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Recently, Green and Losonczy [5, 6] introduced *freely braided* permutations as a special class of restricted permutations that has arisen in the study of Schubert varieties. They suggest as an open problem to enumerate the number of freely braided permutations in S_n. In this paper, we prove that the generating function for the number of freely braided permutations in S_n is given by

$$\frac{1-3x-2x^2+(1+x)\sqrt{1-4x}}{1-4x-x^2+(1-x^2)\sqrt{1-4x}}.$$

Keywords: restricted permutations, freely braided permutations, generating functions

1 Introduction

Let $\alpha \in S_n$ and $\tau \in S_k$ be two permutations. Then α *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $(\alpha_{i_1}, \ldots, \alpha_{i_k})$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*; α *avoids* τ, or is τ-*avoiding*, if α does not contain such a subsequence. The set of all τ-avoiding permutations in S_n is denoted by $S_n(\tau)$. For a collection of patterns T, α *avoids* T if α avoids all $\tau \in T$; the corresponding subset of S_n is denoted by $S_n(T)$.

One important and often difficult problem in the study of restricted permutations is the enumeration problem: given a set T of permutations, enumerate the set $S_n(T)$ consisting of those permutations in S_n which avoid every element of T. The first systematic study was not undertaken until 1985, when Simion and Schmidt [15] solved the enumeration problem for every $T \subseteq S_3$. More recent work on various instances of the enumeration problem may be found in [1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [13], [14] and the references therein, [16], [17], [18], [19], and [20].

Recently, a special class of restricted permutations has arisen in the study of Schubert varieties. Green and Losonczy [5] defined, for any simply laced Coxeter group, a subset of "freely braided elements" (for details, see [5] and [6]), and they suggest as an open problem to enumerate the number of freely-braided permutations in S_n.

Definition 1.1. *A permutation* π *is said to be* freely-braided *if and only if* π *avoids each of the four patterns* 1234, 1243, 1324, *and* 2134. *We denote the set of all freely-braided permutations in* S_n *by* $\mathcal{F}_n$, *i.e.,* $\mathcal{F}_n = S_n(1234, 1243, 1324, 2134)$.

Remark 1.2. *In [5], a permutation* π *is "freely braided" if and only if* π *avoids each of the four patterns* 4321, 3421, 4231, *and* 4312. *Note, however, that a permutation* π *avoids these four patterns if and only if* $r(\pi)$ *avoids each of the four patterns* 1234, 1243, 1324, *and* 2134, *where* $r : \pi_1\pi_2\dots\pi_n \to \pi_n\dots\pi_2\pi_1$. *So, for all* $n \geq 0$,

$$\#S_n(1234, 1243, 1324, 2134) = \#S_n(4321, 4231, 4312, 3421).$$

In this paper we give a complete answer for the number of freely-braided permutations in $\mathcal{F}_n$. The main result of this paper can be formulated as follows.

Theorem 1.3. *The generating function for the number of freely-braided permutations in* $\mathcal{F}_n$ *is given by*

$$\frac{1-3x-2x^2+(1+x)\sqrt{1-4x}}{1-4x-x^2+(1-x^2)\sqrt{1-4x}} = \frac{1}{1-4x-x^2}(1-3x-x^2+x^3-x^2(1+x)^2C(x)),$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ *is the generating function for the Catalan numbers* ($C_n = \frac{1}{n+1}\binom{2n}{n}$).

The proof of the above theorem is presented in Section 2.

2 Proof Theorem 1.3

Given $b_1, b_2, \dots, b_m \in \mathbb{N}$, we define

$$f_n(b_1, b_2, \dots, b_m) = \#\{\pi = \pi_1\pi_2\dots\pi_n \in \mathcal{F}_n \mid \pi_1\pi_2\dots\pi_m = b_1b_2\dots b_m\}.$$

It is natural to extend f_n to the case $m = 0$ by setting $f_n(\varnothing) = f_n = \#\mathcal{F}_n$. The following properties of the numbers $f_n(b_1, \dots, b_m)$ can be deduced easily from the definitions.

Lemma 2.1.

(1) Let $m \geq 1$ *and* $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. *Then, for all* $b_m + 1 \leq j \leq n-2$,

$$f_n(b_1, \dots, b_m, j) = 0.$$

(2) Let $m \geq 2$ *and* $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. *Then, for all* $b_m + 1 \leq j \leq n-1$,

$$f_n(b_1, \dots, b_m, j) = 0.$$

(3) Let $m \geq 1$ *and* $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. *Then*

$$f_n(b_1, \dots, b_m, n) = f_{n-1}(b_1, \dots, b_m).$$

(4) Let $m \geq 1$ *and* $n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1$. *Then*

$$f_n(n, b_1, \dots, b_m) = f_n(n-1, b_1, \dots, b_m) = f_{n-1}(b_1, \dots, b_m).$$

(5) Let $m \geq 1$ and $n-1 \geq b_1 > b_2 > \cdots > b_m \geq 1$. Then

$$f_n(b_1, n, \ldots, b_m) = f_{n-1}(b_1, \ldots, b_m).$$

Proof. For (1), observe that if $\pi \in S_n$ is such that $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m j$, then the entries b_m, j, $n-1$, n give an occurrence of the pattern 1234 or 1243 in π.

For (2), observe that if $\pi \in S_n$ is such that $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m j$, then either the entries b_{m-1}, b_m, $n-1$, n give an occurrence of the pattern 2134 in π or the entries b_m, j, $n-1$, n give an occurrence of 1234 or 1243 in π.

For (3), observe that if $\pi \in S_n$ is such that $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m n$, where $n-2 \geq b_1 > \cdots > b_m \geq 1$, then no occurrence of the patterns 1234, 1243, 1324, 2134 in π can involve the entry $\pi_{m+1} = n$. Hence, there is a bijection between the set of permutations $\pi \in \mathcal{F}_n$ with $\pi_1 \ldots \pi_m \pi_{m+1} = b_1 \ldots b_m n$ and the set of permutations $\sigma \in \mathcal{F}_{n-1}$ such that $\sigma_1 \ldots \sigma_m = b_1 \ldots b_m$.

Using similar arguments as in the proof of (3) we get that (4) and (5) hold. □

Next we introduce objects $A_m(n)$, $B_m(n)$ and $C_m(n)$ which organize suitably the information about the numbers $f_n(b_1, b_2, ..., b_m)$ and play an important role in the proof of the main result.

Definition 2.2. *For $1 \leq m \leq n$ set*

$$\begin{aligned} A_m(n) &= \sum_{n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1} f_n(b_1, b_2, \ldots, b_m), \\ B_m(n) &= \sum_{n-2 \geq b_1 > b_2 > \cdots > b_m \geq 1} f_n(b_1, n-1, b_2, b_3, \ldots, b_m), \quad \textit{and} \\ C_m(n) &= \sum_{n-1 \geq b_1 > b_2 > \cdots > b_m \geq 1} f_n(b_1, b_2, \ldots, b_m). \end{aligned}$$

As before, this definition is extended to
the case $m = 0$ by setting $A_0(n) = C_0(n) = f_n$ and $B_0(n) = 0$.

In the following two subsections we derive expressions for $A_m(n)$ and $B_1(n)$, which are used in subsection 2.3 to complete the proof of Theorem 1.3.

2.1 A recurrence for the f_j, also involving $B_1(n)$

In the following result we derive an expression for $A_m(n)$.

Proposition 2.3.

(1) For all $n \geq 2$,

$$A_1(n) = f_n - 2f_{n-1}.$$

(2) For all $n \geq 3$,

$$A_2(n) = f_n - 3f_{n-1} + f_{n-2} - B_1(n).$$

(3) For all $2 \leq m \leq n-2$,

$$A_m(n) = A_{m+1}(n) + A_m(n-1) + \cdots + A_0(n-1-m).$$

Proof. For (1), Definition 2.2 for $A_1(n)$ gives that

$$A_1(n)=\sum_{b_1=1}^{n-2} f_n(b_1)=f_n-f_n(n-1)-f_n(n).$$

Observe that if $\pi\in S_n$ is such that $\pi_1=n-1$ or $\pi_1=n$, then no occurrence of the patterns 1234, 1243, 1324, 2134 in π can involve the entry π_1. So we get that the number of permutations in $\mathcal{F}_n$ starting with n (resp., $n-1$) is f_{n-1}. Hence, $A_1(n)=f_n-2f_{n-1}$, as claimed in (1).

For (2), Definition 2.2 for $A_1(n)$ and $A_2(n)$ gives that

$$A_1(n)=A_2(n)+\sum_{b_1=1}^{n-2}\sum_{b_2=b_1+1}^{n} f_n(b_1,b_2).$$

Using Lemma 2.1, parts (1) and (5), and Definition 2.2 we obtain that

$$A_1(n)=A_2(n)+B_1(n)+A_1(n-1)+f_{n-1}(n-2).$$

Hence, by using the proof of (1) and Definition 2.2 we get the desired result.

For (3), let $2\le m\le n-2$. Definition 2.2 yields

$$A_m(n)=A_{m+1}(n)+\sum_{n-2\ge b_1>b_2>\cdots>b_m\ge 1}\ \sum_{j=b_m+1}^{n} f_n(b_1,\ldots,b_m,j).$$

Using Lemma 2.1, parts (2) and (3), we have

$$\begin{aligned} A_m(n) &= A_{m+1}(n)+\sum_{n-2\ge b_1>b_2>\cdots>b_m\ge 1} f_n(b_1,b_2,\ldots,b_m,n)\\ &= A_{m+1}(n)+\sum_{n-2\ge b_1>b_2>\cdots>b_m\ge 1} f_{n-1}(b_1,b_2,\ldots,b_m)\\ &= A_{m+1}(n)+C_m(n-1). \end{aligned}$$

Definition 2.2 and Lemma 2.1 (4) give

$$\begin{aligned} C_m(n) &= A_m(n)+\sum_{n-2\ge b_2>\cdots>b_m\ge 1} f_n(n-1,b_2,\ldots,b_m)\\ &= A_m(n)+\sum_{n-2\ge b_2>\cdots>b_m\ge 1} f_{n-1}(b_2,\ldots,b_m) \qquad (2.1)\\ &= A_m(n)+C_{m-1}(n-1). \end{aligned}$$

Hence, by induction on m together with (1) we get the desired result. □

We next find an explicit expression for $A_m(n)$ in terms of $A_0(n)=f_n$, $A_1(n)$ and $A_2(n)$.

Theorem 2.4. *For all $n\ge 5$ and $2\le m\le n-2$,*

$$A_m(n)=\sum_{j\ge 0}(-1)^j\binom{m-1-j}{j}A_2(n-j)-\sum_{j\ge 0}(-1)^j\binom{m-3-j}{j}(A_1(n-2-j)+A_0(n-3-j)),$$

with the usual convention that $\binom{a}{b}=0$ if $a<b$ or $a<0$.

Proof. For $m=2$ we have that $A_2(n)=A_2(n)$, so the theorem holds. Assume the theorem for m and all appropriate n, and let us prove the equality for $m+1$. Using Proposition 2.3(3) we get that

$$A_{m+1}(n)=A_m(n)-\sum_{i=0}^{m}A_{m-i}(n-1-i),$$

and by the induction hypothesis, we arrive at

$$\begin{aligned}
&A_{m+1}(n)\\
&\quad=\sum_{j\geq 0}(-1)^j\binom{m-1-j}{j}A_2(n-j)-\sum_{j\geq 0}(-1)^j\binom{m-3-j}{j}(A_1(n-2-j)+A_0(n-3-j))\\
&\qquad-\sum_{i=0}^{m}\sum_{j\geq 0}(-1)^j\binom{m-1-i-j}{j}A_2(n-1-i-j)\\
&\qquad+\sum_{i=0}^{m}\sum_{j\geq 0}(-1)^j\binom{m-3-i-j}{j}(A_1(n-3-i-j)+A_0(n-4-i-j))\\
&\quad=\sum_{j\geq 0}(-1)^j\binom{m-1-j}{j}A_2(n-j)-\sum_{j\geq 0}(-1)^j\binom{m-3-j}{j}(A_1(n-2-j)+A_0(n-3-j))\\
&\qquad-\sum_{j\geq 0}\left(\sum_{i=0}^{j}(-1)^i\binom{m-1-j}{i}A_2(n-1-j)-\sum_{i=0}^{j}(-1)^i\binom{m-3-j}{i}(A_1(n-3-j)+A_0(n-4-j))\right).
\end{aligned}$$

Using the familiar identity $\binom{p}{0}-\binom{p}{1}+\cdots+(-1)^q\binom{p}{q}=(-1)^q\binom{p-1}{q}$ we obtain that

$$\begin{aligned}
A_{m+1}(n)\quad&=\sum_{j\geq 0}(-1)^j\binom{m-1-j}{j}A_2(n-j)-\sum_{j\geq 0}(-1)^j\binom{m-3-j}{j}(A_1(n-2-j)+A_0(n-3-j))\\
&\quad-\sum_{j\geq 0}(-1)^j\binom{m-2-j}{j}A_2(n-1-j)+\sum_{j\geq 0}(-1)^j\binom{m-4-j}{j}(A_1(n-3-j)+A_0(n-4-j)),
\end{aligned}$$

and by using the identity $\binom{p}{q}=\binom{p-1}{q}+\binom{p-1}{q-1}$ we get that

$$A_{m+1}(n)=\sum_{j\geq 0}(-1)^j\binom{m-j}{j}A_2(n-j)-\sum_{j\geq 0}(-1)^j\binom{m-2-j}{j}(A_1(n-2-j)+A_0(n-3-j)).$$

Hence, by induction on m we get the desired result. □

Using Theorem 2.4 for $m=n-2$, together with Proposition 2.3, parts (1) and (2), and $A_{n-2}(n)=1$ (see Definition 2.2) we get the main result of this subsection.

Theorem 2.5. *For all $n\geq 5$,*

$$\sum_{j\geq 0}(-1)^j\binom{n-3-j}{j}(f_{n-j}-3f_{n-1-j}+f_{n-2-j}-B_1(n-j))=1+\sum_{j\geq 0}(-1)^j\binom{n-5-j}{j}(f_{n-2-j}-f_{n-3-j}).$$

2.2 A recursive formula for $B_1(n)$

We next we find a recurrence for $B_1(n)$ in terms of f_n.

Proposition 2.6. *We have*

$B_1(n)=C_0(n-2)+C_1(n-2)+\cdots+C_{n-2}(n-2)$ *for all* $n\geq 3$.

$B_1(n)-B_1(n-1)=A_0(n-2)+A_1(n-2)+\cdots+A_{n-4}(n-2)$ *for all* $n\geq 4$.

Proof. For (2.6), by Definition 2.2 we get that

$$B_1(n)=f_n(n-2,n-1)+\sum_{n-3\geq b_1\geq 1} f_n(b_1,n-1).$$

Observe that if a permutation $\pi\in S_n$ is such that $\pi_1=n-2$ and $\pi_2=n-1$, then no occurrence of the patterns 1234, 1243, 1324, 2134 can involve either the entry $n-1$ or the entry n. Thus, $f_n(n-2,n-1)=f_{n-2}=C_0(n-2)$, and for all $n\geq 3$, $B_1(n)=C_0(n-2)+L_1(n)$, where we define

$$L_m(n)=\sum_{n-3\geq b_1>\cdots>b_m\geq 1} f_n(b_1,n-1,b_2,\ldots,b_m) \quad \text{for } m\geq 1.$$

Using Lemma 2.1, parts (1) and (2), we get that

$$L_m(n)=L_{m+1}(n)+\sum_{n-3\geq b_1>\cdots>b_m\geq 1} f_n(b_1,n-1,b_2,\ldots,b_m,n),$$

and by Lemma 2.1, parts (3) and (5), together with Definition 2.2 we arrive at

$$L_m(n)=L_{m+1}(n)+C_m(n-2)$$

for all $m\geq 1$. Hence, by induction on m together with the fact that $L_{n-1}(n)=0$ we have

$$B_1(n)=C_0(n-2)+C_1(n-2)+\cdots+C_{n-2}(n-2),$$

as claimed.

For (2.6), using Equation (2.1) together with (2.6) and $C_n(n)=A_{n-1}(n)=0$ (see Definition 2.2) we get the desired result. $\square$

Theorem 2.7. *For all* $n\geq 4$,

$$\begin{aligned}B_1(n)-B_1(n-2)&=f_{n-1}-2f_{n-2}+f_{n-3}\\&\quad-\sum_{j\geq 0}(-1)^j\binom{n-3-j}{j}A_2(n-1-j)+\sum_{j\geq 0}(-1)^j\binom{n-5-j}{j}\left(A_1(n-3-j)+A_0(n-4-j)\right).\end{aligned}$$

Proof. It is easy to check that the theorem holds for $n=4,5,6$. Now, let $n\geq 7$. By using Proposition 2.6 (2.6) and Theorem 2.4 we get that

$$\begin{aligned}B_1(n)-B_1(n-1)&=A_0(n-2)+A_1(n-2)+\sum_{m=2}^{n-4}\sum_{j\geq 0}(-1)^j\binom{m-1-j}{j}A_2(n-2-j)\\&\quad-\sum_{m=3}^{n-4}\sum_{j\geq 0}(-1)^j\binom{m-3-j}{j}\left(A_1(n-4-j)+A_0(n-5-j)\right)\\&=A_0(n-2)+A_1(n-2)-A_2(n-2)+\sum_{m=1}^{n-4}\sum_{j\geq 0}(-1)^j\binom{m-1-j}{j}A_2(n-2-j)\\&\quad-\sum_{m=3}^{n-4}\sum_{j\geq 0}(-1)^j\binom{m-3-j}{j}\left(A_1(n-4-j)+A_0(n-5-j)\right)\\&=A_0(n-2)+A_1(n-2)-A_2(n-2)+\sum_{j\geq 0}\sum_{i=j}^{n-5-j}(-1)^j\binom{i}{j}A_2(n-2-j)\\&\quad-\sum_{j\geq 0}\sum_{i=j}^{n-7-j}(-1)^j\binom{i}{j}\left(A_1(n-4-j)+A_0(n-5-j)\right).\end{aligned}$$

Therefore, using the identity $\binom{p}{p}+\binom{p+1}{p}+\cdots+\binom{q}{p}=\binom{q+1}{p+1}$ gives that

$$\begin{aligned}
&B_1(n)-B_1(n-1)\\
&\quad=A_0(n-2)+A_1(n-2)-A_2(n-2)+A_2(n-1)-A_1(n-3)-A_0(n-4)\\
&\quad-\sum_{j\geq 0}(-1)^j\tbinom{n-3-j}{j}A_2(n-1-j)+\sum_{j\geq 0}(-1)^j\tbinom{n-5-j}{j}(A_1(n-3-j)+A_0(n-4-j)).
\end{aligned}$$

Hence, using Proposition 2.3, parts (1) and (2), we obtain the desired identity.

□

2.3 Proof of Theorem 1.3

We start by showing the following result.

Lemma 2.8. *Let $t(x)$ be the generating function for the sequence $(t_n)_{n\geq 0}$, that is, $t(x)=\sum_{n\geq 0}t_nx^n$. Then*

$$\sum_{n\geq m}\left(x^n\sum_{j\geq 0}(-1)^j\binom{n-m-j}{j}t_{n-s-j}\right)=\frac{x^s}{(1-x)^{m-s}}\left(t(x(1-x))-\sum_{j=0}^{m-s-1}t_jx^j(1-x)^j\right).$$

Proof. We have

$$\begin{aligned}
\sum_{n\geq m}\left(x^n\sum_{j\geq 0}(-1)^j\tbinom{n-m-j}{j}t_{n-s-j}\right)&=\sum_{n\geq 0}\sum_{j=0}^{n}(-1)^j\tbinom{n}{j}x^{n+m+j}t_{n+m-s}\\
&=\sum_{n\geq 0}t_{n+m-s}x^{n+m}(1-x)^n=\tfrac{x^s}{(1-x)^{m-s}}\left(t(x(1-x))-\sum_{j=0}^{m-s-1}t_jx^j(1-x)^j\right),
\end{aligned}$$

as claimed. □

Now we are ready to prove the main result of this paper, namely Theorem 1.3, which is restated here for easy reference.

Theorem 1.3. The generating function for the number of freely-braided permutations in $\mathcal{F}_n$ is given by

$$\frac{1-3x-2x^2+(1+x)\sqrt{1-4x}}{1-4x-x^2+(1-x^2)\sqrt{1-4x}}.$$

Proof. We denote the generating function for the number of freely-braided permutations in $\mathcal{F}_n$ by $F(x)$, that is, $F(x)=\sum_{n\geq 0}f_nx^n$. Also, we denote the generating function for the sequence $\{B_1(n)\}_{n\geq 0}$ by $B(x)$, that is, $B(x)=\sum_{n\geq 0}B_1(n)x^n$.

Theorem 2.5 gives

$$\sum_{j\geq 0}(-1)^j\binom{n-3-j}{j}(f_{n-j}-3f_{n-1-j}+f_{n-2-j}-B_1(n-j))=1+\sum_{j\geq 0}(-1)^j\binom{n-5-j}{j}(f_{n-2-j}-f_{n-3-j}),$$

for all $n\geq 5$. Multiplying by x^n and summing over all $n\geq 5$ together with using Lemma 2.8 we arrive at

$$\begin{aligned}&-x^4+\tfrac{1}{(1-x)^3}\big((1-3x(1-x)+x^2(1-x)^2)F(x(1-x))-1+2x(1-x)-B(x(1-x))\big)\\&=\tfrac{x^5}{1-x}+\tfrac{x^2}{(1-x)^3}(F(x(1-x))-1-x(1-x)-2x^2(1-x)^2)+\tfrac{x^2}{(1-x)^2}(F(x(1-x))-1-x(1-x)),\end{aligned}$$

or equivalently,

$$F(x(1-x))-\frac{1}{(1-x)^3}B(x(1-x))=\frac{1}{1-x}. \tag{2.2}$$

Theorem 2.7 gives

$$\begin{aligned}B_1(n)-B_1(n-2)&=f_{n-1}-2f_{n-2}+f_{n-3}\\&-\sum_{j\geq0}(-1)^j\tbinom{n-3-j}{j}A_2(n-1-j)+\sum_{j\geq0}(-1)^j\tbinom{n-5-j}{j}(A_1(n-3-j)+A_0(n-4-j)),\end{aligned}$$

for all $n\geq4$. Multiplying by x^n and summing over all $n\geq4$ together with using Lemma 2.8 we arrive at

$$\begin{aligned}(1-x^2)B(x)-x^3&=\tfrac{x}{(1-x)^2}F(x)-x+x^2(1-x)\\&-\tfrac{x}{(1-x)^2}((1-3x(1-x)+x^2(1-x)^2)F(x(1-x))-1+2x(1-x)-B(x(1-x)))\\&+\tfrac{x^3}{(1-x)^2}(F(x(1-x))-1-x(1-x))-\tfrac{x^4}{1-x}(F(x(1-x)-1),\end{aligned}$$

or equivalently,

$$(1-x^2)B(x)=x^2-x(1-x)F(x(1-x))+x(1-x)^2F(x)+\frac{x}{(1-x)^2}B(x(1-x)). \tag{2.3}$$

Using Equations 2.2 and 2.3 we get that

$$\begin{cases}B(x(1-x))=(1-x)^3F(x(1-x))-(1-x)^2\\(1+x)B(x)=-x+x(1-x)F(x),\end{cases}$$

or equivalently,

$$\begin{cases}B(x) &=\left(1-\frac{1-\sqrt{1-4x}}{2}\right)^3F(x)-(1-\frac{1-\sqrt{1-4x}}{2})^2\\(1+x)B(x)&=-x+x(1-x)F(x)\end{cases}.$$

The rest is easy to check. □

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