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# On an open problem of Green and Losonczy: exact enumeration of freely braided permutations

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Recently, Green and Losonczy [5, 6] introduced *freely braided* permutations as a special class of restricted permutations that has arisen in the study of Schubert varieties. They suggest as an open problem to enumerate the number of freely braided permutations in  $S_n$ . In this paper, we prove that the generating function for the number of freely braided permutations in  $S_n$  is given by

$$\frac{1 - 3x - 2x^2 + (1+x)\sqrt{1-4x}}{1 - 4x - x^2 + (1-x^2)\sqrt{1-4x}}.$$

**Keywords:** restricted permutations, freely braided permutations, generating functions

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## 1 Introduction

Let  $\alpha \in S_n$  and  $\tau \in S_k$  be two permutations. Then  $\alpha$  *contains*  $\tau$  if there exists a subsequence  $1 \leq i_1 < i_2 < \dots < i_k \leq n$  such that  $(\alpha_{i_1}, \dots, \alpha_{i_k})$  is order-isomorphic to  $\tau$ ; in such a context  $\tau$  is usually called a *pattern*;  $\alpha$  *avoids*  $\tau$ , or is  $\tau$ -*avoiding*, if  $\alpha$  does not contain such a subsequence. The set of all  $\tau$ -avoiding permutations in  $S_n$  is denoted by  $S_n(\tau)$ . For a collection of patterns  $T$ ,  $\alpha$  *avoids*  $T$  if  $\alpha$  avoids all  $\tau \in T$ ; the corresponding subset of  $S_n$  is denoted by  $S_n(T)$ .

One important and often difficult problem in the study of restricted permutations is the enumeration problem: given a set  $T$  of permutations, enumerate the set  $S_n(T)$  consisting of those permutations in  $S_n$  which avoid every element of  $T$ . The first systematic study was not undertaken until 1985, when Simion and Schmidt [15] solved the enumeration problem for every  $T \subseteq S_3$ . More recent work on various instances of the enumeration problem may be found in [1], [2], [3], [4], [7], [8], [9], [10], [11], [12], [13], [14] and the references therein, [16], [17], [18], [19], and [20].

Recently, a special class of restricted permutations has arisen in the study of Schubert varieties. Green and Losonczy [5] defined, for any simply laced Coxeter group, a subset of “freely braided elements” (for details, see [5] and [6]), and they suggest as an open problem to enumerate the number of freely-braided permutations in  $S_n$ .

**Definition 1.1.** A permutation  $\pi$  is said to be freely-braided if and only if  $\pi$  avoids each of the four patterns 1234, 1243, 1324, and 2134. We denote the set of all freely-braided permutations in  $S_n$  by  $\mathcal{F}_n$ , i.e.,  $\mathcal{F}_n = S_n(1234, 1243, 1324, 2134)$ .

**Remark 1.2.** In [5], a permutation  $\pi$  is "freely braided" if and only if  $\pi$  avoids each of the four patterns 4321, 3421, 4231, and 4312. Note, however, that a permutation  $\pi$  avoids these four patterns if and only if  $r(\pi)$  avoids each of the four patterns 1234, 1243, 1324, and 2134, where  $r : \pi_1\pi_2 \dots \pi_n \rightarrow \pi_n \dots \pi_2\pi_1$ . So, for all  $n \geq 0$ ,

$$\#S_n(1234, 1243, 1324, 2134) = \#S_n(4321, 4231, 4312, 3421).$$

In this paper we give a complete answer for the number of freely-braided permutations in  $\mathcal{F}_n$ . The main result of this paper can be formulated as follows.

**Theorem 1.3.** The generating function for the number of freely-braided permutations in  $\mathcal{F}_n$  is given by

$$\frac{1 - 3x - 2x^2 + (1+x)\sqrt{1-4x}}{1 - 4x - x^2 + (1-x^2)\sqrt{1-4x}} = \frac{1}{1 - 4x - x^2} (1 - 3x - x^2 + x^3 - x^2(1+x)^2C(x)),$$

where  $C(x) = \frac{1-\sqrt{1-4x}}{2x}$  is the generating function for the Catalan numbers ( $C_n = \frac{1}{n+1} \binom{2n}{n}$ ).

The proof of the above theorem is presented in Section 2.

## 2 Proof Theorem 1.3

Given  $b_1, b_2, \dots, b_m \in \mathbb{N}$ , we define

$$f_n(b_1, b_2, \dots, b_m) = \#\{\pi = \pi_1\pi_2 \dots \pi_n \in \mathcal{F}_n \mid \pi_1\pi_2 \dots \pi_m = b_1b_2 \dots b_m\}.$$

It is natural to extend  $f_n$  to the case  $m = 0$  by setting  $f_n(\emptyset) = f_n = \#\mathcal{F}_n$ . The following properties of the numbers  $f_n(b_1, \dots, b_m)$  can be deduced easily from the definitions.

**Lemma 2.1.**

(1) Let  $m \geq 1$  and  $n - 2 \geq b_1 > b_2 > \dots > b_m \geq 1$ . Then, for all  $b_m + 1 \leq j \leq n - 2$ ,

$$f_n(b_1, \dots, b_m, j) = 0.$$

(2) Let  $m \geq 2$  and  $n - 2 \geq b_1 > b_2 > \dots > b_m \geq 1$ . Then, for all  $b_m + 1 \leq j \leq n - 1$ ,

$$f_n(b_1, \dots, b_m, j) = 0.$$

(3) Let  $m \geq 1$  and  $n - 2 \geq b_1 > b_2 > \dots > b_m \geq 1$ . Then

$$f_n(b_1, \dots, b_m, n) = f_{n-1}(b_1, \dots, b_m).$$

(4) Let  $m \geq 1$  and  $n - 2 \geq b_1 > b_2 > \dots > b_m \geq 1$ . Then

$$f_n(n, b_1, \dots, b_m) = f_n(n - 1, b_1, \dots, b_m) = f_{n-1}(b_1, \dots, b_m).$$

(5) Let  $m \geq 1$  and  $n - 1 \geq b_1 > b_2 > \dots > b_m \geq 1$ . Then

$$f_n(b_1, n, \dots, b_m) = f_{n-1}(b_1, \dots, b_m).$$

*Proof.* For (1), observe that if  $\pi \in S_n$  is such that  $\pi_1 \dots \pi_m \pi_{m+1} = b_1 \dots b_m j$ , then the entries  $b_m, j, n - 1, n$  give an occurrence of the pattern 1234 or 1243 in  $\pi$ .

For (2), observe that if  $\pi \in S_n$  is such that  $\pi_1 \dots \pi_m \pi_{m+1} = b_1 \dots b_m j$ , then either the entries  $b_{m-1}, b_m, n - 1, n$  give an occurrence of the pattern 2134 in  $\pi$  or the entries  $b_m, j, n - 1, n$  give an occurrence of 1234 or 1243 in  $\pi$ .

For (3), observe that if  $\pi \in S_n$  is such that  $\pi_1 \dots \pi_m \pi_{m+1} = b_1 \dots b_m n$ , where  $n - 2 \geq b_1 > \dots > b_m \geq 1$ , then no occurrence of the patterns 1234, 1243, 1324, 2134 in  $\pi$  can involve the entry  $\pi_{m+1} = n$ . Hence, there is a bijection between the set of permutations  $\pi \in \mathcal{F}_n$  with  $\pi_1 \dots \pi_m \pi_{m+1} = b_1 \dots b_m n$  and the set of permutations  $\sigma \in \mathcal{F}_{n-1}$  such that  $\sigma_1 \dots \sigma_m = b_1 \dots b_m$ .

Using similar arguments as in the proof of (3) we get that (4) and (5) hold. □

Next we introduce objects  $A_m(n), B_m(n)$  and  $C_m(n)$  which organize suitably the information about the numbers  $f_n(b_1, b_2, \dots, b_m)$  and play an important role in the proof of the main result.

**Definition 2.2.** For  $1 \leq m \leq n$  set

$$\begin{aligned} A_m(n) &= \sum_{n-2 \geq b_1 > b_2 > \dots > b_m \geq 1} f_n(b_1, b_2, \dots, b_m), \\ B_m(n) &= \sum_{n-2 \geq b_1 > b_2 > \dots > b_m \geq 1} f_n(b_1, n-1, b_2, b_3, \dots, b_m), \quad \text{and} \\ C_m(n) &= \sum_{n-1 \geq b_1 > b_2 > \dots > b_m \geq 1} f_n(b_1, b_2, \dots, b_m). \end{aligned}$$

As before, this definition is extended to

the case  $m = 0$  by setting  $A_0(n) = C_0(n) = f_n$  and  $B_0(n) = 0$ .

In the following two subsections we derive expressions for  $A_m(n)$  and  $B_1(n)$ , which are used in subsection 2.3 to complete the proof of Theorem 1.3.

### 2.1 A recurrence for the $f_j$ , also involving $B_1(n)$

In the following result we derive an expression for  $A_m(n)$ .

**Proposition 2.3.**

(1) For all  $n \geq 2$ ,

$$A_1(n) = f_n - 2f_{n-1}.$$

(2) For all  $n \geq 3$ ,

$$A_2(n) = f_n - 3f_{n-1} + f_{n-2} - B_1(n).$$

(3) For all  $2 \leq m \leq n - 2$ ,

$$A_m(n) = A_{m+1}(n) + A_m(n-1) + \dots + A_0(n-1-m).$$

*Proof.* For (1), Definition 2.2 for  $A_1(n)$  gives that

$$A_1(n) = \sum_{b_1=1}^{n-2} f_n(b_1) = f_n - f_n(n-1) - f_n(n).$$

Observe that if  $\pi \in S_n$  is such that  $\pi_1 = n-1$  or  $\pi_1 = n$ , then no occurrence of the patterns 1234, 1243, 1324, 2134 in  $\pi$  can involve the entry  $\pi_1$ . So we get that the number of permutations in  $\mathcal{F}_n$  starting with  $n$  (resp.,  $n-1$ ) is  $f_{n-1}$ . Hence,  $A_1(n) = f_n - 2f_{n-1}$ , as claimed in (1).

For (2), Definition 2.2 for  $A_1(n)$  and  $A_2(n)$  gives that

$$A_1(n) = A_2(n) + \sum_{b_1=1}^{n-2} \sum_{b_2=b_1+1}^n f_n(b_1, b_2).$$

Using Lemma 2.1, parts (1) and (5), and Definition 2.2 we obtain that

$$A_1(n) = A_2(n) + B_1(n) + A_1(n-1) + f_{n-1}(n-2).$$

Hence, by using the proof of (1) and Definition 2.2 we get the desired result.

For (3), let  $2 \leq m \leq n-2$ . Definition 2.2 yields

$$A_m(n) = A_{m+1}(n) + \sum_{n-2 \geq b_1 > b_2 > \dots > b_m \geq 1} \sum_{j=b_m+1}^n f_n(b_1, \dots, b_m, j).$$

Using Lemma 2.1, parts (2) and (3), we have

$$\begin{aligned} A_m(n) &= A_{m+1}(n) + \sum_{n-2 \geq b_1 > b_2 > \dots > b_m \geq 1} f_n(b_1, b_2, \dots, b_m, n) \\ &= A_{m+1}(n) + \sum_{n-2 \geq b_1 > b_2 > \dots > b_m \geq 1} f_{n-1}(b_1, b_2, \dots, b_m) \\ &= A_{m+1}(n) + C_m(n-1). \end{aligned}$$

Definition 2.2 and Lemma 2.1 (4) give

$$\begin{aligned} C_m(n) &= A_m(n) + \sum_{n-2 \geq b_2 > \dots > b_m \geq 1} f_n(n-1, b_2, \dots, b_m) \\ &= A_m(n) + \sum_{n-2 \geq b_2 > \dots > b_m \geq 1} f_{n-1}(b_2, \dots, b_m) \\ &= A_m(n) + C_{m-1}(n-1). \end{aligned} \tag{2.1}$$

Hence, by induction on  $m$  together with (1) we get the desired result. □

We next find an explicit expression for  $A_m(n)$  in terms of  $A_0(n) = f_n$ ,  $A_1(n)$  and  $A_2(n)$ .

**Theorem 2.4.** For all  $n \geq 5$  and  $2 \leq m \leq n-2$ ,

$$A_m(n) = \sum_{j \geq 0} (-1)^j \binom{m-1-j}{j} A_2(n-j) - \sum_{j \geq 0} (-1)^j \binom{m-3-j}{j} (A_1(n-2-j) + A_0(n-3-j)),$$

with the usual convention that  $\binom{a}{b} = 0$  if  $a < b$  or  $a < 0$ .

*Proof.* For  $m = 2$  we have that  $A_2(n) = A_2(n)$ , so the theorem holds. Assume the theorem for  $m$  and all appropriate  $n$ , and let us prove the equality for  $m + 1$ . Using Proposition 2.3(3) we get that

$$A_{m+1}(n) = A_m(n) - \sum_{i=0}^m A_{m-i}(n-1-i),$$

and by the induction hypothesis, we arrive at

$$\begin{aligned} A_{m+1}(n) &= \sum_{j \geq 0} (-1)^j \binom{m-1-j}{j} A_2(n-j) - \sum_{j \geq 0} (-1)^j \binom{m-3-j}{j} (A_1(n-2-j) + A_0(n-3-j)) \\ &\quad - \sum_{i=0}^m \sum_{j \geq 0} (-1)^j \binom{m-1-i-j}{j} A_2(n-1-i-j) \\ &\quad + \sum_{i=0}^m \sum_{j \geq 0} (-1)^j \binom{m-3-i-j}{j} (A_1(n-3-i-j) + A_0(n-4-i-j)) \\ &= \sum_{j \geq 0} (-1)^j \binom{m-1-j}{j} A_2(n-j) - \sum_{j \geq 0} (-1)^j \binom{m-3-j}{j} (A_1(n-2-j) + A_0(n-3-j)) \\ &\quad - \sum_{j \geq 0} \left( \sum_{i=0}^j (-1)^i \binom{m-1-j}{i} A_2(n-1-j) - \sum_{i=0}^j (-1)^i \binom{m-3-j}{i} (A_1(n-3-j) + A_0(n-4-j)) \right). \end{aligned}$$

Using the familiar identity  $\binom{p}{0} - \binom{p}{1} + \cdots + (-1)^q \binom{p}{q} = (-1)^q \binom{p-1}{q}$  we obtain that

$$\begin{aligned} A_{m+1}(n) &= \sum_{j \geq 0} (-1)^j \binom{m-1-j}{j} A_2(n-j) - \sum_{j \geq 0} (-1)^j \binom{m-3-j}{j} (A_1(n-2-j) + A_0(n-3-j)) \\ &\quad - \sum_{j \geq 0} (-1)^j \binom{m-2-j}{j} A_2(n-1-j) + \sum_{j \geq 0} (-1)^j \binom{m-4-j}{j} (A_1(n-3-j) + A_0(n-4-j)), \end{aligned}$$

and by using the identity  $\binom{p}{q} = \binom{p-1}{q} + \binom{p-1}{q-1}$  we get that

$$A_{m+1}(n) = \sum_{j \geq 0} (-1)^j \binom{m-j}{j} A_2(n-j) - \sum_{j \geq 0} (-1)^j \binom{m-2-j}{j} (A_1(n-2-j) + A_0(n-3-j)).$$

Hence, by induction on  $m$  we get the desired result.  $\square$

Using Theorem 2.4 for  $m = n - 2$ , together with Proposition 2.3, parts (1) and (2), and  $A_{n-2}(n) = 1$  (see Definition 2.2) we get the main result of this subsection.

**Theorem 2.5.** For all  $n \geq 5$ ,

$$\sum_{j \geq 0} (-1)^j \binom{n-3-j}{j} (f_{n-j} - 3f_{n-1-j} + f_{n-2-j} - B_1(n-j)) = 1 + \sum_{j \geq 0} (-1)^j \binom{n-5-j}{j} (f_{n-2-j} - f_{n-3-j}).$$

## 2.2 A recursive formula for $B_1(n)$

We next we find a recurrence for  $B_1(n)$  in terms of  $f_n$ .

**Proposition 2.6.** We have

$B_1(n) = C_0(n-2) + C_1(n-2) + \cdots + C_{n-2}(n-2)$  for all  $n \geq 3$ .

$B_1(n) - B_1(n-1) = A_0(n-2) + A_1(n-2) + \cdots + A_{n-4}(n-2)$  for all  $n \geq 4$ .

*Proof.* For (2.6), by Definition 2.2 we get that

$$B_1(n) = f_n(n-2, n-1) + \sum_{n-3 \geq b_1 \geq 1} f_n(b_1, n-1).$$

Observe that if a permutation  $\pi \in S_n$  is such that  $\pi_1 = n-2$  and  $\pi_2 = n-1$ , then no occurrence of the patterns 1234, 1243, 1324, 2134 can involve either the entry  $n-1$  or the entry  $n$ . Thus,  $f_n(n-2, n-1) = f_{n-2} = C_0(n-2)$ , and for all  $n \geq 3$ ,  $B_1(n) = C_0(n-2) + L_1(n)$ , where we define

$$L_m(n) = \sum_{n-3 \geq b_1 > \cdots > b_m \geq 1} f_n(b_1, n-1, b_2, \dots, b_m) \quad \text{for } m \geq 1.$$

Using Lemma 2.1, parts (1) and (2), we get that

$$L_m(n) = L_{m+1}(n) + \sum_{n-3 \geq b_1 > \cdots > b_m \geq 1} f_n(b_1, n-1, b_2, \dots, b_m, n),$$

and by Lemma 2.1, parts (3) and (5), together with Definition 2.2 we arrive at

$$L_m(n) = L_{m+1}(n) + C_m(n-2)$$

for all  $m \geq 1$ . Hence, by induction on  $m$  together with the fact that  $L_{n-1}(n) = 0$  we have

$$B_1(n) = C_0(n-2) + C_1(n-2) + \cdots + C_{n-2}(n-2),$$

as claimed.

For (2.6), using Equation (2.1) together with (2.6) and  $C_n(n) = A_{n-1}(n) = 0$  (see Definition 2.2) we get the desired result.  $\square$

**Theorem 2.7.** For all  $n \geq 4$ ,

$$\begin{aligned} B_1(n) - B_1(n-2) &= f_{n-1} - 2f_{n-2} + f_{n-3} \\ &\quad - \sum_{j \geq 0} (-1)^j \binom{n-3-j}{j} A_2(n-1-j) + \sum_{j \geq 0} (-1)^j \binom{n-5-j}{j} (A_1(n-3-j) + A_0(n-4-j)). \end{aligned}$$

*Proof.* It is easy to check that the theorem holds for  $n = 4, 5, 6$ . Now, let  $n \geq 7$ . By using Proposition 2.6 (2.6) and Theorem 2.4 we get that

$$\begin{aligned} B_1(n) - B_1(n-1) &= A_0(n-2) + A_1(n-2) + \sum_{m=2}^{n-4} \sum_{j \geq 0} (-1)^j \binom{m-1-j}{j} A_2(n-2-j) \\ &\quad - \sum_{m=3}^{n-4} \sum_{j \geq 0} (-1)^j \binom{m-3-j}{j} (A_1(n-4-j) + A_0(n-5-j)) \\ &= A_0(n-2) + A_1(n-2) - A_2(n-2) + \sum_{m=1}^{n-4} \sum_{j \geq 0} (-1)^j \binom{m-1-j}{j} A_2(n-2-j) \\ &\quad - \sum_{m=3}^{n-4} \sum_{j \geq 0} (-1)^j \binom{m-3-j}{j} (A_1(n-4-j) + A_0(n-5-j)) \\ &= A_0(n-2) + A_1(n-2) - A_2(n-2) + \sum_{j \geq 0} \sum_{i=j}^{n-5-j} (-1)^j \binom{i}{j} A_2(n-2-j) \\ &\quad - \sum_{j \geq 0} \sum_{i=j}^{n-7-j} (-1)^j \binom{i}{j} (A_1(n-4-j) + A_0(n-5-j)). \end{aligned}$$

Therefore, using the identity  $\binom{p}{p} + \binom{p+1}{p} + \dots + \binom{q}{p} = \binom{q+1}{p+1}$  gives that

$$\begin{aligned} B_1(n) - B_1(n-1) &= A_0(n-2) + A_1(n-2) - A_2(n-2) + A_2(n-1) - A_1(n-3) - A_0(n-4) \\ &\quad - \sum_{j \geq 0} (-1)^j \binom{n-3-j}{j} A_2(n-1-j) + \sum_{j \geq 0} (-1)^j \binom{n-5-j}{j} (A_1(n-3-j) + A_0(n-4-j)). \end{aligned}$$

Hence, using Proposition 2.3, parts (1) and (2), we obtain the desired identity.  $\square$

### 2.3 Proof of Theorem 1.3

We start by showing the following result.

**Lemma 2.8.** *Let  $t(x)$  be the generating function for the sequence  $(t_n)_{n \geq 0}$ , that is,  $t(x) = \sum_{n \geq 0} t_n x^n$ . Then*

$$\sum_{n \geq m} \left( x^n \sum_{j \geq 0} (-1)^j \binom{n-m-j}{j} t_{n-s-j} \right) = \frac{x^s}{(1-x)^{m-s}} \left( t(x(1-x)) - \sum_{j=0}^{m-s-1} t_j x^j (1-x)^j \right).$$

*Proof.* We have

$$\begin{aligned} \sum_{n \geq m} \left( x^n \sum_{j \geq 0} (-1)^j \binom{n-m-j}{j} t_{n-s-j} \right) &= \sum_{n \geq 0} \sum_{j=0}^n (-1)^j \binom{n}{j} x^{n+m+j} t_{n+m-s} \\ &= \sum_{n \geq 0} t_{n+m-s} x^{n+m} (1-x)^n = \frac{x^s}{(1-x)^{m-s}} \left( t(x(1-x)) - \sum_{j=0}^{m-s-1} t_j x^j (1-x)^j \right), \end{aligned}$$

as claimed.  $\square$

Now we are ready to prove the main result of this paper, namely Theorem 1.3, which is restated here for easy reference.

**Theorem 1.3.** The generating function for the number of freely-braided permutations in  $\mathcal{F}_n$  is given by

$$\frac{1 - 3x - 2x^2 + (1+x)\sqrt{1-4x}}{1 - 4x - x^2 + (1-x^2)\sqrt{1-4x}}.$$

*Proof.* We denote the generating function for the number of freely-braided permutations in  $\mathcal{F}_n$  by  $F(x)$ , that is,  $F(x) = \sum_{n \geq 0} f_n x^n$ . Also, we denote the generating function for the sequence  $\{B_1(n)\}_{n \geq 0}$  by  $B(x)$ , that is,  $B(x) = \sum_{n \geq 0} B_1(n) x^n$ .

Theorem 2.5 gives

$$\sum_{j \geq 0} (-1)^j \binom{n-3-j}{j} (f_{n-j} - 3f_{n-1-j} + f_{n-2-j} - B_1(n-j)) = 1 + \sum_{j \geq 0} (-1)^j \binom{n-5-j}{j} (f_{n-2-j} - f_{n-3-j}),$$

for all  $n \geq 5$ . Multiplying by  $x^n$  and summing over all  $n \geq 5$  together with using Lemma 2.8 we arrive at



$$\begin{aligned}
& -x^4 + \frac{1}{(1-x)^3} \left( (1 - 3x(1-x) + x^2(1-x)^2)F(x(1-x)) - 1 + 2x(1-x) - B(x(1-x)) \right) \\
& = \frac{x^5}{1-x} + \frac{x^2}{(1-x)^3} (F(x(1-x)) - 1 - x(1-x) - 2x^2(1-x)^2) + \frac{x^2}{(1-x)^2} (F(x(1-x)) - 1 - x(1-x)),
\end{aligned}$$

or equivalently,

$$F(x(1-x)) - \frac{1}{(1-x)^3} B(x(1-x)) = \frac{1}{1-x}. \quad (2.2)$$

Theorem 2.7 gives

$$\begin{aligned}
B_1(n) - B_1(n-2) &= f_{n-1} - 2f_{n-2} + f_{n-3} \\
& - \sum_{j \geq 0} (-1)^j \binom{n-3-j}{j} A_2(n-1-j) + \sum_{j \geq 0} (-1)^j \binom{n-5-j}{j} (A_1(n-3-j) + A_0(n-4-j)),
\end{aligned}$$

for all  $n \geq 4$ . Multiplying by  $x^n$  and summing over all  $n \geq 4$  together with using Lemma 2.8 we arrive at

$$\begin{aligned}
(1-x^2)B(x) - x^3 &= \frac{x}{(1-x)^2} F(x) - x + x^2(1-x) \\
& - \frac{x}{(1-x)^2} \left( (1 - 3x(1-x) + x^2(1-x)^2)F(x(1-x)) - 1 + 2x(1-x) - B(x(1-x)) \right) \\
& + \frac{x^3}{(1-x)^2} (F(x(1-x)) - 1 - x(1-x)) - \frac{x^4}{1-x} (F(x(1-x)) - 1),
\end{aligned}$$

or equivalently,

$$(1-x^2)B(x) = x^2 - x(1-x)F(x(1-x)) + x(1-x)^2 F(x) + \frac{x}{(1-x)^2} B(x(1-x)). \quad (2.3)$$

Using Equations 2.2 and 2.3 we get that

$$\begin{cases} B(x(1-x)) = (1-x)^3 F(x(1-x)) - (1-x)^2 \\ (1+x)B(x) = -x + x(1-x)F(x), \end{cases}$$

or equivalently,

$$\begin{cases} B(x) &= \left(1 - \frac{1-\sqrt{1-4x}}{2}\right)^3 F(x) - \left(1 - \frac{1-\sqrt{1-4x}}{2}\right)^2 \\ (1+x)B(x) &= -x + x(1-x)F(x) \end{cases}.$$

The rest is easy to check.  $\square$

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