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# A note on $t$ -designs with $t$ intersection numbers

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We use the Ray-Chaudhuri and Wilson inequality for a 0-design with  $t$  intersection numbers to prove that ‘For a fixed block size  $k$ , there exist finitely many parametrically feasible  $t$ -designs with  $t$  intersection numbers and  $\lambda > 1$ ’.

**Keywords:**  $t$ -designs

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## 1 Introduction

Let  $X$  be a finite set of  $v$  elements, called *points*, and let  $\beta$  be a finite family of distinct  $k$ -subsets of  $X$ , called *blocks*. Then the pair  $D = (X, \beta)$  is called a  $t$ -design with parameters  $(v, k, \lambda)$  if any  $t$ -subset of  $X$  is contained in exactly  $\lambda$  members of  $\beta$ . If  $\lambda_i$  denotes the number of blocks containing  $i$  points,  $i = 0, 1, 2, \dots, t-1$ , then  $\lambda_i$  is independent of the choice of the  $i$  points and  $\lambda_i \binom{k-i}{t-i} = \lambda \binom{v-i}{t-i}$ . In particular,  $b = \lambda_0$  is the number of blocks, and  $\lambda_1 = r$  is the number of blocks through any point of  $D$ . A 0-design is a pair  $(X, \beta)$  where  $\beta$  is a collection of  $k$ -subsets of  $X$ . For  $0 \leq x < k$ ,  $x$  is called *intersection number of  $D$*  if there exists  $B, B' \in \beta$  such that  $|B \cap B'| = x$ . A 2-design with two intersection numbers is said to be a *quasi-symmetric design*.

For a 0-design with  $t$  intersection numbers, Ray-Chaudhuri and Wilson proved that  $b \leq \binom{v}{t}$  (see [1]). This result is used by Sane and Shrikhande [7] to prove that, for a fixed value of the block size  $k$ , there exist finitely many quasi-symmetric designs with  $\lambda > 1$ .

In the literature many finiteness results for quasi-symmetric designs and quasi-symmetric 3-designs are proved by using this result by Sane and Shrikhande (see [5], [6], [7], [8]).

Our main aim in this paper is to extend the result by Sane and Shrikhande to  $t$ -designs with  $t$  intersection numbers. More specifically, we obtain a relation between the parameters of a  $t$ -design with  $t$  intersection numbers, and we use it to show that, for a fixed value of the block size  $k$ ,  $v$  takes finitely many values. Finally, we use the result by Ray-Chaudhuri and Wilson to complete the proof.

## 2 Main Results

**Theorem 2.1 (Ray-Chaudhuri and Wilson)** *Let  $D$  be a 0-design with  $t$  ( $0 < t \leq k \leq v - t$ ) intersection numbers  $x_1, x_2, \dots, x_t$  ( $0 \leq x_1 < x_2 < \dots < x_t < k$ ). Then  $b \leq \binom{v}{t}$ .*

**Lemma 2.2** *Let  $D$  be a  $t$ -design with  $t$  intersection numbers  $x_1, x_2, \dots, x_t$  ( $0 \leq x_1 < x_2 < \dots < x_t < k$ ). Then the following relation holds:*

$$\begin{vmatrix} x_2 - x_1 & \cdots & x_t - x_1 & x_1(b-1) - (r-1) \\ \binom{x_2}{2} - \binom{x_1}{2} & \cdots & \binom{x_t}{2} - \binom{x_1}{2} & \binom{x_1}{2}(b-1) - \binom{k}{2}(\lambda_2 - 1) \\ \vdots & \vdots & \vdots & \vdots \\ \binom{x_2}{t} - \binom{x_1}{t} & \cdots & \binom{x_t}{t} - \binom{x_1}{t} & \binom{x_1}{t}(b-1) - \binom{k}{t}(\lambda_t - 1) \end{vmatrix} = 0. \tag{1}$$

**Proof:** Let  $a_i, 2 \leq i \leq t$ , be the number of blocks intersecting  $B_0$  in  $x_i$  points. Fix a block  $B_0$  and count in two ways the number of  $(j + 1)$ -tuples  $(\{p_1, p_2, \dots, p_j\}, B)$ , where  $B$  is a block of  $D$  other than  $B_0$ , and where  $p_1, p_2, \dots, p_j$  are distinct points of  $D$  contained in  $B \cap B_0$ . For  $j = 1, 2, \dots, t$ , we have

$$\sum_{i=2}^t \binom{x_i}{j} a_i + \binom{x_1}{j} \left( b - 1 - \sum_{i=2}^t a_i \right) - \binom{k}{j} (\lambda_j - 1) = 0.$$

We rewrite these equations as follows:

$$\sum_{i=2}^t \left[ \binom{x_i}{j} - \binom{x_1}{j} \right] a_i + \binom{x_1}{j} (b - 1) - \binom{k}{j} (\lambda_j - 1) = 0.$$

These are  $t$  equations in  $t - 1$  unknowns  $a_2, a_3, \dots, a_t$ . Since  $a_2, a_3, \dots, a_t, 1$  cannot be simultaneously zero, the coefficient matrix is singular. Hence the determinant given in equation (1) must be zero.  $\square$

We would like to point out that equation (1) is an extension of equation (1) of [7]. The latter is found to be useful in the study of quasi-symmetric designs. An immediate application of (1) is given in the following theorem.

**Theorem 2.3** *For a fixed block size  $k$ , there exist finitely many parametrically feasible  $t$ -designs with  $t$  intersection numbers  $x_1, x_2, \dots, x_t$  ( $0 \leq x_1 < x_2 < \dots < x_t < k$ ) and  $\lambda > 1$ .*

**Proof:** By Theorem 2.1, the inequality  $b \leq \binom{v}{t}$  holds for any  $t$ -design with  $t$  intersection numbers. Hence it suffices to show that for a fixed  $k, v$  takes finitely many values.

We divide the proof in two parts, by considering the cases  $x_1 \neq 0$  and  $x_1 = 0$  separately.

For  $x_1 \neq 0$ , we have the equality

$$\sum_{i=2}^t (x_i - x_1) a_i = k(r - 1) - x_1(b - 1).$$

This implies  $k(r - 1) - x_1(b - 1) > 0$ . Hence,  $x_1 < \frac{k(r-1)}{b-1} < \frac{kr}{b} = \frac{k^2}{v}$ . Now it is easy to see that  $v < \frac{k^2}{x_1}$ .

For  $x_1 = 0$ , by Lemma 2.2 we have

$$\begin{vmatrix} x_2 & \cdots & x_t & k(r-1) \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} & \binom{k}{2}(\lambda_2-1) \\ \vdots & \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} & \binom{k}{t-1}(\lambda_{t-1}-1) \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} & \binom{k}{t}(\lambda_t-1) \end{vmatrix} = 0.$$

Now we put  $\lambda_j = \binom{v-j}{t-j} \lambda / \binom{k-j}{t-j}$ ,  $j = 1, 2, \dots, t-1$ , to get

$$A \begin{vmatrix} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{vmatrix} = \lambda A \begin{vmatrix} k \binom{v-1}{t-1} / \binom{k-1}{t-1} \\ \binom{k}{2} \binom{v-2}{t-2} / \binom{k}{t-1} \\ \vdots \\ \binom{k}{t-1} \binom{v-t+1}{1} / \binom{k-t+1}{1} \\ \binom{k}{t} \end{vmatrix},$$

where

$$A = \begin{bmatrix} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} \end{bmatrix}.$$

We simplify the above equation as follows:

$$A \begin{vmatrix} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{vmatrix} = \lambda A \begin{vmatrix} k \frac{(v-1)(v-2)\cdots(v-t+2)(v-t+1)}{(k-1)(k-2)\cdots(k-t+2)(k-t+1)} \\ \binom{k}{2} \frac{(v-2)(v-3)\cdots(v-t+2)(v-t+1)}{(k-2)(k-3)\cdots(k-t+2)(k-t+1)} \\ \vdots \\ \binom{k}{t-1} \frac{v-t+1}{k-t+1} \\ \binom{k}{t} \end{vmatrix}.$$

Now we multiply both sides by  $(k-1)(k-2)\cdots(k-t+2)(k-t+1)$ , to get

$$\begin{aligned} & \left| \begin{array}{c} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{array} \right| (k-1)(k-2)\cdots(k-t+2)(k-t+1) \\ &= \lambda \left| \begin{array}{c} k(v-1)(v-2)\cdots(v-t+2)(v-t+1) \\ \binom{k}{2}(v-2)(v-3)\cdots(v-t+2)(v-t+1)(k-1) \\ \vdots \\ \binom{k}{t-1}(v-t+1)(k-1)(k-2)\cdots(k-t+2) \\ \binom{k}{t}(k-1)(k-2)\cdots(k-t+2)(k-t+1) \end{array} \right|. \end{aligned}$$

Note that  $\lambda$  divides the left hand side. Hence  $\lambda$  takes finitely many values only. It is clear from the above equation, by expanding the right hand side determinant with respect to the last column of the matrix, that  $v-t+1$  divides

$$(k-1)(k-2)\cdots(k-t+1) \left( \left| \begin{array}{c} k \\ \binom{k}{2} \\ \vdots \\ \binom{k}{t-1} \\ \binom{k}{t} \end{array} \right| - \lambda \binom{k}{t} \left| \begin{array}{ccc} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-1} & \cdots & \binom{x_t}{t-1} \end{array} \right| \right).$$

For a fixed  $k$ , this is a finite number. If it is non-zero, then  $v$  takes only finitely many values, otherwise  $v-t+2$  divides

$$\binom{k}{t-1}(k-1)(k-2)\cdots(k-t+2) \left| \begin{array}{ccc} x_2 & \cdots & x_t \\ \binom{x_2}{2} & \cdots & \binom{x_t}{2} \\ \vdots & \vdots & \vdots \\ \binom{x_2}{t-2} & \cdots & \binom{x_t}{t-2} \\ \binom{x_2}{t} & \cdots & \binom{x_t}{t} \end{array} \right|.$$

This can be simplified to

$$\binom{k}{t-1} (k-1)(k-2)\cdots(k-t+2) \frac{x_2 x_3 \cdots x_t}{2! 3! \cdots (t-2)! t!} \begin{vmatrix} 1 & \cdots & 1 \\ x_2 & \cdots & x_t \\ \vdots & \vdots & \vdots \\ x_2^{t-3} & \cdots & x_t^{t-3} \\ x_2^{t-1} & \cdots & x_t^{t-1} \end{vmatrix}.$$

It is easy to check that the above determinant is non-zero. This completes the proof. □

In [7], Sane and Shrikhande proved analogues of Theorem 2.3 for quasi-symmetric designs. As in our paper, they also divide their proof in two parts, depending on whether or not the smaller intersection number,  $x$ , is non-zero or not. The case  $x_1 \neq 0$  of Theorem 2.3 is similar to that of  $x \neq 0$  of [7]. But to prove their result for the case  $x = 0$ , they use a rather long argument, whereas our proof is elementary and short.

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