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# Improved Expansion of Random Cayley Graphs

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Alon and Roichman (1994) proved that for every  $\epsilon > 0$  there is a finite  $c(\epsilon)$  such that for any sufficiently large group  $G$ , the expected value of the second largest (in absolute value) eigenvalue of the normalized adjacency matrix of the Cayley graph with respect to  $c(\epsilon)\log|G|$  random elements is less than  $\epsilon$ . We reduce the number of elements to  $c(\epsilon)\log D(G)$  (for the same  $c$ ), where  $D(G)$  is the sum of the dimensions of the irreducible representations of  $G$ . In sufficiently non-abelian families of groups (as measured by these dimensions),  $\log D(G)$  is asymptotically  $(1/2)\log|G|$ . As is well known, a small eigenvalue implies large graph expansion (and conversely); see Tanner (1984) and Alon and Milman (1984, 1985). For any specified eigenvalue or expansion, therefore, random Cayley graphs (of sufficiently non-abelian groups) require only half as many edges as was previously known.

**Keywords:** expander graphs, Cayley graphs, second eigenvalue, logarithmic generators

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## 1 Introduction

All groups considered in this paper are finite.

**Definition 1** Let  $G$  be a group, and  $S \subset G$  be a multiset. The **Cayley graph**  $X(G, S)$  is the multigraph on vertex set  $G$ , with  $n$  undirected edges connecting  $g$  and  $tg$  if  $t$  appears  $n$  times in the multiset union  $S \sqcup S^{-1}$ , where  $S^{-1}$  is the multiset  $\{s^{-1} : s \in S\}$ . The **normalized adjacency matrix**  $A_{X(G, S)}^*$  is  $1/(2|S|)$  times the adjacency matrix of  $X(G, S)$ .

**Definition 2** Let  $M$  be an  $n \times n$  matrix with real eigenvalues  $x_1, \dots, x_n$ , where  $|x_1| \geq \dots \geq |x_n|$ . Define  $\lambda(M) = |x_1|$  and  $\mu(M) = |x_2|$ . Write  $\mu(X(G, S))$  for  $\mu(A_{X(G, S)}^*)$ .

**Definition 3** Let  $D(G)$  be the sum of the dimensions of the irreducible representations of  $G$ .

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Observe that  $|G|^{1/2} < D(G) \leq |G|$ . The upper bound is met only by abelian groups but is approached also by other groups whose irreducible representations are mostly low-dimensional, such as dihedral groups. The lower bound is approached, in the sense that  $\log D(G) \rightarrow (1/2) \log |G|$ , by a variety of families of groups possessing mostly high-dimensional irreducible representations.

Examples:

- (a) The affine group  $A_p$  over the prime field  $GF(p)$ .  $|A_p| = p(p-1)$ , while  $D(A_p) = 2p-2$ .
- (b) The symmetric group  $S_n$ .  $|S_n| = n!$ , hence  $\log |S_n| \in n \log n - O(n)$ , while  $D(S_n) \in e^{O(\sqrt{n})} \sqrt{n!}$ , hence  $\log D(S_n) \in (1/2)n \log n + O(\sqrt{n})$ .

(For the upper bound on  $D(S_n)$ , take the number of irreducible representations of  $S_n$  times the maximum of their dimensions. The first of these is  $p(n)$ , the number of partitions of  $n$ , which has the asymptotic behavior  $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}$ . The second was shown by Vershik and Kerov (1985) to be bounded above by  $e^{-k\sqrt{n}} \sqrt{n!}$  for a positive constant  $k$ .)

**Theorem 1** *For any  $\varepsilon > 0$  the following holds for every sufficiently large group  $G$ . Let  $S$  be a multiset of  $c(\varepsilon) \log D(G)$  uniformly and independently sampled elements of  $G$ , for  $c(\varepsilon) = 4e/\varepsilon^2$ . Then we have  $\mathbb{E}[\mu(X(G, S))] < (1 + o(1))\varepsilon$ .*

(Here and throughout  $o(1)$  allows for a quantity tending to 0 for large  $|G|$ .) Russell and Landau (2004) have independently obtained a similar result.

As a detail note that in Alon and Roichman (1994),  $S$  is generated by sampling without repetition (i.e.,  $S$  is a set), while we employ sampling with repetition. The principal benefit of this is to simplify the argument, but it also leads to some sharpening: the value of  $c(\varepsilon)$  obtained in Alon and Roichman (1994) is slightly larger than given here, while substituting sampling with repetition into their argument leads to the same  $c(\varepsilon)$ .

## 2 Proof

The combinatorial outline of the proof follows that of Alon and Roichman; the heart of the improvement lies in taking a certain union bound over the irreducible representations, rather than over the entire regular representation, of the group.

### 2.1 Decomposition into irreducible representations

Fix a group  $G$ , and let  $S$  be a multiset of  $N$  elements of  $G$ . Let  $T = S \sqcup S^{-1}$ ; let  $\alpha$  be the element in the group algebra  $\mathbb{C}[G]$  defined by:

$$\alpha = \sum_{t \in T} \frac{1}{|T|} t.$$

Let the operator  $L$  be the left-action of  $\alpha$  on  $\mathbb{C}[G]$ . Its matrix representation with respect to the standard basis is the normalized adjacency matrix of  $X(G, S)$ . The Fourier Transform  $\mathcal{F}$  is an algebra isomorphism from  $\mathbb{C}[G]$  to  $\bigoplus_{r=1}^R \mathcal{M}_r$ , where  $R$  is the number of irreducible representations of  $G$ , and  $\mathcal{M}_r = \text{Mat}_{d_r \times d_r}(\mathbb{C})$ . Hence the eigenvalues of  $L$  are the same as the eigenvalues of the left-action of  $\mathcal{F}(\alpha)$  on  $\bigoplus \mathcal{M}_r$ . Explicitly,

$$\mathcal{F}(\alpha) = \bigoplus_{r=1}^R \left( \sum_{t \in T} \frac{1}{|T|} \rho_r(t) \right),$$

where  $\rho_r : G \rightarrow \mathcal{M}_r$  are the (unitary) irreducible representations, expressed with respect to fixed bases. Focus on an arbitrary component  $r$  of  $\mathcal{F}(\alpha)$ : let  $\Psi_r = (1/|T|) \sum_{t \in T} \rho_r(t)$ .

Since  $\Psi_r$  is an average of unitary matrices, its eigenvalues are bounded in absolute value by 1.

Let  $\rho_1$  be the one-dimensional trivial representation  $\rho_1 : G \mapsto \mathbb{C}$ . Then for any  $S$ ,  $\Psi_1 = 1$ . Therefore,  $\mu(X(G, S)) = \lambda(A)$ , where  $A$  is the following block-diagonal matrix:

$$A = \begin{pmatrix} \Psi_2 & 0 & \dots & 0 \\ 0 & \Psi_3 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \Psi_R \end{pmatrix}.$$

### 2.2 From eigenvalues to random walks

**Fact 1** Let  $M$  be a square matrix with real eigenvalues. Then for every positive integer  $m$ ,

$$\lambda(M) \leq (\text{Tr}(M^{2m}))^{1/2m}.$$

Because of the symmetric construction of  $T$ ,  $A$  is Hermitian. By convexity,

$$E[\mu(X(G, S))] \leq (E[\text{Tr}(A^{2m})])^{1/2m}.$$

Since  $A$  is block-diagonal,  $A^{2m}$  shares the same block structure, with blocks  $\Psi_i^{2m}$  ( $2 \leq i \leq R$ ).

$$\begin{aligned} \text{Tr}(A^{2m}) &= \sum_{r=2}^R \text{Tr}(\Psi_r^{2m}) \\ &= \sum_{r=2}^R \left( \sum_{t_1, \dots, t_{2m} \in T} \frac{\chi_r(t_1 \cdots t_{2m})}{|T|^{2m}} \right) \\ &= \sum_{r=2}^R \sum_{g \in G} \chi_r(g) \frac{N_g}{|T|^{2m}}, \end{aligned}$$

where  $\chi_r$  is the character of  $\rho_r$  and  $N_g$  is the number of ways to produce  $g$  as a product of  $2m$  (not necessarily distinct) elements of  $T$ .

**Definition 4** Let  $RW$  denote the following random walk process.

- (1) Choose a uniform random word of length  $2m$  from the free monoid on the  $N$  letters  $\{a_1, a_2, \dots, a_N\}$  (e.g.,  $a_2 a_5 a_5^{-1} a_1^{-1} a_7 a_3$ ).
- (2) Reduce the word in the free group (e.g.,  $a_2 a_5 a_5^{-1} a_1^{-1} a_7 a_3 \rightarrow a_2 a_1^{-1} a_7 a_3$ ).

- (3) Uniformly and independently assign (not necessarily distinct) group elements to the letters that appear in the remaining word, and evaluate the product in  $G$ .

Let  $\mathbf{RW}_g$  be the event that the result is  $g$ .  $\Pr(\mathbf{RW}_g) = N_g/|T|^{2m}$ , so

$$\mathbb{E}[\mathrm{Tr}(A^{2m})] = \sum_{g \in G} \Pr(\mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g).$$

### 2.3 Mixing in the random walk

**Definition 5** Let  $\omega$  be a reduced word as obtained via step (2) of process  $\mathbf{RW}$  (definition 4). Say that  $\omega$  has a **singleton** if there is an  $i$  such that the number of occurrences of  $a_i$  in  $\omega$  plus the number of occurrences of  $a_i^{-1}$  in  $\omega$  is exactly one.

Let  $\Omega$  be the event that the reduced word has a singleton. Now:

$$\begin{aligned} & \sum_{g \in G} \Pr(\mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) \\ = & \sum_{g \in G} \Pr(\Omega \wedge \mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) + \sum_{g \in G} \Pr(\overline{\Omega} \wedge \mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) \\ \leq & \sum_{g \in G} \Pr(\Omega \wedge \mathbf{RW}_g) \sum_{r=2}^R \mathrm{Re} \chi_r(g) + \Pr(\overline{\Omega}) D(G). \end{aligned} \tag{1}$$

**Lemma 1**  $\Pr(\mathbf{RW}_g | \Omega) = 1/|G|$ .

**Proof:** In step (3) of  $\mathbf{RW}$  (definition 4), assign the singleton element last; then, there will exist a unique group element that makes  $\omega$  evaluate to  $g$ .  $\square$

Comment: This lemma replaces an upper bound of  $1/|G| + O(m/G^2)$  in Alon and Roichman (1994), the additional term being the result of their requiring distinct assignments in step (3). This additional term leads in turn to an extra summand of  $e^{-b}$  in the analogue, in their work, of the center expression in Inequality (2).

By Lemma 1 and the orthogonality of characters, the first term of Bound (1) vanishes. Combining our inequalities:

$$\mathbb{E}[\mu(X(G, S))] \leq (\mathbb{E}[\mathrm{Tr}(A^{2m})])^{1/2m} \leq \Pr(\overline{\Omega})^{1/2m} D(G)^{1/2m}.$$

To bound  $\Pr(\overline{\Omega})$ , we follow the spirit of Alon and Roichman (1994) and define the following two events in terms of the quantity  $M = 2m(1 - \log \log 2m / \log 2m)$ :

- (A) After step (2) of  $\mathbf{RW}$  (definition 4), the length of the reduced word is less than  $M$ .
- (B) After step (2) of  $\mathbf{RW}$  (definition 4), the length of the reduced word is at least  $M$ , but there are no singletons.

Clearly,  $\Pr(\overline{\Omega}) \leq \Pr(A) + \Pr(B)$ . Alon and Roichman (1994) produced these bounds:

$$\begin{aligned}\Pr(A) &\leq 2^{2m} (2/N)^{m \log \log 2m / \log 2m} \\ \Pr(B) &\leq 2^M (m/N)^{M/2}.\end{aligned}$$

Substituting  $N = c(\varepsilon) \log D(G)$  and  $2m = (1/b) \log D(G)$ , for any constant  $b$ , we obtain an expression almost identical to one of Alon and Roichman (1994), except that  $|G|$ 's are replaced by  $D(G)$ 's:

$$\Pr(\overline{\Omega})^{1/2m} D(G)^{1/2m} \leq (1 + o(1)) e^b \sqrt{\frac{2}{bc(\varepsilon)}} \leq (1 + o(1)) \varepsilon \quad (2)$$

where we use the choices  $c(\varepsilon) = 4e/\varepsilon^2$  and  $b = 1/2$ . □

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