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Improved Expansion of Random Cayley Graphs

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Alon and Roichman (1994) proved that for every $\epsilon > 0$ there is a finite $c(\epsilon)$ such that for any sufficiently large group G , the expected value of the second largest (in absolute value) eigenvalue of the normalized adjacency matrix of the Cayley graph with respect to $c(\epsilon)\log|G|$ random elements is less than ϵ . We reduce the number of elements to $c(\epsilon)\log D(G)$ (for the same c), where $D(G)$ is the sum of the dimensions of the irreducible representations of G . In sufficiently non-abelian families of groups (as measured by these dimensions), $\log D(G)$ is asymptotically $(1/2)\log|G|$. As is well known, a small eigenvalue implies large graph expansion (and conversely); see Tanner (1984) and Alon and Milman (1984, 1985). For any specified eigenvalue or expansion, therefore, random Cayley graphs (of sufficiently non-abelian groups) require only half as many edges as was previously known.

Keywords: expander graphs, Cayley graphs, second eigenvalue, logarithmic generators

1 Introduction

All groups considered in this paper are finite.

Definition 1 Let G be a group, and $S \subset G$ be a multiset. The **Cayley graph** $X(G, S)$ is the multigraph on vertex set G , with n undirected edges connecting g and tg if t appears n times in the multiset union $S \sqcup S^{-1}$, where S^{-1} is the multiset $\{s^{-1} : s \in S\}$. The **normalized adjacency matrix** $A_{X(G, S)}^*$ is $1/(2|S|)$ times the adjacency matrix of $X(G, S)$.

Definition 2 Let M be an $n \times n$ matrix with real eigenvalues x_1, \dots, x_n , where $|x_1| \geq \dots \geq |x_n|$. Define $\lambda(M) = |x_1|$ and $\mu(M) = |x_2|$. Write $\mu(X(G, S))$ for $\mu(A_{X(G, S)}^*)$.

Definition 3 Let $D(G)$ be the sum of the dimensions of the irreducible representations of G .

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Observe that $|G|^{1/2} < D(G) \leq |G|$. The upper bound is met only by abelian groups but is approached also by other groups whose irreducible representations are mostly low-dimensional, such as dihedral groups. The lower bound is approached, in the sense that $\log D(G) \rightarrow (1/2) \log |G|$, by a variety of families of groups possessing mostly high-dimensional irreducible representations.

Examples:

- (a) The affine group A_p over the prime field $GF(p)$. $|A_p| = p(p-1)$, while $D(A_p) = 2p-2$.
- (b) The symmetric group S_n . $|S_n| = n!$, hence $\log |S_n| \in n \log n - O(n)$, while $D(S_n) \in e^{O(\sqrt{n})} \sqrt{n!}$, hence $\log D(S_n) \in (1/2)n \log n + O(\sqrt{n})$.

(For the upper bound on $D(S_n)$, take the number of irreducible representations of S_n times the maximum of their dimensions. The first of these is $p(n)$, the number of partitions of n , which has the asymptotic behavior $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$. The second was shown by Vershik and Kerov (1985) to be bounded above by $e^{-k\sqrt{n}} \sqrt{n!}$ for a positive constant k .)

Theorem 1 *For any $\varepsilon > 0$ the following holds for every sufficiently large group G . Let S be a multiset of $c(\varepsilon) \log D(G)$ uniformly and independently sampled elements of G , for $c(\varepsilon) = 4e/\varepsilon^2$. Then we have $\mathbb{E}[\mu(X(G, S))] < (1 + o(1))\varepsilon$.*

(Here and throughout $o(1)$ allows for a quantity tending to 0 for large $|G|$.) Russell and Landau (2004) have independently obtained a similar result.

As a detail note that in Alon and Roichman (1994), S is generated by sampling without repetition (i.e., S is a set), while we employ sampling with repetition. The principal benefit of this is to simplify the argument, but it also leads to some sharpening: the value of $c(\varepsilon)$ obtained in Alon and Roichman (1994) is slightly larger than given here, while substituting sampling with repetition into their argument leads to the same $c(\varepsilon)$.

2 Proof

The combinatorial outline of the proof follows that of Alon and Roichman; the heart of the improvement lies in taking a certain union bound over the irreducible representations, rather than over the entire regular representation, of the group.

2.1 Decomposition into irreducible representations

Fix a group G , and let S be a multiset of N elements of G . Let $T = S \sqcup S^{-1}$; let α be the element in the group algebra $\mathbb{C}[G]$ defined by:

$$\alpha = \sum_{t \in T} \frac{1}{|T|} t.$$

Let the operator L be the left-action of α on $\mathbb{C}[G]$. Its matrix representation with respect to the standard basis is the normalized adjacency matrix of $X(G, S)$. The Fourier Transform \mathcal{F} is an algebra isomorphism from $\mathbb{C}[G]$ to $\bigoplus_{r=1}^R \mathcal{M}_r$, where R is the number of irreducible representations of G , and $\mathcal{M}_r = \text{Mat}_{d_r \times d_r}(\mathbb{C})$. Hence the eigenvalues of L are the same as the eigenvalues of the left-action of $\mathcal{F}(\alpha)$ on $\bigoplus \mathcal{M}_r$. Explicitly,

$$\mathcal{F}(\alpha) = \bigoplus_{r=1}^R \left(\sum_{t \in T} \frac{1}{|T|} \rho_r(t) \right),$$

where $\rho_r : G \rightarrow \mathcal{M}_r$ are the (unitary) irreducible representations, expressed with respect to fixed bases. Focus on an arbitrary component r of $\mathcal{F}(\alpha)$: let $\Psi_r = (1/|T|) \sum_{t \in T} \rho_r(t)$.

Since Ψ_r is an average of unitary matrices, its eigenvalues are bounded in absolute value by 1.

Let ρ_1 be the one-dimensional trivial representation $\rho_1 : G \mapsto \mathbb{C}$. Then for any S , $\Psi_1 = 1$. Therefore, $\mu(X(G, S)) = \lambda(A)$, where A is the following block-diagonal matrix:

$$A = \begin{pmatrix} \Psi_2 & 0 & \dots & 0 \\ 0 & \Psi_3 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \Psi_R \end{pmatrix}.$$

2.2 From eigenvalues to random walks

Fact 1 *Let M be a square matrix with real eigenvalues. Then for every positive integer m ,*

$$\lambda(M) \leq (\text{Tr}(M^{2m}))^{1/2m}.$$

Because of the symmetric construction of T , A is Hermitian. By convexity,

$$E[\mu(X(G, S))] \leq (E[\text{Tr}(A^{2m})])^{1/2m}.$$

Since A is block-diagonal, A^{2m} shares the same block structure, with blocks Ψ_i^{2m} ($2 \leq i \leq R$).

$$\begin{aligned} \text{Tr}(A^{2m}) &= \sum_{r=2}^R \text{Tr}(\Psi_r^{2m}) \\ &= \sum_{r=2}^R \left(\sum_{t_1, \dots, t_{2m} \in T} \frac{\chi_r(t_1 \cdots t_{2m})}{|T|^{2m}} \right) \\ &= \sum_{r=2}^R \sum_{g \in G} \chi_r(g) \frac{N_g}{|T|^{2m}}, \end{aligned}$$

where χ_r is the character of ρ_r and N_g is the number of ways to produce g as a product of $2m$ (not necessarily distinct) elements of T .

Definition 4 *Let \mathbf{RW} denote the following random walk process.*

- (1) *Choose a uniform random word of length $2m$ from the free monoid on the N letters $\{a_1, a_2, \dots, a_N\}$ (e.g., $a_2 a_5 a_5^{-1} a_1^{-1} a_7 a_3$).*
- (2) *Reduce the word in the free group (e.g., $a_2 a_5 a_5^{-1} a_1^{-1} a_7 a_3 \rightarrow a_2 a_1^{-1} a_7 a_3$).*

- (3) Uniformly and independently assign (not necessarily distinct) group elements to the letters that appear in the remaining word, and evaluate the product in G .

Let \mathbf{RW}_g be the event that the result is g . $\Pr(\mathbf{RW}_g) = N_g/|T|^{2m}$, so

$$\mathbb{E}[\mathrm{Tr}(A^{2m})] = \sum_{g \in G} \Pr(\mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g).$$

2.3 Mixing in the random walk

Definition 5 Let ω be a reduced word as obtained via step (2) of process \mathbf{RW} (definition 4). Say that ω has a **singleton** if there is an i such that the number of occurrences of a_i in ω plus the number of occurrences of a_i^{-1} in ω is exactly one.

Let Ω be the event that the reduced word has a singleton. Now:

$$\begin{aligned} & \sum_{g \in G} \Pr(\mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) \\ = & \sum_{g \in G} \Pr(\Omega \wedge \mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) + \sum_{g \in G} \Pr(\bar{\Omega} \wedge \mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) \\ \leq & \sum_{g \in G} \Pr(\Omega \wedge \mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) + \Pr(\bar{\Omega}) D(G). \end{aligned} \tag{1}$$

Lemma 1 $\Pr(\mathbf{RW}_g | \Omega) = 1/|G|$.

Proof: In step (3) of \mathbf{RW} (definition 4), assign the singleton element last; then, there will exist a unique group element that makes ω evaluate to g . \square

Comment: This lemma replaces an upper bound of $1/|G| + O(m/G^2)$ in Alon and Roichman (1994), the additional term being the result of their requiring distinct assignments in step (3). This additional term leads in turn to an extra summand of e^{-b} in the analogue, in their work, of the center expression in Inequality (2).

By Lemma 1 and the orthogonality of characters, the first term of Bound (1) vanishes. Combining our inequalities:

$$\mathbb{E}[\mu(X(G, S))] \leq (\mathbb{E}[\mathrm{Tr}(A^{2m})])^{1/2m} \leq \Pr(\bar{\Omega})^{1/2m} D(G)^{1/2m}.$$

To bound $\Pr(\bar{\Omega})$, we follow the spirit of Alon and Roichman (1994) and define the following two events in terms of the quantity $M = 2m(1 - \log \log 2m / \log 2m)$:

- (A) After step (2) of \mathbf{RW} (definition 4), the length of the reduced word is less than M .
- (B) After step (2) of \mathbf{RW} (definition 4), the length of the reduced word is at least M , but there are no singletons.

Clearly, $\Pr(\overline{\Omega}) \leq \Pr(A) + \Pr(B)$. Alon and Roichman (1994) produced these bounds:

$$\begin{aligned}\Pr(A) &\leq 2^{2m} (2/N)^{m \log \log 2m / \log 2m} \\ \Pr(B) &\leq 2^M (m/N)^{M/2}.\end{aligned}$$

Substituting $N = c(\varepsilon) \log D(G)$ and $2m = (1/b) \log D(G)$, for any constant b , we obtain an expression almost identical to one of Alon and Roichman (1994), except that $|G|$'s are replaced by $D(G)$'s:

$$\Pr(\overline{\Omega})^{1/2m} D(G)^{1/2m} \leq (1 + o(1)) e^b \sqrt{\frac{2}{bc(\varepsilon)}} \leq (1 + o(1)) \varepsilon \quad (2)$$

where we use the choices $c(\varepsilon) = 4e/\varepsilon^2$ and $b = 1/2$. □

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