On Linear Layouts of Graphs

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On Linear Layouts of Graphs†

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In a total order of the vertices of a graph, two edges with no endpoint in common can be *crossing*, *nested*, or *disjoint*. A *k-stack* (respectively, *k-queue*, *k-arch*) *layout* of a graph consists of a total order of the vertices, and a partition of the edges into k sets of pairwise non-crossing (respectively, non-nested, non-disjoint) edges. Motivated by numerous applications, stack layouts (also called *book embeddings*) and queue layouts are widely studied in the literature, while this is the first paper to investigate arch layouts.

Our main result is a characterisation of k-arch graphs as the *almost* $(k+1)$*-colourable* graphs; that is, the graphs G with a set S of at most k vertices, such that $G\setminus S$ is $(k+1)$-colourable. In addition, we survey the following fundamental questions regarding each type of layout, and in the case of queue layouts, provide simple proofs of a number of existing results. How does one partition the edges given a fixed ordering of the vertices? What is the maximum number of edges in each type of layout? What is the maximum chromatic number of a graph admitting each type of layout? What is the computational complexity of recognising the graphs that admit each type of layout?

A comprehensive bibliography of all known references on these topics is included.

Keywords: graph layout, graph drawing, stack layout, queue layout, arch layout, book embedding, queue-number, stack-number, page-number, book-thickness.

Mathematics Subject Classification: 05C62 (graph representations)

1 Introduction

We consider undirected, finite, simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n=|V(G)|$ and $m=|E(G)|$. The subgraph of G induced by a set of vertices $S\subseteq V(G)$ is denoted by $G[S]$. $G\setminus S$ denotes $G[V(G)\setminus S]$, and $G\setminus v$ denotes $G\setminus\{v\}$ for each vertex $v\in V(G)$.

A *vertex ordering* of an n-vertex graph G is a bijection $\sigma: V(G)\rightarrow\{1,2,\dots,n\}$. We write $v<_\sigma w$ to mean that $\sigma(v)<\sigma(w)$. Thus $<_\sigma$ is a total order on $V(G)$. We say G (or $V(G)$) is *ordered by* $<_\sigma$. At times, it will be convenient to express σ by the list $(v_1,v_2,\dots,v_n)$, where $\sigma(v_i)=i$. These notions extend

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to subsets of vertices in the natural way. Suppose that $V_1, V_2, \dots, V_k$ are disjoint sets of vertices, such that each V_i is ordered by $<_i$. Then $(V_1, V_2, \dots, V_k)$ denotes the vertex ordering σ such that $v <_\sigma w$ whenever $v \in V_i$ and $w \in V_j$ with $i < j$, or $v \in V_i$, $w \in V_i$, and $v <_i w$. We write $V_1 <_\sigma V_2 <_\sigma \cdots <_\sigma V_k$.

In a vertex ordering σ of a graph G, let $L(e)$ and $R(e)$ denote the endpoints of each edge $e \in E(G)$ such that $L(e) <_\sigma R(e)$. Consider two edges $e, f \in E(G)$ with no common endpoint. There are the following three possibilities for the relative positions of the endpoints of e and f in σ. Without loss of generality $L(e) <_\sigma L(f)$.

- e and f *cross*: $L(e) <_\sigma L(f) <_\sigma R(e) <_\sigma R(f)$.
- e and f *nest* and f is *nested inside* e: $L(e) <_\sigma L(f) <_\sigma R(f) <_\sigma R(e)$
- e and f are *disjoint*: $L(e) <_\sigma R(e) <_\sigma L(f) <_\sigma R(f)$

Edges with a common endpoint do not cross, do not nest, and are not disjoint.

A *stack* (respectively, *queue*, *arch*) in σ is a set of edges $F \subseteq E(G)$ such that no two edges in F are crossing (respectively, nested, disjoint) in σ. Observe that when traversing σ, edges in a stack appear in LIFO order, and edges in a queue appear in FIFO order — hence the names.

A linear layout of a graph G is a pair $(\sigma, \{E_1, E_2, \dots, E_k\})$ where σ is a vertex ordering of G, and $\{E_1, E_2, \dots, E_k\}$ is a partition of $E(G)$. A *k-stack* (respectively, *queue*, *arch*) *layout* of G is a linear layout $(\sigma, \{E_1, E_2, \dots, E_k\})$ such that each E_ℓ is a *stack* (respectively, *queue*, *arch*) in σ. At times we write $\mathsf{stack}(e) = \ell$ (or $\mathsf{queue}(e) = \ell$, $\mathsf{arch}(e) = \ell$) if $e \in E_\ell$. Layouts of K_6 of each type are illustrated in Figure 1.

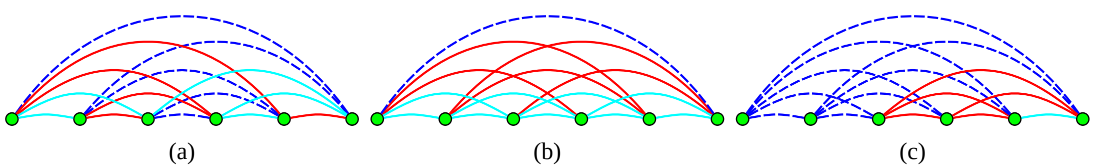

Fig. 1: Layouts of K_6: (a) 3-stack, (b) 3-queue, (c) 3-arch.

A graph admitting a k-stack (respectively, queue, arch) layout is called a *k-stack* (respectively, *queue*, *arch*) *graph*. The *stack-number* (respectively, *queue-number*, *arch-number*) of a graph G, denoted by $\mathsf{sn}(G)$ (respectively, $\mathsf{qn}(G)$, $\mathsf{an}(G)$), is the minimum k such that G is a k-stack (respectively, k-queue, k-arch) graph.

Stack and queue layouts were respectively introduced by Ollmann [85] and Heath *et al.* [55, 59]. As far as we are aware, arch layouts have not previously been studied, although Dan Archdeacon‡ suggests doing so.

Stack layouts are more commonly called *book embeddings*, and stack-number has been called *book-thickness*, *fixed outer-thickness*, and *page-number*. Applications of stack layouts include sorting permutations [36, 49, 86, 89, 102], fault tolerant VLSI design [17, 92, 93, 94], complexity theory [38, 39, 66], compact graph encodings [63, 82], compact routing tables [45], and graph drawing [6, 24, 108, 109]. Numerous other aspects of stack layouts have been studied in the literature [7, 8, 10, 11, 14, 15, 16, 18,

‡ `http://www.emba.uvm.edu/~archdeac/problems/stackq.htm`

20, 22, 33, 34, 35, 37, 40, 42, 51, 52, 53, 54, 55, 59, 60, 62, 64, 65, 67, 69, 70, 74, 75, 76, 77, 79, 80, 81, 84, 88, 91, 95, 96, 97, 98, 99, 100, 103, 106, 111]. Stack layouts of directed graphs [23, 50, 57, 58] and posets [2, 3, 56, 73, 83, 101] have also been investigated.

Applications of queue layouts include sorting permutations [36, 61, 86, 89, 102], parallel process scheduling [5], matrix computations [88], and graph drawing [25, 27, 110]. Other aspects of queue layouts have been studied in the literature [29, 30, 41, 91, 95]. Queue layouts of directed graphs [57, 58] and posets [56] have also been investigated.

Table 1 summarises some of the known bounds on the stack-number and queue-number of various classes of graphs. A blank entry indicates that a more general result provides the best known bound.

Tab. 1: Upper bounds on the stack-number and queue-number.

graph family	stack-number	reference	queue-number	reference
n vertices	$\lceil \frac{n}{2} \rceil$	[4, 17, 48]	$\lfloor \frac{n}{2} \rfloor$	[59]
m edges	$O(\sqrt{m})$	[76]	$\mathbf{e}\sqrt{m}$	Theorem 4
proper minor-closed	bounded	[9, 11]		
genus γ	$O(\sqrt{\gamma})$	[75]		
tree-width w	$w+1$	[42]	$3^w \cdot 6^{(4^w-3w-1)/9} - 1$	[27, 31]
tree-width w, max. degree Δ			$36\Delta w$	[110]
path-width p			p	[110]
band-width b	$b-1$	[100]	$\lceil \frac{b}{2} \rceil$	[59]
track-number t			$t-1$	[27, 30, 110]
toroidal	7	[33]		
planar	4	[111]		
bipartite planar	2	[21, 87]		
2-trees	2	[91]	3	[28, 91]
Halin	2	[41]	3	[41]
X-trees	2	[17]	2	[59]
outerplanar	1	[4]	2	[55]
arched levelled planar	2	[55]	1	[55]
trees	1	[17]	1	[59]

Consider a vertex ordering $\sigma = (v_1, v_2, \dots, v_n)$ of a graph G. For each edge $v_iv_j \in E(G)$, let the *width* of v_iv_j in σ be $|i-j|$, and let the *midpoint* of v_iv_j be $\frac{1}{2}(i+j)$. The *band-width* of σ is the maximum width of an edge of G in σ. The *band-width* of G, denoted by $\mathsf{bw}(G)$, is the minimum band-width over all vertex orderings of G. Consider the two fundamental observations:

Observation 1 ([59]). *Edges whose widths differ by at most one are not nested.*

Observation 2. *Distinct edges with the same midpoint are nested.*

Observation 1 was made by Heath and Rosenberg [59]. Remarkably, Observation 2 seems to have gone unnoticed in the literature on queue layouts.

Our main result is a characterisation of k-arch graphs, given in Section 3. We also survey various fundamental questions regarding each type of layout, and in the case of queue layouts, provide new and simple proofs (based on Observation 2) of a number of existing results. In Section 2 we consider how to partition the edges given a fixed vertex ordering. In Section 4 we analyse the computational complexity of recognising the graphs that admit each type of layout. In Section 5 we consider the extremal questions: what is the maximum number of edges in each type of layout, and what is the maximum chromatic number of a graph admitting each type of layout? Section 6 considers how to produce a queue layout of a graph G given queue layouts of the biconnected components of G. In Section 7 we give a simple proof of the known result that queue-number is in $O(\sqrt{m})$.

2 Fixed Vertex Orderings

Consider the problem of assigning the edges of a graph G to the minimum number of stacks given a fixed vertex ordering σ of G. This problem is equivalent to colouring a circle graph with the minimum number of colours. (A *circle graph* is the intersection graph of a set of chords of a circle.) As illustrated in Figure 2(a), a *twist* in σ is a matching $\{v_iw_i \in E(G) : 1 \leq i \leq k\}$ such that

$$v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_1 <_\sigma w_2 <_\sigma \cdots <_\sigma w_k \quad .$$

A vertex ordering with a k-edge twist needs at least k stacks, since each edge of a twist must be in a distinct stack. However, the converse is not true. There exist vertex orderings with no $(k+1)$-edge twist that require $\Omega(k \log k)$ stacks [71]. Moreover, it is $\mathcal{NP}$-complete to test if a fixed vertex ordering of a graph admits a k-stack layout [43]§. On the other hand, Kostochka [72] proved that a vertex ordering with no 3-edge twist admits a 5-stack layout, and Ageev [1] proved that 5-stacks are sometimes necessary in this case. In general, Kostochka and Kratochvíl [71] proved that a vertex ordering with no $(k+1)$-edge twist admits a 2^{k+6}-stack layout¶, thus improving on previous bounds by Gyárfás [46, 47]. Hence the stack-number of a graph G is bounded by the minimum, taken over all vertex orderings σ of G, of the maximum number of edges in a twist in σ.

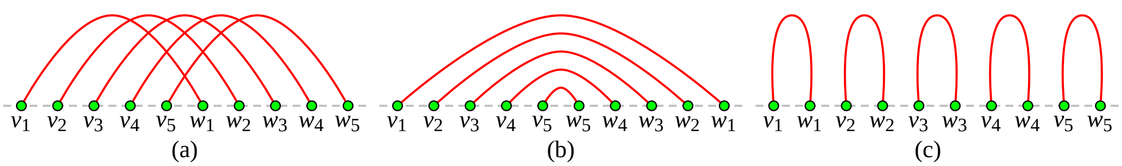

Fig. 2: (a) 5-edge twist, (b) 5-edge rainbow, (c) 5-edge necklace.

Now consider the analogous problem for queue layouts: assign the edges of a graph G to the minimum number of queues given a fixed vertex ordering σ of G. As illustrated in Figure 2(b), a *rainbow* in σ is a matching $\{v_iw_i \in E(G) : 1 \leq i \leq k\}$ such that

$$v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_k <_\sigma w_{k-1} <_\sigma \cdots <_\sigma w_1 \quad .$$

The rainbow $\{v_iw_i : 2 \leq i \leq k\}$ is said to be *inside* v_1w_1. We now give a simple proof of a result by Heath and Rosenberg [59].

§ Unger [104, 105] claimed that it is $\mathcal{NP}$-complete to determine whether a given vertex ordering of a graph G admits a 4-stack layout, and that there is a $O(n \log n)$ time algorithm in the case of 3-stack layouts. Crucial details are missing from these papers.

¶ Unger [104] claimed without proof that a vertex ordering with no $(k+1)$-edge twist admits a $2k$-stack layout. This claim is refuted by Ageev [1] in the case of $k = 2$.

Lemma 1 ([59]). *A vertex ordering of a graph G admits a k-queue layout of G if and only if it has no $(k+1)$-edge rainbow.*

Proof. A k-queue layout has no $(k+1)$-edge rainbow since each edge of a rainbow must be in a distinct queue. Conversely, suppose we have a vertex ordering with no $(k+1)$-edge rainbow. For every edge $vw \in E(G)$, let $\mathsf{queue}(vw)$ be the maximum number of edges in a rainbow inside vw plus one. If vw is nested inside xy then $\mathsf{queue}(vw) < \mathsf{queue}(xy)$. Hence we have a valid queue assignment. The number of queues is at most k. □

Heath and Rosenberg [59] presented a $O(m \log\log n)$ time algorithm that, given a fixed vertex ordering of a graph G with no $(k+1)$-edge rainbow, assigns the edges of G to k queues. Lemma 1 implies that the queue-number of G is the minimum, taken over all vertex orderings σ of G, of the maximum number of edges in a rainbow in σ. Hence determining the queue-number of a graph is no more than the question of finding the right vertex ordering.

Now consider the problem of assigning the edges of a graph G to the minimum number of arches given a fixed vertex ordering σ of G. As illustrated in Figure 2(c), a *necklace* in σ is a matching $\{v_i w_i : 1 \leq i \leq k\}$ such that

$$v_1 <_\sigma w_1 <_\sigma v_2 <_\sigma w_2 <_\sigma \cdots <_\sigma v_k <_\sigma w_k \enspace .$$

The necklace $\{v_i w_i : 1 \leq i \leq k-1\}$ is said to *precede* the edge $v_k w_k$.

Lemma 2. *A vertex ordering of a graph G admits a k-arch layout of G if and only if it has no $(k+1)$-edge necklace.*

Proof. A k-arch layout has no $(k+1)$-edge necklace, since each edge of a necklace must be assigned to a distinct arch. Conversely, suppose we have a vertex ordering with no $(k+1)$-edge necklace. For every edge $vw \in E(G)$, let $\mathsf{arch}(vw)$ be the maximum number of edges in a necklace that precedes vw plus one. If vw and xy are disjoint then, without loss of generality, vw is in a necklace that precedes xy, and thus $\mathsf{arch}(vw) < \mathsf{arch}(xy)$. Hence we have a valid arch assignment. The number of arches is at most k. □

Lemma 2 implies that the arch-number of a graph G is the minimum, taken over all vertex orderings σ of G, of the maximum number of edges in a necklace in σ. For example, $\mathsf{an}(K_n) = \lfloor \frac{n}{2} \rfloor$. Now consider the following algorithm.

Algorithm ASSIGNARCHES(G, σ)
1. **let** $k_0 = 0$
2. **let** $(v_1, v_2, \dots, v_n) = \sigma$
3. **for** $i = 1, 2, \dots, n$ **do**
4.   **for** each edge $v_i v_j \in E(G)$ with $i < j$, **let** $\mathsf{arch}(v_i v_j) = 1 + k_{i-1}$
5.   **let** $k_i = k_{i-1}$
6.   **for** each edge $v_i v_j \in E(G)$ with $j < i$, **let** $k_i = \max\{k_i, 1 + k_{j-1}\}$

Lemma 3. *Given a vertex ordering σ of an n-vertex m-edge graph G, the algorithm* ASSIGNARCHES(G, σ) *assigns the edges of G to the minimum number of arches with respect to σ in $O(n+m)$ time.*

Proof. It is easily verified that the algorithm maintains the invariant that k_i is the maximum number of edges in a necklace in the vertex ordering $(v_1, v_2, \dots, v_i)$. Hence, for every edge $vw \in E(G)$, $\mathsf{arch}(vw)$ is the maximum number of edges in a necklace that precedes vw plus one. Thus, as in Lemma 2, we have an assignment of the edges to the minimum number of arches. $\square$

The proofs of Lemmata 1 and 2 hide an application of the easy half of Dilworth's Theorem [26] for partitioning a poset into k antichains, where k is the maximum size of a chain. In Lemma 1, $e \prec f$ if e is nested inside f, and in Lemma 2, $e \prec f$ if $R(e) <_\sigma L(f)$. The problem of assigning edges to queues in a fixed vertex ordering is equivalent to colouring a permutation graph [32]. Assigning edges to arches corresponds to partitioning an interval graph into cliques.

3 Arch Layout Characterisation

A graph G is *almost k-colourable* if there is a set $S \subseteq V(G)$ of at most $k-1$ vertices such that $G \setminus S$ is k-colourable.

Theorem 1. *A graph G has arch-number $\mathsf{an}(G) \leq k$ if and only if G is almost $(k+1)$-colourable.*

Proof. ($\Longleftarrow$) First suppose that G is almost $(k+1)$-colourable. Then there is a set of vertices $S = \{x_1, x_2, \dots, x_k\} \subseteq V(G)$ such that $G \setminus S$ is $(k+1)$-colourable. Let $V_1, V_2, \dots, V_{k+1}$ be the colour classes in such a colouring. Let σ be a vertex ordering such that

$$V_1 <_\sigma x_1 <_\sigma V_2 <_\sigma x_2 <_\sigma \cdots <_\sigma V_k <_\sigma x_k <_\sigma V_{k+1} \ .$$

Clearly every necklace in σ has at most k edges. By Lemma 2, σ admits a k-arch layout of G.

($\Longrightarrow$) The proof is by induction on k. For $k = 0$, the result is trivial. Suppose that $\mathsf{an}(G) \leq k-1$ implies G is almost k-colourable. Let G be a k-arch graph with vertex ordering $\sigma = (v_1, v_2, \dots, v_n)$.

Let $V_{\leq p} = (v_1, v_2, \dots, v_p)$ and $V_{>p} = (v_{p+1}, v_{p+2}, \dots, v_n)$. It is simple to verify that the maximum number of edges in a necklace in $V_{\leq p}$ is equal to, or one less than, the maximum number of edges in a necklace in $V_{\leq p+1}$, for all $1 \leq p \leq n-1$. Consequently, there is maximum number i such that $V_{\leq i}$ admits a $(k-1)$-arch layout. By the maximality of i, $V_{\leq i+1}$ contains a k-edge necklace. Therefore $V_{>i+1}$ is an independent set of G, otherwise an edge of $G[V_{>i+1}]$ together with the k-edge necklace of $V_{\leq i+1}$ would comprise a $(k+1)$-edge necklace. Therefore, $G[V_{>i}]$ is a forest, at most one component of which is a star centred at v_{i+1}, and the remaining components are isolated vertices.

By the induction hypothesis there is a set S_{k-1} of $k-1$ vertices such that $G[V_{\leq i} \setminus S_{k-1}]$ is k-colourable. Since $G[V_{>i}]$ is a star centred at v_{i+1} with some isolated vertices, it follows that for $S_k = S_{k-1} \cup \{v_{i+1}\}$, the induced subgraph $G[V \setminus S_k]$ is $(k+1)$-colourable. Therefore, G is almost $(k+1)$-colourable. $\square$

Arch-number and chromatic number are tied in the following strong sense.

Theorem 2. *The arch-number of every graph G satisfies:*

$$\mathsf{an}(G) + 1 \leq \chi(G) \leq 2\,\mathsf{an}(G) + 1.$$

Proof. By Theorem 1, G is almost $(\mathsf{an}(G)+1)$-colourable. Thus it is $(2\,\mathsf{an}(G)+1)$-colourable. Conversely, G is almost $\chi(G)$-colourable, and $\mathsf{an}(G) \leq \chi(G) - 1$ by Theorem 1. $\square$

Theorem 2 implies that any graph family that has bounded chromatic-number also has bounded arch-number. Examples include graphs with bounded maximum degree, graphs with bounded tree-width, and graphs with an excluded clique minor, and so on.

Lemma 4. *Planar graphs have arch-number at most three and this bound is tight.*

Proof. The Four Colour Theorem and Theorem 1 imply that all planar graphs have arch-number at most three. Any planar graph G containing three vertex-disjoint K_4 subgraphs is not almost 3-colourable. By Theorem 1, $\mathsf{an}(G) = 3$. $\square$

4 Computational Complexity

The 1-stack graphs are precisely the outerplanar graphs [4], and thus can be recognised in $O(n)$ time [78]. 2-stack graphs are characterised as the subgraphs of planar Hamiltonian graphs [4], which implies that it is $\mathcal{NP}$-complete to test if $\mathsf{sn}(G) \leq 2$ [107]. Heath and Rosenberg [59] characterised 1-queue graphs as the 'arched levelled' planar graphs, and proved that it is $\mathcal{NP}$-complete to recognise such graphs.

Lemma 5. *There is a $O(n(n+m))$ time algorithm to determine if a given n-vertex m-edge graph G has arch-number* $\mathsf{an}(G) \leq 1$.

Proof. By Theorem 1, $\mathsf{an}(G) \leq 1$ if and only if there is a vertex v such that $G \setminus v$ is bipartite. The result follows since bipartiteness can be tested in $O(n+m)$ time by breadth-first search. $\square$

Note that almost bipartite graphs have been studied by Prömel *et al.* [90].

Open Problem 1. Is there a sub-quadratic time algorithm for determining whether $\mathsf{an}(G) \leq 1$?

Theorem 3. *Given a graph G and an integer $k \geq 2$, it is $\mathcal{NP}$-complete to determine if G has arch-number* $\mathsf{an}(G) \leq k$.

Proof. The problem is clearly in $\mathcal{NP}$. The remainder of the proof is a reduction from the graph k-colourability problem: given a graph G and an integer k, is $\chi(G) \leq k$? Let G' be the graph comprised of k components each isomorphic to G. We claim that $\chi(G) \leq k$ if and only if G' is almost k-colourable. The result will follow from Theorem 1 and since graph k-colourability is $\mathcal{NP}$-complete [68].

If G is k-colourable then so is G', and thus G' is almost k-colourable. Conversely, if G' is almost k-colourable then there is a set S of at most $k-1$ vertices such that $\chi(G' \setminus S) \leq k$. Since $|S| \leq k-1$, $G' \setminus S$ contains a component isomorphic to G, and thus G is k-colourable. $\square$

The next result follows from the reduction in Theorem 3 and since it is $\mathcal{NP}$-complete to determine if a 4-regular planar graph is 3-colourable [19, 44].

Corollary 1. *It is $\mathcal{NP}$-complete to determine if a given 4-regular planar graph G has arch-number* $\mathsf{an}(G) \leq 2$. $\square$

5 Extremal Questions

In this section we consider the extremal questions:

- what is the maximum number of edges in a particular type of layout?
- what is the maximum chromatic number of a graph admitting a particular type of layout?

The answer to the first question for stack layouts has been observed by many authors.

Lemma 6 ([4, 18, 69]). *Every s-stack n-vertex graph has at most $(s+1)n-3s$ edges, and this bound is tight for all even $n \geq 4$ and all $1 \leq s \leq \frac{n}{2}$.*

Proof. It will be beneficial to view the vertex ordering $(v_0, v_1, \dots, v_{n-1})$ as circular. Each edge $v_i v_{(i+1) \bmod n}$ is said to be a *boundary* edge. Each stack has at most $2n-3$ edges, since a 1-stack graph is outerplanar. Every boundary edge can be assigned to any stack. Thus there are at most $n-3$ non-boundary edges in each stack, and at most n boundary edges, giving a total of at most $s(n-3)+n = (s+1)n-3s$ edges.

Now for the lower bound. As illustrated in Figure 3(a), for each $0 \leq i \leq s-1$, let

$$E_i = \{v_j v_k : \lceil \tfrac{1}{2}(j+k) \rceil \equiv i \pmod{\tfrac{n}{2}}\} \ .$$

Then $E_0, E_1, \dots, E_{s-1}$ are edge-disjoint paths, each of which is a stack of $n-3$ non-boundary edges. Adding the boundary edges to any stack, we obtain an s-stack graph with the desired number of edges. Note that with $s = \frac{n}{2}$, we obtain an $\frac{n}{2}$-stack layout of K_n. □

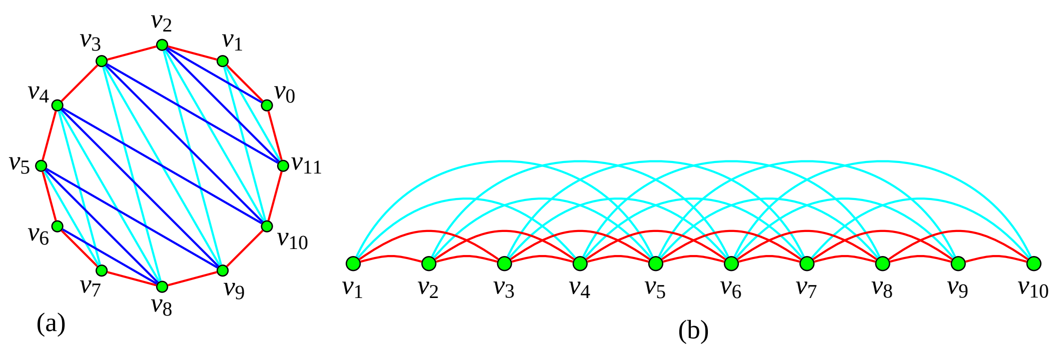

Fig. 3: Edge-maximal layouts: (a) 2-stack, (b) 2-queue.

As observed by Bernhart and Kainen [4], Lemma 6 implies that (every induced subgraph of) an s-stack graph has a vertex of degree less than $2s+2$, and is therefore vertex $(2s+2)$-colourable by the minimum-degree-greedy algorithm. This result can be improved for small s. 1-stack graphs are outerplanar, which are 3-colourable, and 2-stack graphs are planar, which are 4-colourable.

Open Problem 2. What is the maximum chromatic number χ of the s-stack graphs? In general, $\chi \in \{2s, 2s+1, 2s+2\}$ since $\chi(K_n) = 2\,\mathsf{sn}(K_n)$ for even n.

Now consider the maximum number of edges in a k-queue graph. The answer for $k = 1$ was given by Heath and Rosenberg [59] and Pemmaraju [88]. We now give a simple proof for this case. The proof by Heath and Rosenberg [59] is based on the characterisation of 1-queue graphs as the arched levelled planar graphs. The proof by Pemmaraju [88] is based on a relationship between queue layouts and 'staircase covers of matrices'. The observant reader will notice parallels between the following proof and that of the lower bound on the volume of three-dimensional drawings due to Bose *et al.* [13].

Lemma 7. *A queue in a graph with n vertices has at most* $2n-3$ *edges.*

Proof. By Observation 2, distinct edges with the same midpoint are nested. Since every midpoint is in $\{\frac{3}{2}, 2, 2\frac{1}{2}, \dots, n-\frac{1}{2}\}$, there are at most $2n-3$ midpoints. The result follows since no two edges in a queue are nested. □

An immediate generalisation of Lemma 7 is that every k-queue graph has at most $k(2n-3)$ edges [59]. The following improved upper bound was first discovered by Pemmaraju [88] with a longer proof. That this bound is tight for all values of n and k is new.

Lemma 8. *Every n-vertex graph with queue-number k has at most* $2kn - k(2k+1)$ *edges. For every k and* $n \geq 2k$*, there exists an n-vertex graph with queue-number k and* $2kn - k(2k+1)$ *edges*

Proof. First we prove the upper bound. Note that $n \geq 2k$, since by Lemma 1, the corresponding vertex ordering has a k-edge rainbow. By Observation 2, distinct edges with the same midpoint are nested. Since at most k edges are pairwise nested in a k-queue layout, at most k edges have the same midpoint. Moreover, for all integers $1 \leq i \leq k$, at most $i-1$ edges have a midpoint of i, and at most $i-1$ edges have a midpoint of $i-\frac{1}{2}$. At the other end of the vertex ordering, for all integers $1 \leq i \leq k-1$, at most i edges have a midpoint of $n-i$, and at most i edges have a midpoint of $n-i+\frac{1}{2}$. Since $n \geq 2k$ we are not double counting here. It follows that the number of edges is at most

$$2\sum_{i=1}^{k}(i-1) + (2n-4k+1)k + 2\sum_{i=1}^{k-1} i \;=\; 2kn - k(2k+1) \;\; .$$

We now prove the lower bound. As illustrated in Figure 3(b), let P_n^s denote the s^{th} power of the n-vertex path P_n. That is, P_n^s has $V(P_n^s) = \{v_1, v_2, \dots, v_n\}$ and $E(P_n^s) = \{v_i v_j : |i-j| \leq s\}$. Heath and Rosenberg [59] proved that $\mathsf{qn}(P_n^{2k}) = k$ for $n \geq 2k$, where for each $1 \leq \ell \leq k$, the set of edges $\{v_i v_j : 2\ell - 1 \leq |i-j| \leq 2\ell\}$ is a queue in the vertex ordering $(v_1, v_2, \dots, v_n)$. (This is Observation 1.) Swaminathan *et al.* [100] proved that P_n^{2k} has $2kn - k(2k+1)$ edges. (P_n^s appears in [59, 100] with regard to the relationship between band-width and queue- and stack-number, respectively.) □

Lemma 8 implies that (every induced subgraph of) a k-queue graph has a vertex of degree less than $4k$, and is therefore $4k$-colourable by the minimum-degree-greedy algorithm.

Open Problem 3. What is the maximum chromatic number χ of a k-queue graph? We know that $\chi \in \{2k+1, 2k+2, \dots, 4k\}$ since $\chi(K_n) = 2\,\mathsf{qn}(K_n) + 1$ for odd n (by Lemma 1). Note that the extremal example P_n^{2k} in Lemma 8 also has chromatic number $2k+1$.

We now prove that the lower bound in Open Problem 3 is attainable in the case of $k = 1$.

Lemma 9. *Every* 1*-queue graph G is* 3*-colourable.*

Proof. Let σ be the vertex ordering in a 1-queue layout of a graph G. Partition the vertices into independent sets $V_1, V_2, \dots, V_k$ such that $\sigma = (V_1, V_2, \dots, V_k)$, and for all $1 \le i \le k-1$, there exists an edge in $G[V_i \cup V_{i+1}]$. Such a partition can be computed by starting with each vertex in its own set, and repeatedly merging consecutive sets that have no edge between them. For all $s \ge 3$, there is no edge in any $G[V_i \cup V_{i+s}]$, as otherwise it would be nested with an edge in $G[V_{i+1} \cup V_{i+2}]$. Thus for each $0 \le j \le 2$, $W_j = \bigcup\{V_i : i \equiv j \pmod 3\}$ is an independent set, and $\{W_0, W_1, W_2\}$ is a 3-colouring of G. □

The next lemma shows that, in terms of the maximum number of edges, arch layouts behave very differently from stack and queue layouts. Even 1-arch layouts may have a quadratic number of edges.

Lemma 10. *The maximum number of edges in a k-arch layout with n vertices is at most*

$$\frac{kn(n+2) - k(2k+1)}{2(k+1)} \quad , \tag{1}$$

which is attainable whenever $k+1$ divides $n-k$.

Proof. Let G be a k-arch graph with n vertices and the maximum number of edges. By Theorem 1, G is almost $(k+1)$-colourable. That is, there is a set $S \subseteq V(G)$ of at most k vertices such that $G \setminus S$ is $(k+1)$-colourable. Each vertex $v \in S$ is adjacent to every other vertex in G, as otherwise G is not edge-maximal. The $(k+1)$-colourable graph with the maximum number of edges is the complete $(k+1)$-partite graph with partitions whose sizes are as equal as possible. Thus $G \setminus S$ is this graph. When the partitions have the same size we obtain the most edges. Here, $G \setminus S$ is obtained from K_{n-k} by removing $k+1$ vertex-disjoint copies of the complete graph on $(n-k)/(k+1)$ vertices. Thus the number of edges is

$$\binom{n}{2} - (k+1)\binom{(n-k)/(k+1)}{2} \quad ,$$

which is easily seen to reduce to (1). □

6 Biconnected Components

Clearly the stack-number (respectively, queue-number) of a graph is at most the maximum stack-number (queue-number) of its connected components. Bernhart and Kainen [4] proved that the stack-number of a graph is at most the maximum stack-number of its maximal biconnected components (*blocks*). We now prove an analogous result for queue layouts.

Lemma 11. *Every graph G has queue-number $\mathsf{qn}(G) \le 1 + \max\{\mathsf{qn}(B) : B$ is a block of $G\}$.*

Proof. Clearly we can assume that G is connected. Let T be the *block-cut-tree* of G. That is, there is a node in T for each block and for each cut-vertex of G. Two nodes of T are adjacent if and only if one corresponds to a block B, and the other corresponds to a cut-vertex v, and $v \in V(B)$. T is a tree, as otherwise a cycle in T would correspond to a single block in G. Root T at a node r corresponding to an arbitrary block R of G.

Consider a cut-vertex v of G. The block containing v that corresponds to the parent node of v in T is called the *parent* block of v. The other blocks containing v are called *child* blocks of v.

There are no nested edges in any breadth-first vertex ordering of T [17]. Let σ be a breadth-first vertex ordering of T starting at R, such that cut-nodes of T with a common parent block B are ordered in σ according to their order in the given queue layout of B.

Now create a vertex ordering π of G from σ, as illustrated in Figure 4. Specifically, delete from σ all the nodes corresponding to cut vertices of G; replace r by the given queue layout of R; and for each block $B \neq R$ whose parent node in T corresponds to a cut vertex v of G, replace the node in σ that corresponds to B by the given queue layout of $B \setminus v$.

An edge of G that is incident to a cut-vertex v and is contained in a child block of v is called a *jump edge*. According to the above algorithm, a cut-vertex of G is positioned within its parent block in π. Thus, if two non-jump edges are nested, then they are in the same block B, and are nested in the given queue layout of B. Since the blocks are separated, non-jump edges can inherit their queue assignment from the queue layout of their block.

Since the edges of T are not nested, and by the choice of ordering for cut-nodes of T with a common parent block, jump edges are not nested, and thus can form one new queue. Thus the total number of queues is as claimed. □

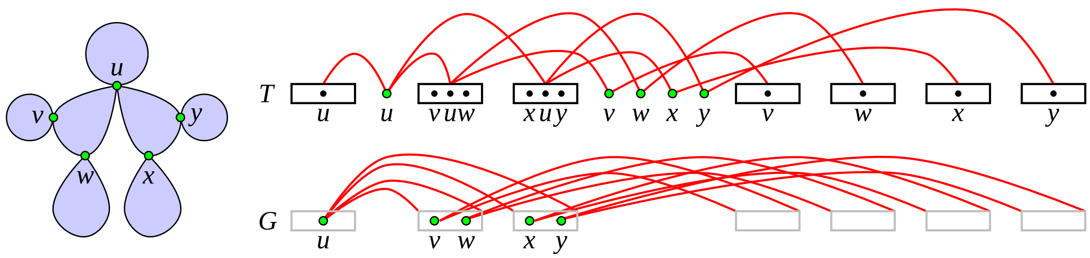

Fig. 4: Constructing a queue layout of G from queue layouts of the biconnected components of G.

7 A Bound on the Queue-Number

Malitz [76] proved that the stack-number of an m-edge graph is $O(\sqrt{m})$. The exact bound is in fact $72\sqrt{m}$. Heath and Rosenberg [59] (see also Shahrokhi and Shi [95]) observed that an analogous method proves that queue-number is $O(\sqrt{m})$. We now establish this result using a simplified version of the argument of Malitz [76] and with an improved constant.

Theorem 4. *Every graph G with m edges has queue-number $\mathsf{qn}(G) < \boldsymbol{e}\sqrt{m}$, where $\boldsymbol{e}$ is the base of the natural logarithm.*

Proof. Let $n = |V(G)|$. Let σ be a random vertex ordering of G. For all positive integers k, let A_k be the event that there exists a k-edge rainbow in σ. Then the probability

$$\mathbf{Pr}\{A_k\} \leq \underbrace{\binom{m}{k}}_{(1)} \cdot \underbrace{\binom{n}{2k}}_{(2)} \cdot \underbrace{\frac{2^k k!(n-2k)!}{n!}}_{(3)} ,$$

where:

(1) is an upper bound on the number of k-edge matchings M,

(2) is the number of vertex positions in σ for M, and

(3) is the probability that M with fixed vertex positions is a rainbow.

Thus

$$\mathbf{P}\{A_k\} \leq \frac{m^k}{k!} \cdot \frac{n!}{(2k)!(n-2k)!} \cdot \frac{2^k\, k!(n-2k)!}{n!} = \frac{(2m)^k}{(2k)!} \ .$$

By Stirling's formula, $\mathbf{P}\{A_k\} < \left(\frac{\mathbf{e}^2 m}{2k^2}\right)^k$. Let $k_0 = \lceil \mathbf{e}\sqrt{m} \rceil$. Thus, $\mathbf{P}\{\overline{A_{k_0}}\} > 1 - \left(\frac{1}{2}\right)^{\lceil \mathbf{e}\sqrt{m} \rceil} > 0$. That is, with positive probability a random vertex ordering has no k_0-edge rainbow. Hence there exists a vertex ordering with no k_0-edge rainbow. By Lemma 1, G has a queue layout with $k_0 - 1 < \mathbf{e}\sqrt{m}$ queues. $\square$

References

[1] A. A. AGEEV, A triangle-free circle graph with chromatic number 5. *Discrete Math.*, **152(1-3)**:295–298, 1996.

[2] M. ALZOHAIRI AND I. RIVAL, Series-parallel planar ordered sets have pagenumber two. In S. NORTH, ed., *Proc. 4th International Symp. on Graph Drawing (GD '96)*, vol. 1190 of *Lecture Notes in Comput. Sci.*, pp. 11–24, Springer, 1997.

[3] M. ALZOHAIRI, I. RIVAL, AND A. KOSTOCHKA, The pagenumber of spherical lattices is unbounded. *Arab J. Math. Sci.*, **7(1)**:79–82, 2001.

[4] F. R. BERNHART AND P. C. KAINEN, The book thickness of a graph. *J. Combin. Theory Ser. B*, **27(3)**:320–331, 1979.

[5] S. N. BHATT, F. R. K. CHUNG, F. T. LEIGHTON, AND A. L. ROSENBERG, Scheduling tree-dags using FIFO queues: A control-memory trade-off. *J. Parallel Distrib. Comput.*, **33**:55–68, 1996.

[6] T. C. BIEDL, T. SHERMER, S. WHITESIDES, AND S. WISMATH, Bounds for orthogonal 3-D graph drawing. *J. Graph Algorithms Appl.*, **3(4)**:63–79, 1999.

[7] T. BILSKI, Embedding graphs in books with prespecified order of vertices. *Studia Automat.*, **18**:5–12, 1993.

[8] T. BILSKI, Optimum embedding of complete graphs in books. *Discrete Math.*, **182(1-3)**:21–28, 1998.

[9] R. BLANKENSHIP, *Book Embeddings of Graphs*. Ph.D. thesis, Department of Mathematics, Louisiana State University, U.S.A., 2003.

[10] R. BLANKENSHIP AND B. OPOROWSKI, Drawing subdivisions of complete and complete bipartite graphs on books. Tech. Rep. 1999-4, Department of Mathematics, Louisiana State University, 1999.

[11] R. BLANKENSHIP AND B. OPOROWSKI, Book embeddings of graphs and minor-closed classes. In *Proc. 32nd Southeastern International Conf. on Combinatorics, Graph Theory and Computing*, Department of Mathematics, Louisiana State University, 2001.

[12] H. BODLAENDER, ed., *Proc. 29th Workshop on Graph Theoretic Concepts in Computer Science (WG'03)*, vol. 2880 of *Lecture Notes in Comput. Sci.*, Springer, 2003.

[13] P. BOSE, J. CZYZOWICZ, P. MORIN, AND D. R. WOOD, The maximum number of edges in a three-dimensional grid-drawing. In *Proc. 19th European Workshop on Computational Geometry*, pp. 101–103, University of Bonn, Germany, 2003.

[14] J. BUSS AND P. SHOR, On the pagenumber of planar graphs. In *Proc. 16th ACM Symp. on Theory of Computing (STOC '84)*, pp. 98–100, ACM, 1984.

[15] J. F. BUSS, A. L. ROSENBERG, AND J. D. KNOTT, Vertex types in book-embeddings. *SIAM J. Discrete Math.*, **2(2)**:156–175, 1989.

[16] Y.-C. CHEN, H.-L. FU, AND I.-F. SUN, A study of typenumber in book-embedding. *Ars Combin.*, **62**:97–103, 2002.

[17] F. R. K. CHUNG, F. T. LEIGHTON, AND A. L. ROSENBERG, Embedding graphs in books: a layout problem with applications to VLSI design. *SIAM J. Algebraic Discrete Methods*, **8(1)**:33–58, 1987.

[18] G. A. COTTAFAVA AND O. D'ANTONA, Book-thickness and book-coarseness of graphs. In *Proc. 5th International Symp. on Network Theory (Sarajevo)*, pp. 337–340, 1984.

[19] D. P. DAILEY, Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete. *Discrete Math.*, **30**:289–293, 1980.

[20] E. DAMIANI, O. D'ANTONA, AND P. SALEMI, An upper bound to the crossing number of the complete graph drawn on the pages of a book. *J. Combin. Inform. System Sci.*, **19(1-2)**:75–84, 1994.

[21] H. DE FRAYSSEIX, P. OSSONA DE MENDEZ, AND J. PACH, A left-first search algorithm for planar graphs. *Discrete Comput. Geom.*, **13(3-4)**:459–468, 1995.

[22] A. M. DEAN AND J. P. HUTCHINSON, Relations among embedding parameters for graphs. In *Graph theory, combinatorics, and applications, Vol. 1*, pp. 287–296, Wiley, 1991.

[23] E. DI GIACOMO, W. DIDIMO, G. LIOTTA, AND S. WISMATH, Book embeddings and point-set embeddings of series-parallel digraphs. In M. T. GOODRICH AND S. G. KOBOUROV, eds., *Proc. 10th International Symp. on Graph Drawing (GD '02)*, vol. 2528 of *Lecture Notes in Comput. Sci.*, pp. 162–173, Springer, 2002.

[24] E. DI GIACOMO, W. DIDIMO, G. LIOTTA, AND S. K. WISMATH, Drawing planar graphs on a curve. In [12], pp. 192–204.

[25] E. DI GIACOMO AND H. MEIJER, Track drawings of graphs with constant queue number. In G. LIOTTA, ed., *Proc. 11th International Symp. on Graph Drawing (GD '03)*, vol. 2912 of *Lecture Notes in Comput. Sci.*, pp. 214–225, Springer, 2004.

[26] R. P. DILWORTH, A decomposition theorem for partially ordered sets. *Ann. of Math. (2)*, **51**:161–166, 1950.

[27] V. DUJMOVIĆ, P. MORIN, AND D. R. WOOD, Layout of graphs with bounded tree-width, 2002, submitted.

[28] V. DUJMOVIĆ AND D. R. WOOD, Tree-partitions of k-trees with applications in graph layout. Tech. Rep. TR-2002-03, School of Computer Science, Carleton University, Ottawa, Canada, 2002.

[29] V. DUJMOVIĆ AND D. R. WOOD, Stacks, queues and tracks: Layouts of graph subdivisions, 2003, submitted. See Tech. Rep. TR-2003-08, School of Computer Science, Carleton University, Ottawa, Canada, 2003.

[30] V. DUJMOVIĆ AND D. R. WOOD, Track layouts of graphs, 2003, submitted. See Tech. Rep. TR-2003-07, School of Computer Science, Carleton University, Ottawa, Canada, 2003.

[31] V. DUJMOVIĆ AND D. R. WOOD, Tree-partitions of k-trees with applications in graph layout. In [12], pp. 205–217.

[32] B. DUSHNIK AND E. W. MILLER, Partially ordered sets. *Amer. J. Math.*, **63**:600–610, 1941.

[33] T. ENDO, The pagenumber of toroidal graphs is at most seven. *Discrete Math.*, **175(1-3)**:87–96, 1997.

[34] H. ENOMOTO, T. NAKAMIGAWA, AND K. OTA, On the pagenumber of complete bipartite graphs. *J. Combin. Theory Ser. B*, **71(1)**:111–120, 1997.

[35] D. EPPSTEIN, Separating geometric thickness from book thickness. arXiv.org:math.CO/0109195, 2001.

[36] S. EVEN AND A. ITAI, Queues, stacks, and graphs. In Z. KOHAVI AND A. PAZ, eds., *Proc. International Symp. on Theory of Machines and Computations*, pp. 71–86, Academic Press, 1971.

[37] H.-L. FU AND I.-F. SUN, The typenumber of trees. *Discrete Math.*, **253(1-3)**:3–10, 2002.

[38] Z. GALIL, R. KANNAN, AND E. SZEMERÉDI, On 3-pushdown graphs with large separators. *Combinatorica*, **9(1)**:9–19, 1989.

[39] Z. GALIL, R. KANNAN, AND E. SZEMERÉDI, On nontrivial separators for k-page graphs and simulations by nondeterministic one-tape Turing machines. *J. Comput. System Sci.*, **38(1)**:134–149, 1989.

[40] R. A. GAMES, Optimal book embeddings of the FFT, Benes, and barrel shifter networks. *Algorithmica*, **1(2)**:233–250, 1986.

[41] J. L. GANLEY, Stack and queue layouts of Halin graphs, 1995, manuscript.

[42] J. L. GANLEY AND L. S. HEATH, The pagenumber of k-trees is $O(k)$. *Discrete Appl. Math.*, **109(3)**:215–221, 2001.

[43] M. R. GAREY, D. S. JOHNSON, G. L. MILLER, AND C. H. PAPADIMITRIOU, The complexity of coloring circular arcs and chords. *SIAM J. Algebraic Discrete Methods*, **1(2)**:216–227, 1980.

[44] M. R. GAREY, D. S. JOHNSON, AND L. STOCKMEYER, Some simplified *NP*-complete graph problems. *Theoret. Comput. Sci.*, **1**:237–267, 1976.

[45] C. GAVOILLE AND N. HANUSSE, Compact routing tables for graphs of bounded genus. In J. WIEDERMANN, P. VAN EMDE BOAS, AND M. NIELSEN, eds., *Proc. 26th International Colloquium on Automata, Languages and Programming (ICALP '99)*, vol. 1644 of *Lecture Notes in Computer Science*, pp. 351–360, Springer, 1999.

[46] A. GYÁRFÁS, On the chromatic number of multiple interval graphs and overlap graphs. *Discrete Math.*, **55(2)**:161–166, 1985.

[47] A. GYÁRFÁS, Corrigendum: "On the chromatic number of multiple interval graphs and overlap graphs". *Discrete Math.*, **62(3)**:333, 1986.

[48] A. GYÁRFÁS AND J. LEHEL, Covering and coloring problems for relatives of intervals. *Discrete Math.*, **55(2)**:167–180, 1985.

[49] T. HARJU AND L. ILIE, Forbidden subsequences and permutations sortable on two parallel stacks. In *Where mathematics, computer science, linguistics and biology meet*, pp. 267–275, Kluwer, 2001.

[50] T. HASUNUMA, Embedding iterated line digraphs in books. *Networks*, **40(2)**:51–62, 2002.

[51] T. HASUNUMA AND Y. SHIBATA, Embedding de Bruijn, Kautz and shuffle-exchange networks in books. *Discrete Appl. Math.*, **78(1-3)**:103–116, 1997.

[52] L. S. HEATH, Embedding planar graphs in seven pages. In *Proc. 25th Annual Symp. on Foundations of Comput. Sci. (FOCS '84)*, pp. 74–83, IEEE, 1984.

[53] L. S. HEATH, Embedding outerplanar graphs in small books. *SIAM J. Algebraic Discrete Methods*, **8(2)**:198–218, 1987.

[54] L. S. HEATH AND S. ISTRAIL, The pagenumber of genus g graphs is $O(g)$. *J. Assoc. Comput. Mach.*, **39(3)**:479–501, 1992.

[55] L. S. HEATH, F. T. LEIGHTON, AND A. L. ROSENBERG, Comparing queues and stacks as mechanisms for laying out graphs. *SIAM J. Discrete Math.*, **5(3)**:398–412, 1992.

[56] L. S. HEATH AND S. V. PEMMARAJU, Stack and queue layouts of posets. *SIAM J. Discrete Math.*, **10(4)**:599–625, 1997.

[57] L. S. HEATH AND S. V. PEMMARAJU, Stack and queue layouts of directed acyclic graphs. II. *SIAM J. Comput.*, **28(5)**:1588–1626, 1999.

[58] L. S. HEATH, S. V. PEMMARAJU, AND A. N. TRENK, Stack and queue layouts of directed acyclic graphs. I. *SIAM J. Comput.*, **28(4)**:1510–1539, 1999.

[59] L. S. HEATH AND A. L. ROSENBERG, Laying out graphs using queues. *SIAM J. Comput.*, **21(5)**:927–958, 1992.

[60] R. A. HOCHBERG AND M. F. STALLMANN, Optimal one-page tree embeddings in linear time. *Inform. Process. Lett.*, **87**:59–66, 2003.

[61] A. IMAMIYA AND A. NOZAKI, Generating and sorting permutations using restricted-deques. *Information Processing in Japan*, **17**:80–86, 1977.

[62] M. ISHIWATA, Book presentation of the complete *n*-partite graphs. *Kobe J. Math.*, **13(1)**:27–48, 1996.

[63] G. JACOBSON, Space-efficient static trees and graphs. In *Proc. 30th Annual Symp. Foundations of Comput. Sci. (FOCS '89)*, pp. 549–554, IEEE, 1989.

[64] P. C. KAINEN, The book thickness of a graph. II. In *Proc. 20th Southeastern Conference on Combinatorics, Graph Theory, and Computing*, vol. 71 of *Congr. Numer.*, pp. 127–132, 1990.

[65] P. C. KAINEN AND S. OVERBAY, Book embeddings of graphs and a theorem of Whitney. Submitted.

[66] R. KANNAN, Unraveling k-page graphs. *Inform. and Control*, **66(1-2)**:1–5, 1985.

[67] N. KAPOOR, M. RUSSELL, I. STOJMENOVIC, AND A. Y. ZOMAYA, A genetic algorithm for finding the pagenumber of interconnection networks. *J. Parallel Distrib. Comput.*, **62(2)**:267–283, 2002.

[68] R. M. KARP, Reducibility among combinatorial problems. In R. E. MILLER AND J. W. THATCHER, eds., *Complexity of Computer Communications*, pp. 85–103, Plenum Press, 1972.

[69] C. D. KEYS, Graphs critical for maximal bookthickness. *Pi Mu Epsilon J.*, **6**:79–84, 1975.

[70] K. KOBAYASHI, Book presentation and local unknottedness of spatial graphs. *Kobe J. Math.*, **10(2)**:161–171, 1993.

[71] A. KOSTOCHKA AND J. KRATOCHVÍL, Covering and coloring polygon-circle graphs. *Discrete Math.*, **163(1-3)**:299–305, 1997.

[72] A. V. KOSTOCHKA, Upper bounds on the chromatic number of graphs. *Trudy Inst. Mat.*, **10**:204–226, 1988.

[73] Q. H. LE TU, A planar poset which requires four pages. *Ars Combin.*, **35**:291–302, 1993.

[74] Y. LIN, Cutwidth and related parameters of graphs. *J. Zhengzhou Univ. Nat. Sci. Ed.*, **34(1)**:1–5, 2002.

[75] S. M. MALITZ, Genus g graphs have pagenumber $O(\sqrt{g})$. *J. Algorithms*, **17(1)**:85–109, 1994.

[76] S. M. MALITZ, Graphs with E edges have pagenumber $O(\sqrt{E})$. *J. Algorithms*, **17(1)**:71–84, 1994.

[77] S. MASUYAMA AND S. NAITO, Deciding whether graph G has page number one is in NC. *Inform. Process. Lett.*, **44(1)**:7–10, 1992.

[78] S. L. MITCHELL, Linear algorithms to recognize outerplanar and maximal outerplanar graphs. *Inform. Process. Lett.*, **9(5)**:229–232, 1979.

[79] S. MORAN AND Y. WOLFSTAHL, One-page book embedding under vertex-neighborhood constraints. *SIAM J. Discrete Math.*, **3(3)**:376–390, 1990.

[80] S. MORAN AND Y. WOLFSTHAL, Two-page book embedding of trees under vertex-neighborhood constraints. *Discrete Appl. Math.*, **43(3)**:233–241, 1993.

[81] D. J. MUDER, M. L. WEAVER, AND D. B. WEST, Pagenumber of complete bipartite graphs. *J. Graph Theory*, **12(4)**:469–489, 1988.

[82] J. I. MUNRO AND V. RAMAN, Succinct representation of balanced parentheses and static trees. *SIAM J. Comput.*, **31(3)**:762–776, 2001.

[83] R. NOWAKOWSKI AND A. PARKER, Ordered sets, pagenumbers and planarity. *Order*, **6(3)**:209–218, 1989.

[84] B. OBRENIĆ, Embedding de Bruijn and shuffle-exchange graphs in five pages. *SIAM J. Discrete Math.*, **6(4)**:642–654, 1993.

[85] L. T. OLLMANN, On the book thicknesses of various graphs. In F. HOFFMAN, R. B. LEVOW, AND R. S. D. THOMAS, eds., *Proc. 4th Southeastern Conference on Combinatorics, Graph Theory and Computing*, vol. VIII of *Congressus Numerantium*, p. 459, 1973.

[86] E. T. ORDMAN AND W. SCHMITT, Permutations using stacks and queues. In *Proc. of 24th Southeastern International Conference on Combinatorics, Graph Theory, and Computing*, vol. 96 of *Congr. Numer.*, pp. 57–64, 1993.

[87] S. OVERBAY, *Generalized Book Embeddings*. Ph.D. thesis, Colorado State University, U.S.A., 1998.

[88] S. V. PEMMARAJU, *Exploring the Powers of Stacks and Queues via Graph Layouts*. Ph.D. thesis, Virginia Polytechnic Institute and State University, U.S.A., 1992.

[89] V. R. PRATT, Computing permutations with double-ended queues. Parallel stacks and parallel queues. In *Proc. 5th Annual ACM Symp. on Theory of Computing (STOC '73)*, pp. 268–277, ACM, 1973.

[90] H. J. PRÖMEL, T. SCHICKINGER, AND A. STEGER, A note on triangle-free and bipartite graphs. *Discrete Math.*, **257(2-3)**:531–540, 2002.

[91] S. RENGARAJAN AND C. E. VENI MADHAVAN, Stack and queue number of 2-trees. In D. DING-ZHU AND L. MING, eds., *Proc. 1st Annual International Conf. on Computing and Combinatorics (COCOON '95)*, vol. 959 of *Lecture Notes in Comput. Sci.*, pp. 203–212, Springer, 1995.

[92] A. L. ROSENBERG, The DIOGENES approach to testable fault-tolerant arrays of processors. *IEEE Trans. Comput.*, **C-32**:902–910, 1983.

[93] A. L. ROSENBERG, Book embeddings and wafer-scale integration. In *Proc. 17th Southeastern International Conference on Combinatorics, Graph Theory, and Computing*, vol. 54 of *Congr. Numer.*, pp. 217–224, 1986.

[94] A. L. ROSENBERG, DIOGENES, circa 1986. In *Proc. VLSI Algorithms and Architectures*, vol. 227 of *Lecture Notes in Comput. Sci.*, pp. 96–107, Springer, 1986.

[95] F. SHAHROKHI AND W. SHI, On crossing sets, disjoint sets, and pagenumber. *J. Algorithms*, **34(1)**:40–53, 2000.

[96] F. SHAHROKHI, L. A. SZÉKELY, O. SÝKORA, AND I. VRŤO, The book crossing number of a graph. *J. Graph Theory*, **21(4)**:413–424, 1996.

[97] E. STÖHR, A trade-off between page number and page width of book embeddings of graphs. *Inform. and Comput.*, **79(2)**:155–162, 1988.

[98] E. STÖHR, Optimal book embeddings in depth-k K_n-cylinders. *J. Information Processing and Cybernetics EIK*, **26**, 1990.

[99] E. STÖHR, The pagewidth of trivalent planar graphs. *Discrete Math.*, **89(1)**:43–49, 1991.

[100] R. P. SWAMINATHAN, D. GIRIRAJ, AND D. K. BHATIA, The pagenumber of the class of bandwidth-k graphs is $k-1$. *Inform. Process. Lett.*, **55(2)**:71–74, 1995.

[101] M. M. SYSŁO, Bounds to the page number of partially ordered sets. In M. NAGL, ed., *Proc. 15th International Workshop in Graph-Theoretic Concepts in Computer Science (WG '89)*, vol. 411 of *Lecture Notes in Comput. Sci.*, pp. 181–195, Springer, 1990.

[102] R. TARJAN, Sorting using networks of queues and stacks. *J. Assoc. Comput. Mach.*, **19**:341–346, 1972.

[103] M. TOGASAKI AND K. YAMAZAKI, Pagenumber of pathwidth-k graphs and strong pathwidth-k graphs. *Discrete Math.*, **259(1-3)**:361–368, 2002.

[104] W. UNGER, On the k-colouring of circle-graphs. In M. W. R. CORI, ed., *Proc. 5th International Symp. on Theoretical Aspects of Computer Science (STACS '88)*, vol. 294 of *Lecture Notes in Comput. Sci.*, pp. 61–72, Springer, 1988.

[105] W. UNGER, The complexity of colouring circle graphs. In A. FINKEL AND M. JANTZEN, eds., *Proc. 9th International Symp. on Theoretical Aspects of Computer Science (STACS '92)*, vol. 577 of *Lecture Notes in Comput. Sci.*, pp. 389–400, Springer, 1992.

[106] M. J. WANG, Some results on embedding grid graphs in books. *J. Zhengzhou Univ. Nat. Sci. Ed.*, **29(2)**:31–34, 1997.

[107] A. WIGDERSON, The complexity of the Hamiltonian circuit problem for maximal planar graphs. Tech. Rep. EECS 198, Princeton University, U.S.A., 1982.

[108] D. R. WOOD, Bounded degree book embeddings and three-dimensional orthogonal graph drawing. In P. MUTZEL, M. JÜNGER, AND S. LEIPERT, eds., *Proc. 9th International Symp. on Graph Drawing (GD '01)*, vol. 2265 of *Lecture Notes in Comput. Sci.*, pp. 312–327, Springer, 2002.

[109] D. R. WOOD, Degree constrained book embeddings. *J. Algorithms*, **45(2)**:144–154, 2002.

[110] D. R. WOOD, Queue layouts, tree-width, and three-dimensional graph drawing. In M. AGRAWAL AND A. SETH, eds., *Proc. 22nd Foundations of Software Technology and Theoretical Computer Science (FST TCS '02)*, vol. 2556 of *Lecture Notes in Comput. Sci.*, pp. 348–359, Springer, 2002.

[111] M. YANNAKAKIS, Embedding planar graphs in four pages. *J. Comput. System Sci.*, **38**:36–67, 1986.