{p-1} \sum_{k \geq 0} \binom{n - (p-1)k - i}{k} q^{p \binom{k+1}{2} + i(k+1)} (-1)^k \\
&= E_n - \sum_{k \geq 0} (-1)^k q^{p \binom{k+1}{2}} \sum_{i=1}^{p-1} \binom{n - (p-1)k - i}{k} q^{i(k+1)} \\
&= E_n - \sum_{k \geq 0} (-1)^k q^{p \binom{k+1}{2}} \left\{ q^{k+1} \binom{n - (p-1)k}{k+1} - q^{p(k+1)} \binom{n - (p-1)(k+1)}{k+1} \right\} \\
&= 1 - \sum_{k \geq 0} (-1)^k q^{p \binom{k+1}{2}} q^{k+1} \binom{n - (p-1)k}{k+1},
\end{aligned}$$

which is the announced formula after a simple change of variable. Note that in the penultimate step the telescoping property of the lemma has been used.

### 3 Computer algebra proofs

The polynomial families  $(E_n)$  and  $(D_n)$  give rise to the following study with respect to possible computer proofs. Let us take as input our sum representations of  $E_n$  and  $D_n$ :

$$\begin{aligned} E_n &= \sum_{k \geq 0} \binom{n-(p-1)k}{k} q^{p \binom{k+1}{2}} (-1)^k, \\ D_n &= \sum_{k \geq 0} \binom{n-(p-1)(k-1)}{k} q^{k+p \binom{k}{2}} (-1)^k. \end{aligned} \quad (3.1)$$

Then, if  $p$  is chosen as a specific positive integer, Riese's package `qZeil` [7] returns the recurrences  $S_n = S_{n-1} - q^n S_{n-p}$  ( $n \geq p$ ) together with a certificate function `Cert` for independent verification. Despite the fact that for general "generic" integer parameter  $p$  there is no algorithm available, a general pattern can be easily guessed from running the algorithm for  $p = 1$ ,  $p = 2$ , and  $p = 3$ , say.

For example, let  $F(n, k)$  be the  $k$ th summand in our sum representation (3.1) of  $E_n$ , then the recurrence for  $E_n$  can be refined to the following statement.

**Theorem 3.1.** *For  $n \geq p$  and  $\delta_k f(n, k) = f(n, k) - f(n, k-1)$ , we have*

$$F(n, k) - F(n-1, k) + q^n F(n-p, k) = \delta_k \text{Cert}(n, k) F(n, k), \quad (3.2)$$

where

$$\text{Cert}(n, k) = q^n \frac{(q^{n-p(k+1)+1})_p}{(q^{n-(p-1)(k+1)})_p}.$$

**Proof.** After dividing both sides of (3.2) by  $F(n, k)$  the proof reduces to checking equality of rational functions. Namely, note that

$$\begin{aligned} \frac{F(n-1, k)}{F(n, k)} &= \frac{1 - q^{n-pk}}{1 - q^{n-(p-1)k}}, \\ \frac{F(n, k-1)}{F(n, k)} &= -\frac{q^{pk}}{1 - q^k} \frac{(q^{n-pk+1})_p}{(q^{n-(p-1)k+1})_{p-1}}, \end{aligned}$$

and

$$\frac{F(n-p, k)}{F(n, k)} = q^{-n} \text{Cert}(n, k).$$

□

Analogously, there is a refined version of the recurrence for  $D_n$ . The certificate in this case is

$$\text{Cert}(n, k) = q^n \frac{(q^{n-pk})_p}{(q^{n-(p-1)k})_p}.$$

Summarizing, with the sum representation for  $E_n$  and  $D_n$  in hand, the corresponding recurrences follow immediately by summing both sides of the computer recurrences (3.2) over all  $k \geq 0$ .

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