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On the maximum average degree and the incidence chromatic number of a graph

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We prove that the incidence chromatic number of every 3-degenerated graph G is at most $\Delta(G)+4$. It is known that the incidence chromatic number of every graph G with maximum average degree $mad(G) < 3$ is at most $\Delta(G)+3$. We show that when $\Delta(G) \geq 5$, this bound may be decreased to $\Delta(G)+2$. Moreover, we show that for every graph G with $mad(G) < 22/9$ (resp. with $mad(G) < 16/7$ and $\Delta(G) \geq 4$), this bound may be decreased to $\Delta(G)+2$ (resp. to $\Delta(G)+1$).

Keywords: incidence coloring, k-degenerated graph, planar graph, maximum average degree

1 Introduction

The concept of incidence coloring was introduced by Brualdi and Massey (3) in 1993.

Let $G = (V(G), E(G))$ be a graph. An *incidence* in G is a pair (v, e) with $v \in V(G)$, $e \in E(G)$, such that v and e are incident. We denote by $I(G)$ the set of all incidences in G. For every vertex v, we denote by I_v the set of incidences of the form (v, vw) and by A_v the set of incidences of the form (w, wv). Two incidences (v, e) and (w, f) are *adjacent* if one of the following holds: (i) $v = w$, (ii) $e = f$ or (iii) the edge vw equals e or f.

A k*-incidence coloring* of a graph G is a mapping σ of $I(G)$ to a set C of k colors such that adjacent incidences are assigned distinct colors. The *incidence chromatic number* $\chi_i(G)$ of G is the smallest k such that G admits a k-incidence coloring.

For a graph G, let $\Delta(G)$, $\delta(G)$ denote the maximum and minimum degree of G respectively. It is easy to observe that for every graph G we have $\chi_i(G) \geq \Delta(G)+1$ (for a vertex v of degree $\Delta(G)$ we must use $\Delta(G)$ colors for coloring I_v and at least one additional color for coloring A_v). Brualdi and Massey proved the following upper bound:

Theorem 1 (3) *For every graph G, $\chi_i(G) \leq 2\Delta(G)$.*

Guiduli (4) showed that the concept of incidence coloring is a particular case of directed star arboricity, introduced by Algor and Alon (1). Following an example from (1), Guiduli proved that there exist graphs G with $\chi_i(G) \geq \Delta(G) + \Omega(\log \Delta(G))$. He also proved that For every graph G, $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$.

Concerning the incidence chromatic number of special classes of graphs, the following is known:

- For every $n \geq 2$, $\chi_i(K_n) = n = \Delta(K_n) + 1$ (3).
- For every $m \geq n \geq 2$, $\chi_i(K_{m,n}) = m + 2 = \Delta(K_{m,n}) + 2$ (3).
- For every tree T of order $n \geq 2$, $\chi_i(T) = \Delta(T) + 1$ (3).
- For every Halin graph G with $\Delta(G) \geq 5$, $\chi_i(G) = \Delta(G) + 1$ (8).
- For every k-degenerated graph G, $\chi_i(G) \leq \Delta(G) + 2k - 1$ (5).
- For every K_4-minor free graph G, $\chi_i(G) \leq \Delta(G) + 2$ and this bound is tight (5).
- For every cubic graph G, $\chi_i(G) \leq 5$ and this bound is tight (6).
- For every planar graph G, $\chi_i(G) \leq \Delta(G) + 7$ (5).

The *maximum average degree* of a graph G, denoted by $mad(G)$, is defined as the maximum of the average degrees $ad(H) = 2 \cdot |E(H)|/|V(H)|$ taken over all the subgraphs H of G.

In this paper we consider the class of 3-degenerated graphs (recall that a graph G is k-degenerated if $\delta(H) \leq k$ for every subgraph H of G), which includes for instance the class of triangle-free planar graphs and the class of graphs with maximum average degree at most 3. More precisely, we shall prove the following:

1. If G is a 3-degenerated graph, then $\chi_i(G) \leq \Delta(G) + 4$ (Theorem 2).
2. If G is a graph with $mad(G) < 3$, then $\chi_i(G) \leq \Delta(G) + 3$ (Corollary 5).
3. If G a graph with $mad(G) < 3$ and $\Delta(G) \geq 5$, then $\chi_i(G) \leq \Delta(G) + 2$ (Theorem 8).
4. If G is a graph with $mad(G) < 22/9$, then $\chi_i(G) \leq \Delta(G) + 2$ (Theorem 11).
5. If G is a graph with $mad(G) < 16/7$ and $\Delta(G) \geq 4$, then $\chi_i(G) = \Delta(G) + 1$ (Theorem 13).

In fact we shall prove something stronger, namely that one can construct for these classes of graphs incidence colorings such that for every vertex v, the number of colors that are used on the incidences of the form (w, wv) is bounded by some fixed constant not depending on the maximum degree of the graph.

More precisely, we define a (k, ℓ)*-incidence coloring* of a graph G as a k-incidence coloring σ of G such that for every vertex $v \in V(G)$, $|\sigma(A_v)| \leq \ell$.

We end this section by introducing some notation that we shall use in the rest of the paper.

Let G be a graph. If v is a vertex in G and vw is an edge in G, we denote by $N_G(v)$ the set of neighbors of v, by $d_G(v) = |N_G(v)|$ the degree of v, by $G \setminus v$ the graph obtained from G by deleting the vertex v and by $G \setminus vw$ the graph obtained from G by deleting the edge vw.

Let G be a graph and σ' a *partial* incidence coloring of G, that is an incidence coloring only defined on some subset I of $I(G)$. For every uncolored incidence $(v, vw) \in I(G) \setminus I$, we denote by $F_G^{\sigma'}(v, vw)$ the set of *forbidden colors* of (v, vw), that is:

$$F_G^{\sigma'}(v, vw) \;=\; \sigma'(A_v) \cup \sigma'(I_v) \cup \sigma'(I_w).$$

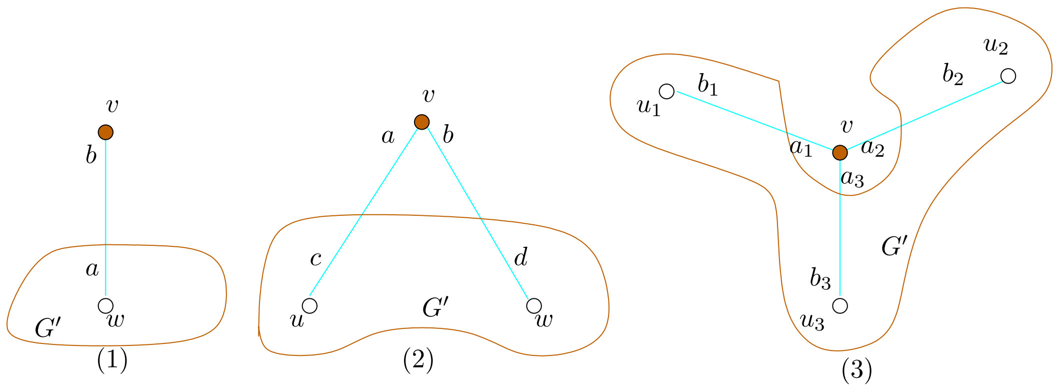


Fig. 1: Configurations for the proof of Theorem 2

We shall often say that we extend such a partial incidence coloring σ' to some incidence coloring σ of G. In that case, it should be understood that we set $\sigma(v, vw) = \sigma'(v, vw)$ for every incidence $(v, vw) \in I$.

We shall make extensive use of the fact that every (k, ℓ)-incidence coloring may be viewed as a (k', ℓ)-incidence coloring for any $k' > k$.

Drawing convention. In a figure representing a forbidden configuration, all the neighbors of "black" or "grey" vertices are drawn, whereas "white" vertices may have other neighbors in the graph.

2 3-degenerated graphs

In this section, we prove the following:

Theorem 2 *Every 3-degenerated graph G admits a $(\Delta(G)+4, 3)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 4$.*

Proof: Let G be a 3-degenerated graph. Observe first that if $\Delta(G) \leq 3$ then, by Theorem 1, $\chi_i(G) \leq 2\Delta(G) < \Delta(G) + 4 \leq 7$ and every $(\Delta(G) + 4)$-incidence coloring of G is obviously a $(\Delta(G) + 4, 3)$-incidence coloring.

Therefore, we assume $\Delta(G) \geq 4$ and we prove the theorem by induction on the number of vertices of G. If G has at most 5 vertices then $G \subseteq K_5$. Since for every $k > 0$, $\chi_i(K_n) = n$, we obtain $\chi_i(G) \leq \chi_i(K_5) = \Delta(K_5) + 1 = 5$, and every 5-incidence coloring of G is obviously a $(\Delta(G) + 4, 3)$-incidence coloring. We assume now that G has $n + 1$ vertices, $n \geq 5$, and that the theorem is true for all 3-degenerated graphs with at most n vertices.

Let v be a vertex of G with minimum degree. Since G is 3-degenerated, we have $d_G(v) \leq 3$. We consider three cases according to $d_G(v)$.

$d_G(v) = 1$**:**

Let w denote the unique neighbor of v in G (see Figure 1.(1)). Due to the induction hypothesis, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 4, 3)$-incidence coloring σ'. We extend σ' to a $(\Delta(G) + 4, 3)$-incidence coloring of G. Since $|F_G^{\sigma'}(w, wv)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G) - 1 + 3 = \Delta(G) + 2$, there is a color a such that $a \notin F_G^{\sigma'}(w, wv)$. We then set $\sigma(w, wv) = a$ and $\sigma(v, vw) = b$, for any color b in $\sigma'(A_w)$.

$d_G(v) = 2$**:**

Let u, w be the two neighbors of v in G (see Figure 1.(2)). Due to the induction hypothesis, the graph $G' = G \setminus v$ admits a $(\Delta(G)+4, 3)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+4, 3)$-incidence coloring σ of G as follows. We first set $\sigma(v, vu) = a$ for a color $a \in \sigma(A_u)$ (if $d_G(u) = 1$, we have the case 1). Now, if $|\sigma'(A_w)| \geq 2$, there is a color $b \in \sigma'(A_w) \setminus \{a\}$ and if $|\sigma'(A_w)| = 1$, since $|F_G^{\sigma}(v, vw)| = |\sigma'(I_w) \cup \{a\}| \leq \Delta(G) - 1 + 1 = \Delta(G)$, there is a color b distinct from a such that $b \notin F_G^{\sigma}(v, vw)$. We set $\sigma(v, vw) = b$.

We still have to color the two incidences (u, uv) and (w, wv). Since $a \in \sigma'(A_u)$, we have $|F_G^{\sigma}(u, uv)| = |\sigma'(I_u) \cup \sigma'(A_u) \cup \{a, b\}| \leq \Delta(G) - 1 + 3 + 2 - 1 = \Delta(G) + 3$. Therefore, there is a color c such that $c \notin F_G^{\sigma}(u, uv)$. Similarly, since $b \in \sigma(A_w)$, we have $|F_G^{\sigma}(w, wv)| \leq \Delta(G) + 3$ and there exists a color d such that $d \notin F_G^{\sigma}(w, wv)$. We can extend σ' to a $(\Delta(G)+4, 3)$-incidence coloring σ of G by setting $\sigma(u, uv) = c$ and $\sigma(w, wv) = d$.

$d_G(v) = 3$**:**

Let u_1, u_2 and u_3 be the three neighbors of v in G (see Figure 1.(3)). Due to the induction hypothesis, the graph $G' = G \setminus v$ admits a $(\Delta(G)+4, 3)$-incidence coloring σ'.

Observe first that for every i, $1 \leq i \leq 3$, since $|F_G^{\sigma'}(v, vu_i)| \leq \Delta(G) - 1$ and since we have $\Delta(G)+4$ colors, we have at least five colors which are not in $F_G^{\sigma'}(v, vu_i)$. Moreover, if $|A_{u_i}| < 3$ then any of these five colors may be assigned to the incidence (v, vu_i) whereas we have only three possible choices (among these five) if $|A_{u_i}| = 3$. In the following, we shall see that having only three available colors is enough, and therefore assume that $|\sigma'(A_{u_i})| = 3$ for every i, $1 \leq i \leq 3$.

We define the sets B and $B_{i,j}$ as follows:

- $\forall\, i, j$, $1 \leq i, j \leq 3$, $i \neq j$, $B_{i,j} := (\sigma'(I_{u_i}) \cup \sigma'(A_{u_i})) \cap \sigma'(A_{u_j})$

- $B := \bigcup_{1 \leq i,j \leq 3} B_{i,j}$, $i \neq j$.

We consider now four subcases according to the degrees of u_1, u_2 and u_3:

1. $\forall\, i$, $1 \leq i \leq 3$, $d_G(u_i) < \Delta(G)$.

   In this case, since we have 3 colors for the incidence (v, vu_i) for every i, $1 \leq i \leq 3$, we can find 3 distinct colors a_1, a_2, a_3 such that $a_i \notin F_G^{\sigma'}(v, vu_i)$. We set $\sigma(v, vu_i) = a_i$ for every i, $1 \leq i \leq 3$.

   We still have to color the three incidences (u_i, u_iv), $1 \leq i \leq 3$. Since $a_i \in \sigma(A_{u_i})$, we have $|F_G^{\sigma}(u_i, u_iv)| = |\sigma(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}| \leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3$ for every i, $1 \leq i \leq 3$. So, there exist three colors b_1, b_2, b_3 such that $b_i \notin F_G^{\sigma}(u_i, u_iv)$, $1 \leq i \leq 3$. We can extend σ' to a $(\Delta(G)+4, 3)$-incidence coloring σ of G by setting $\sigma(u_i, u_iv) = b_i$ for every i, $1 \leq i \leq 3$.

2. Only one of the vertices u_i is of degree $\Delta(G)$.

   We can suppose without loss of generality that $d_G(u_1)$, $d_G(u_2) < \Delta(G)$ and $d_G(u_3) = \Delta(G)$.

Since $|\sigma'(I_{u_3}) \cup \sigma'(A_{u_3})| = \Delta(G) - 1 + 3 = \Delta(G) + 2$ and $|\sigma'(A_{u_1})| = 3$, we have $B_{3,1} \neq \emptyset$. Let $a_1 \in B_{3,1}$. Since $|\sigma'(A_{u_i})| = 3$ for every i, $1 \leq i \leq 3$, there exist two distinct colors a_2 and a_3 distinct from a_1 such that $a_2 \in \sigma'(A_{u_2})$ and $a_3 \in \sigma'(A_{u_3})$. We set $\sigma(v, vu_i) = a_i$ for every i, $1 \leq i \leq 3$.

We still have to color the three incidences of form $(u_i, u_i v)$. Since $a_1 \in B_{3,1}$ and $a_3 \in \sigma'(A_{u_3})$ we have:

$$
\begin{aligned}
|F_G^{\sigma}(u_3, u_3 v)| &= |\sigma'(I_{u_3}) \cup \sigma(A_{u_3}) \cup \{a_1, a_2, a_3\}| \\
&\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta + 3
\end{aligned}
$$

and since $a_i \in \sigma'(A_{u_i})$ for every $i = 1, 2$ we have:

$$
\begin{aligned}
|F_G^{\sigma'}(u_i, u_i v)| &= |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}| \\
&\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta + 3.
\end{aligned}
$$

Therefore, there exist three colors b_1, b_2, b_3 such that $b_i \notin F_G^{\sigma}(u_i, u_i v) \cup \{a_1, a_2, a_3\}$, $1 \leq i \leq 3$. We can extend σ' to a $(\Delta(G) + 4, 3)$-incidence coloring σ of G by setting $\sigma(u_i, u_i v) = b_i$ for every i, $1 \leq i \leq 3$.

3. Only one vertex among the u_i's is of degree less than $\Delta(G)$.

   We can suppose without loss of generality that $d_G(u_1) < \Delta(G)$ and $d_G(u_2) = d_G(u_3) = \Delta(G)$.

   Similarly to the previous case, we have $B_{2,1} \neq \emptyset$ and $B_{3,2} \neq \emptyset$. We consider two cases:

   $B_{2,1} \neq B_{3,2}$

   Let $a_1 \in B_{2,1}$, $a_2 \in B_{3,2} \setminus \{a_1\}$ and $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$. We set $\sigma(v, vu_i) = a_i$ for every i, $1 \leq i \leq 3$.

   We still have to color the three incidences $(u_i, u_i v)$, $1 \leq i \leq 3$. Since $a_1 \in \sigma'(A_{u_1})$ we have:

$$
\begin{aligned}
|F_G^{\sigma}(u_1, u_1 v)| &= |\sigma'(I_{u_1}) \cup \sigma(A_{u_1}) \cup \{a_1, a_2, a_3\}| \\
&\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3
\end{aligned}
$$

   and since $a_i \in B_{i+1,i}$ for $i = 1, 2$ and $a_j \in \sigma'(A_{u_j})$ for $j = 2, 3$, we have:

$$
\begin{aligned}
|F_G^{\sigma}(u_i, u_i v)| &= |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}| \\
&\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3.
\end{aligned}
$$

   Therefore, there exist three colors b_1, b_2, b_3 such that $b_i \notin F_G^{\sigma}(u_i, u_i v)$, $1 \leq i \leq 3$. We can extend σ' to a $(\Delta(G) + 4, 3)$-incidence coloring σ of G by setting $\sigma(u_i, u_i v) = b_i$ for every i, $1 \leq i \leq 3$.

   $B_{2,1} = B_{3,2}$

   Let $a_1 \in B_{2,1} = B_{3,2}$, $a_2 \in \sigma'(A_{u_2}) \setminus \{a_1\}$ and $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$. We set $\sigma(v, vu_i) = a_i$ for every i, $1 \leq i \leq 3$.

We still have to color the three incidences (u_i, u_iv), $1 \leq i \leq 3$. Since $a_1 \in \sigma'(A_{u_1})$ we have:

$$|F^\sigma_G(u_1, u_1v)| = |\sigma'(I_{u_1}) \cup \sigma(A_{u_1}) \cup \{a_1, a_2, a_3\}|$$
$$\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3$$

and since $a_1 \in B_{2,1} = B_{3,2}$ and $a_j \in \sigma'(A_{u_j})$ for $j = 2, 3$, we have:

$$|F^\sigma_G(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$
$$\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3.$$

Therefore, there exist three colors b_1, b_2, b_3 such that $b_i \notin F^\sigma_G(u_i, u_iv)$, $1 \leq i \leq 3$. We can extend σ' to a $(\Delta(G) + 4, 3)$-incidence coloring σ of G by setting $\sigma(u_i, u_iv) = b_i$ for every i, $1 \leq i \leq 3$.

4. $d_G(u_1) = d_G(u_2) = d_G(u_3) = \Delta(G)$.

Similarly to the case (b) we have $B_{i,j} \neq \emptyset$ for every i and j, $1 \leq i, j \leq 3$ and thus $|B| \geq 1$. We prove first that in this case $|B| \geq 2$. Suppose that $|B| = |\{x\}| = 1$; in other words, $((\sigma'(I_{u_i}) \cup A'_{u_i}) \cap A'_{u_j}) = \{x\}$ for every i and j, $1 \leq i, j \leq 3$. Thus we have:

$$|\sigma'(A_{u_1}) \cup \sigma'(I_{u_1}) \cup \sigma'(A_{u_2}) \cup \sigma'(A_{u_3})| = \Delta(G) - 1 + 3 + 3 + 3 - 1 - 1$$
$$= \Delta(G) + 6. \tag{1}$$

But the relation (1) is in contradiction with the fact that σ' is a $(\Delta(G) + 4, 3)$-incidence coloring and we then get $|B| \geq 2$.

Let a_1 and a_2 be two distinct colors in B. We can suppose without loss of generality that $a_1 \in B_{2,1}$ and $a_2 \in B_{3,2}$.

We consider the two following subcases:

$B_{1,3} \setminus \{a_1, a_2\} \neq \emptyset$

Let a_3 be a color in $B_{1,3} \setminus \{a_1, a_2\}$. We set $\sigma(v, vu_i) = a_i$ for every i, $1 \leq i \leq 3$.
Since $a_i \in B_{j,i} = (\sigma'(I_{u_j}) \cup \sigma'(A_{u_j})) \cap \sigma'(A_{u_i})$, $j = i + 1$ mod 3, and $a_i \in \sigma'(A_{u_i})$ for every i, $1 \leq i \leq 3$, we have:

$$|F^\sigma_G(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$
$$\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3.$$

Therefore, there exist three colors b_1, b_2, b_3 such that $b_i \notin F^\sigma_G(u_i, u_iv)$, $1 \leq i \leq 3$. We can extend σ' to a $(\Delta(G) + 4, 3)$-incidence coloring σ of G by setting $\sigma(u_i, u_iv) = b_i$ for every i, $1 \leq i \leq 3$.

$B_{1,3} \setminus \{a_1, a_2\} = \emptyset$

Since $B_{1,3} \neq \emptyset$ we can suppose without loss of generality that $a_2 \in B_{1,3}$. Let $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$. We set $\sigma(v, vu_i) = a_i$ for every i, $1 \leq i \leq 3$.
Since $a_i \in B_{j,i} = (\sigma'(I_{u_j}) \cup \sigma'(A_{u_j})) \cap \sigma'(A_{u_i})$, $j = i + 1$ mod 3, and $a_i \in \sigma'(A_{u_i})$ for $i = 1, 2$, we have:

$$|F^\sigma_G(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$

$$\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3$$

and since $a_2 \in \sigma'(I_{u_1}) \cup \sigma'(A_{u_1})$ and $a_1 \in \sigma'(A - u_1)$ we have:

$$|F_G^\sigma(u_1, u_1v)| = |\sigma'(I_{u_1}) \cup \sigma'(A_{u_1}) \cup \{a_1, a_2, a_3\}|$$

$$\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3.$$

Therefore, there exist three colors b_1, b_2, b_3 such that $b_i \notin F_G^\sigma(u_i, u_iv)$, $1 \leq i \leq 3$. We can extend σ' to a $(\Delta(G) + 4, 3)$-incidence coloring σ of G by setting $\sigma(u_i, u_iv) = b_i$ for every i, $1 \leq i \leq 3$.

It is easy to check that in all cases we obtain a $(\Delta(G) + 4, 3)$-incidence coloring of G and the theorem is proved. □

Since every triangle free planar graph is 3-degenerated, we have:

Corollary 3 *For every triangle free planar graph G, $\chi_i(G) \leq \Delta(G) + 4$.*

3 Graphs with bounded maximum average degree

In this section we study the incidence chromatic number of graphs with bounded maximum average degree. The following result has been proved in (5).

Theorem 4 *Every k-degenerated graph G admits a $(\Delta(G) + 2k - 1, k)$-incidence coloring.*

Since every graph G with $mad(G) < 3$ is 2-degenerated, we get the following:

Corollary 5 *Every graph G with $mad(G) < 3$ admits a $(\Delta(G) + 3, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 3$.*

Concerning planar graphs, we have the following:

Observation 6 (2) *For every planar graph G with girth at least g, $mad(G) < 2g/(g - 2)$.*

Hence, we obtain:

Corollary 7 *Every planar graph G with girth $g \geq 6$ admits a $(\Delta(G) + 3, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 3$.*

Proof: By Observation 6 we have $mad(G) < 2g/(g - 2) \leq (2 \times 6)/(6 - 2) = 3$ and we get the result from Corollary 5. □

If the graph has maximum degree at least 5, the previous result can be improved:

Theorem 8 *Every graph G with $mad(G) < 3$ and $\Delta(G) \geq 5$ admits a $(\Delta(G)+2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.*

Proof: Suppose that the theorem is false and let G be a minimal counter-example (with respect to the number of vertices). We first show that G must avoid all the configurations depicted in Fig. 2.

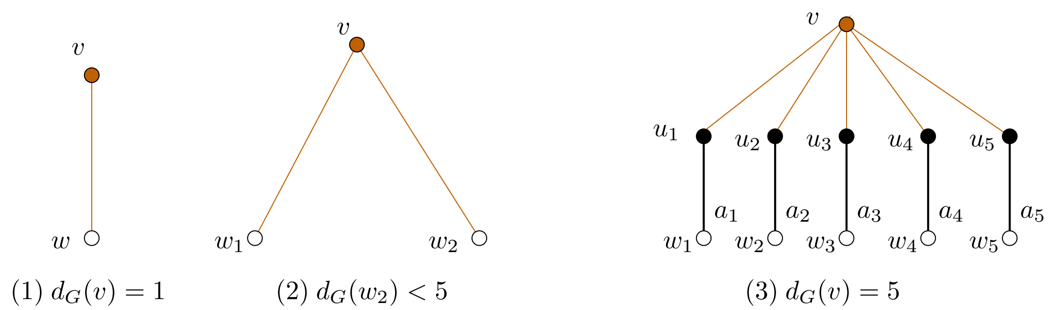


Fig. 2: Forbidden configurations for the proof of Theorem 8

(1) Let w denote the unique neighbor of v in G. Due to the minimality of G, the graph $G' = G \setminus v$ admits a $(\Delta(G)+2, 2)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+2, 2)$-incidence coloring σ of G. Since $|F_G^\sigma(w, wv)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$, there is a color a such that $a \notin F_G^\sigma(w, wv)$. We set $\sigma(w, wv) = a$ and $\sigma(v, vw) = b$, for any color b in $\sigma'(A_w)$.

(2) Let w_1, w_2 denote the two neighbors of v in G. Due to the minimality of G, the graph $G' = G \setminus v$ admits a $(\Delta(G)+2, 2)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+2, 2)$-incidence coloring σ of G.

Since $|F_G^\sigma(w_1, w_1v)| = |\sigma'(I_{w_1}) \cup \sigma'(A_{w_1})| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$ and since we have $\Delta(G) + 2$ possible colors, there is a color a such that $a \notin F_G^\sigma(w_1, w_1v)$. We set $\sigma(w_1, w_1v) = a$. If $|\sigma'(A_{w_2}) \setminus \{a\}| \geq 1$ then there is a color $b \in \sigma'(A_{w_2}) \setminus \{a\}$ and if $\sigma'(A_{w_2}) = \{a\}$, since $|F_G^\sigma(v, vw_2)| = |\sigma'(I_{w_2}) \cup \{a\}| \leq 3+1 = 4 \leq \Delta(G) - 1$, there is a color b such that $b \notin F_G^\sigma(v, vw_2)$. We set $\sigma(v, vw_2) = b$.

Now, if $|\sigma'(A_{w_1}) \setminus \{b\}| \geq 1$ then there is a color $c \in \sigma'(A_{w_1}) \setminus \{b\}$ and if $\sigma'(A_{w_1}) = \{b\}$, since $|F_G^\sigma(v, vw_1)| = |\sigma(I_{w_1}) \cup \{b\}| \leq \Delta(G) + 1$, there is a color c such that $c \notin F_G^\sigma(v, vw_1)$. We set $\sigma(v, vw_1) = c$.

Since $|F_G^\sigma(w_2, w_2v)| = |\sigma'(I_{w_2}) \cup \sigma(A_{w_2}) \cup \{c\}| \leq 3 + 2 + 1 = 6 \leq \Delta(G) + 1$, there is a color d such that $d \notin F_G^\sigma(w_2, w_2v)$ and we set $\sigma(w_2, w_2v) = d$.

(3) Let u_i, $1 \leq i \leq 5$, denote the five neighbors of v and w_i denote the other neighbor of u_i in G (see Figure 2.(3)). Due to the minimality of G, the graph $G' = G \setminus v$ admits a $(\Delta(G)+2, 2)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+2, 2)$-incidence coloring σ of G.

Let $a_i = \sigma'(w_i, w_iu_i)$, $1 \leq i \leq 5$. Since we have $\Delta(G) + 2 \geq 7$ colors, there is a color x distinct from a_i for every i, $1 \leq i \leq 5$.

Since $|F_G^{\sigma'}(u_i, u_iw_i)| = |\sigma'(I_{w_i})| \leq \Delta(G)$ we have two possible colors for the incidence (u_i, u_iw_i) for every i, $1 \leq i \leq 5$. So, we can suppose that $\sigma'(u_i, u_iw_i) \neq x$ for every i, $1 \leq i \leq 5$. We set $\sigma(u_i, u_iv) = x$ for every i, $1 \leq i \leq 5$.

Since $F_G^\sigma(v, vu_i) = \{x, \sigma'(u_i, u_iw_i)\}$ for every i, $1 \leq i \leq 5$, and since we have at least 7 colors, there is 5 distinct colors c_1, c_2 ,..., c_5 such that $c_i \notin \{x, \sigma'(u_i, u_iw_i)\}$, $1 \leq i \leq 5$, and we set $\sigma(v, vu_i) = c_i$ for every i, $1 \leq i \leq 5$.

It is easy to check that in every case we have obtained a $(\Delta(G)+2,2)$-incidence coloring of G, which contradicts our assumption.

We now associate with each vertex v of G an initial charge $d(v) = d_G(v)$, and we use the following discharging procedure: each vertex of degree at least 5 gives $1/2$ to each of its 2-neighbors.

We shall prove that the modernized degree d^* of each vertex of G is at least 3 which contradicts the assumption $mad(G) < 3$ (since $\sum_{u \in G} d^*(u) = \sum_{u \in G} d(u)$). Let v be a vertex of G; we consider the possible cases for old degree $d_G(v)$ of v (since G does not contain the configuration $2(1)$, we have $d_G(v) \geq 2$):

$d_G(v) = 2$.
Since G does not contain the configuration $2(2)$ the two neighbors of v are of degree at least 5. Therefore, v receives $1/2$ from each of its neighbors so that $d^*(v) = 2 + 1/2 + 1/2 = 3$.

$3 \leq d_G(v) \leq 4$.
In this case we have $d^*(v) = d_G(v) \geq 3$.

$d_G(v) = 5$.
Since G does not contain the configuration $2(3)$ at least one of the neighbors of v is of degree at least 3 and v gives at most $4 \times 1/2 = 2$. We obtain $d^*(v) \geq 5 - 2 = 3$.

$d_G(v) = k \geq 6$.
In this case v gives at most $k \times (1/2)$ so that $d^*(v) \geq k - k/2 = k/2 \geq 6/2 = 3$.

Therefore, every vertex in G gets a modernized degree of at least 3 and the theorem is proved. $\square$

Remark 9 The previous result also holds for graphs with maximum degree 2 and for graphs with maximum degree 3 (by the result from (6)) but the question remains open for graphs with maximal degree 4.

As previously, for planar graphs we obtain:

Corollary 10 *Every planar graph G of girth $g \geq 6$ with $\Delta(G) \geq 5$ admits a $(\Delta(G)+2,2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G)+2$.*

For graphs with maximum average degree less than 22/9, we have:

Theorem 11 *Every graph G with $mad(G) < 22/9$ admits a $(\Delta(G)+2,2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G)+2$.*

Proof: It is enough to consider the case of graphs with maximum degree at most 4, since for graphs with maximum degree at least 5 the theorem follows from Theorem 8. Suppose that the theorem is false and let G be a minimal counter-example (with respect to the number of vertices and edges). Observe first that we have $\Delta(G) \geq 3$ since otherwise we obtain by Theorem 1 that $\chi_i(G) \leq 2\Delta(G) \leq \Delta(G)+2$ and every $(\Delta(G)+2)$-incidence coloring of G is obviously a $(\Delta(G)+2,2)$-incidence coloring.

We first show that G cannot contain any of the configurations depicted in Figure 3.

(1) This case is similar to case 1 of Theorem 8.

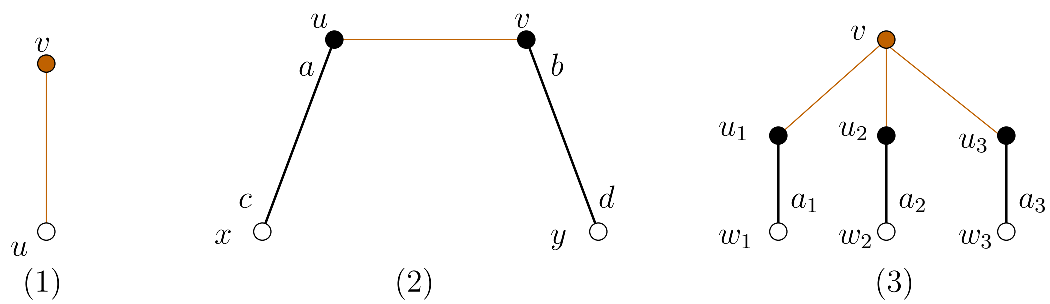


Fig. 3: Forbidden configurations for the proof of Theorem 11

(2) Let x (resp. y) denote the other neighbor of u (resp. v) in G. Due to the minimality of G, the graph $G' = G \backslash uv$ admits a $(\Delta(G)+2, 2)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+2, 2)$-incidence coloring σ of G.

Suppose $\sigma'(u, ux) = a$, $\sigma'(v, vy) = b$, $\sigma'(x, xu) = c$ and $\sigma'(y, yv) = d$.

Suppose first that $|\{a, b, c, d\}| = 4$. In that case, we set $\sigma(u, uv) = d$ and $\sigma(v, vu) = c$.

Now, if $|\{a, b, c, d\}| \leq 3$, we set $\sigma(u, uv) = e$ and $\sigma(v, vu) = f$ for any $e, f \notin \{a, b, c, d\}$.

(3) Let u_1, u_2 and u_3 denote the three neighbors of v and w_i denotes the other neighbor of u_i, $1 \leq i \leq 3$, in G. Due to the minimality of G, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 2, 2)$-incidence coloring σ'. We extend σ' to a $(\Delta(G) + 2, 2)$-incidence coloring σ of G.

Suppose that $a_i = \sigma'(w_i, w_i u_i)$, $1 \leq i \leq 3$. Since we have $\Delta(G) + 2 \geq 5$ colors, there is a color x distinct from a_i for every i, $1 \leq i \leq 3$.

Since $|F_G^{\sigma'}(u_i, u_i w_i)| = |\sigma'(I_{w_i})| \leq \Delta(G)$ we have at least two colors for the incidence $(u_i, u_i w_i)$ for every i, $1 \leq i \leq 3$. Thus, we can suppose $\sigma'(u_i, u_i w_i) \neq x$ for every i, $1 \leq i \leq 3$. We then set $\sigma(u_i, u_i v) = x$ for every i, $1 \leq i \leq 3$.

Since $F_G^{\sigma}(v, vu_i) = \{x, \sigma'(u_i, u_i w_i)\}$ for every i, $1 \leq i \leq 3$, and since we have at least 5 colors, there are 3 distinct colors c_1, c_2 et c_3 such that $c_i \notin \{x, \sigma'(u_i, u_i w_i)\}$, $1 \leq i \leq 3$. We then set $\sigma(v, vu_i) = c_i$ for every i, $1 \leq i \leq 3$.

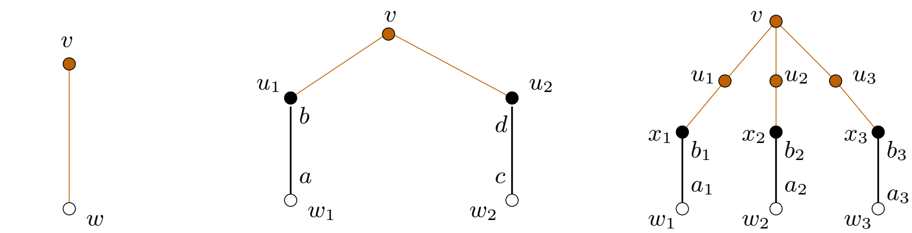
Therefore, in all cases we obtain a $(\Delta(G) + 2, 2)$-incidence coloring of G, which contradicts our assumption.

We now associate with each vertex v of G an initial charge $d(v) = d_G(v)$, and we use the following discharging procedure: each vertex of degree at least 3 gives $2/9$ to each of its 2-neighbors.

We shall prove that the modernized degree d^* of each vertex of G is at least $22/9$ which contradicts the assumption $mad(G) < 22/9$. Let v be a vertex of G; we consider the possible cases for old degree $d_G(v)$ of v (since G does not contain the configuration $3(1)$, we have $d_G(v) \geq 2$):

$d_G(v) = 2$**.**

Since G does not contain the configuration $3(2)$ the two neighbors of v are of degree at least 3. Therefore, v receives then $2/9$ from each of its neighbors so that $d^*(v) = 2 + 2/9 + 2/9 = 22/9$.



(1) $d_G(v) = 1$ (2) $d_G(w_1) < \Delta(G)$ (3) $d_G(w_2) = d_G(w_3) = 3$

Fig. 4: Forbidden configurations for the proof of Theorem 13

$d_G(v) = 3$.
Since G does not contain the configuration $3(3)$, v is adjacent to at most two 2-vertices and v gives at most $2 \times 2/9 = 4/9$. We obtain $d^*(v) \geq 3 - 4/9 = 23/9 \geq 22/9$.

$d_G(v) = 4$.
In this case, v gives at most $4 \times 2/9 = 8/9$ so that $d^*(v) \geq 4 - 8/9 = 28/9 \geq 22/9$.

Therefore, every vertex in G gets a modernized degree of at least 3 and the theorem is proved. $\square$

By considering cycles of length $\ell \not\equiv 0 \pmod 3$, we get that the upper bound of Theorem 11 is tight. As previously, for planar graphs we obtain:

Corollary 12 *Every planar graph G of girth $g \geq 11$ admits a $(\Delta(G) + 2, 2)$-incidence coloring. Therefore, $\chi_i(G) \leq \Delta(G) + 2$.*

Finally, for graphs with maximum average degree less than 16/7, we have:

Theorem 13 *Every graph G with $mad(G) < 16/7$ and $\Delta(G) \geq 4$ admits a $(\Delta(G) + 1, 1)$-incidence coloring. Therefore, $\chi_i(G) = \Delta(G) + 1$.*

Proof: Since for every graph G, $\chi_i(G) \geq \Delta(G) + 1$, it is enough to prove that G admits a $(\Delta(G) + 1, 1)$-incidence coloring. Suppose that the theorem is false and let G be a minimal counter-example (with respect to the number of vertices). We first show that G cannot contain any of the configurations depicted in Figure 4.

(1) This case is similar to case 1 of Theorem 8.

(2) Let u_i, $i = 1, 2$, be the two neighbors of v and w_i denote the other neighbor of u_i in G. Due to the minimality of G, the graph $G' = G \setminus v$ admits a $(\Delta(G) + 1, 1)$-incidence coloring σ'. We extend σ' to a $(\Delta(G) + 1, 1)$-incidence coloring σ of G.

Suppose that $\sigma'(w_1, w_1u_1) = a$, $\sigma'(u_1, u_1w_1) = b$, $\sigma'(w_2, w_2u_2) = c$ and $\sigma'(u_2, u_2w_2) = d$. Since $|F^{\sigma'}_{G'}(w_1, w_1u_1) \cup \{c\}| = |\sigma'(I_{w_1}) \setminus \{a\} \cup \sigma'(A_{w_1}) \cup \{c\}| \leq \Delta(G) - 2 + 1 + 1 = \Delta(G)$, we can suppose that $a \neq c$. We then set $\sigma(v, vu_1) = a$ and $\sigma(v, vu_2) = c$.

Now, since $F^{\sigma}_G(u_1, u_1v) \cup F^{\sigma}_G(u_2, u_2v) = \{a, b, c, d\}$ and since we have at least $\Delta(G) + 1 \geq 5$ colors, there is a color x such that $x \notin \{a, b, c, d\}$. We then set $\sigma(u_1, u_1v) = \sigma(u_2, u_2v) = x$.

(3) Let u_i, $1 \leq i \leq 3$ be the three neighbors of v, x_i denote the other neighbor of u_i and w_i denote the other neighbor of x_i in G. Due to the minimality of G, the graph $G' = G \setminus \{v, u_1, u_2, u_3\}$ admits a $(\Delta(G)+1, 1)$-incidence coloring σ'. We extend σ' to a $(\Delta(G)+1, 1)$-incidence coloring σ of G.

Suppose that $\sigma'(w_i, w_ix_i) = a_i$ and $\sigma'(x_i, x_iw_i) = b_i$ for every i, $1 \leq i \leq 3$. Since $|F_G^{\sigma'}(w_i, w_ix_i) \cup \{b_1\}| = |\sigma'(I_{w_i}) \setminus \{a_i\} \cup \{b_i, b_1\}| \leq 2+2 = 4$ for $i = 2, 3$, and since we have $\Delta(G)+1 \geq 5$ colors, we can suppose that $a_2 \neq b_1 \neq a_3$. We then set $\sigma(u_i, u_ix_i) = a_i$ and $\sigma(u_i, u_iv) = b_1$ for every i, $1 \leq i \leq 3$.

Since $F_G^{\sigma}(v, vu_j) \cup F_G^{\sigma}(x_j, x_ju_j) = \{b_1, b_j, a_j\}$ for $j = 2, 3$, there are two distinct colors c_2 and c_3 such that $c_j \notin \{b_1, b_j, a_j\}$, $j = 2, 3$. We set $\sigma(v, vu_j) = \sigma(x_j, x_ju_j) = c_j$, $j = 2, 3$.

Now, since $F_G^{\sigma'}(v, vu_1) \cup F_G^{\sigma'}(x_1, x_1u_1) = \{a_1, b_1, c_2, c_3\}$ and since we have at least 5 colors, there is a color c_1 such that $c_1 \notin \{a_1, b_1, c_2, c_3\}$. We then set $\sigma(v, vu_1) = \sigma(x_1, x_1u_1) = c_1$.

Therefore, in all cases we obtain a $(\Delta(G)+1, 1)$-incidence coloring of G, which contradicts our assumption.

We now associate with each vertex v of G an initial charge $d(v) = d_G(v)$, and we use the following discharging procedure:

(R1) each vertex of degree 3 gives $2/7$ to each of its 2-neighbors which has a 2-neighbor adjacent to a 3-vertex and gives 1/7 to its other 2-neighbors.

(R2) each vertex of degree at least 4 gives 2/7 to each of its 2-neighbors and gives 1/7 to each 2-vertex which is adjacent to one of its 2-neighbors.

We shall prove that the modernized degree d^* of each vertex of G is at least $16/7$ which contradicts the assumption $mad(G) < 16/7$. Let v be a vertex of G, we consider the possible cases for old degree $d_G(v)$ of v (since G does not contain the configuration 4(1), we have $d_G(v) \geq 2$):

$d_G(v) = 2$**.** In this case we consider five subcases:

1. v has two 2-neighbors, say z_1 and z_2. Let y_i be the other neighbor of z_i, $i = 1, 2$, in G. Since G does not contain the configuration 4(2), y_i is of degree $\Delta(G) \geq 4$ for $i = 1, 2$. Each y_i, $i = 1, 2$, gives $1/7$ to v so that $d^*(v) = 2 + 1/7 + 1/7 = 16/7$.
2. v is adjacent to a 3-vertex z_1 and a 2-vertex which is itself adjacent to a 3-vertex. In this case v receives $2/7$ from z_1 and we have $d^*(v) = 2 + 2/7 = 16/7$.
3. v is adjacent to a 3-vertex z_1 and a 2-vertex which is itself adjacent to a vertex z_2 of degree at least 4. In this case v receives $1/7$ from z_1 and $1/7$ from z_2 so that $d^*(v) = 2 + 1/7 + 1/7 = 16/7$.
4. v is adjacent to two 3-vertices that both gives $1/7$ to v so that $d^*(v) = 2 + 1/7 + 1/7 = 16/7$.
5. One of the two neighbors of v is of degree at least 4. In this case v receives at least $2/7$ so that $d^*(v) \geq 2 + 2/7 = 16/7$.

$d_G(v) = 3$**.**

Let u_1, u_2 and u_3 be the three neighbors of v. We consider two subcases according to the degrees of u_i's.

1. One of the u_i's is of degree at least 3, say u_1. In this case v gives at most $2/7$ to u_2 and $2/7$ to u_3 so that $d^*(v) \geq 3 - 2/7 - 2/7 = 17/7 \geq 16/7$.
2. All the u_i's are of degree 2. Let x_i be the other neighbor of u_i in G, $1 \leq i \leq 3$.
   (a) One of the x_i's is of degree at least 3, say x_1. In this case v gives $1/7$ to u_1, at most $2/7$ to u_2 and at most $2/7$ to u_3. We then have $d^*(v) \geq 3 - 1/7 - 2/7 - 2/7 = 16/7$.
   (b) All the x_i's are of degree 2. Let w_i be the other neighbor of x_i in G, $1 \leq i \leq 3$. Since G does not contain the configuration 4(2) we have $d_G(w_i) \geq 3$ for every i, $1 \leq i \leq 3$, and since G does not contain the configuration 4(3), at most one of the w_i's, $1 \leq i \leq 3$, can be of degree 3. Thus, we can suppose without loss of generality that $d_G(w_1)$ and $d_G(w_2) \geq 4$. In this case, v gives $1/7$ to w_1, $1/7$ to w_2 and at most $2/7$ to w_3. We then have $d^*(v) \geq 3 - 1/7 - 1/7 - 2/7 = 17/7 \geq 16/7$.

$d_G(v) = k \geq 4$.

In this case, v gives at most $k \times (2/7 + 1/7) = 3k/7$ so that $d^*(v) \geq k - 3k/7 = 4k/7 \geq 16/7$.

Therefore, every vertex in G gets a modernized degree of at least $16/7$ and the theorem is proved. □

Considering the lower bound discussed in Section 1, we get that the upper bound of Theorem 13 is tight.

Remark 14 For every graph G, the *square* of G, denoted by G^2, is the graph obtained from G by linking any two vertices at distance at most 2. It is easy to observe that providing a $(k, 1)$-incidence coloring of G is the same as providing a proper k-vertex-colouring of G^2, for every k (by identifying for every vertex v the color of A_v in G with the color of v in G^2). By considering the cycle C_4 on 4 four vertices (note that $C_4^2 = K_4$) we get that the previous result cannot be extended to the case $\Delta = 2$. Consider now the graph H obtained from the cycle C_5 on five vertices by adding one pending edge with a new vertex. Since H^2 contains a subgraph isomorphic to K_5, we similarly get that the previous result cannot be extended to the case $\Delta = 3$.

As previously, for planar graphs we obtain:

Corollary 15 *Every planar graph G of girth $g \geq 16$ and with $\Delta(G) \geq 4$ admits a $(\Delta(G) + 1, 1)$-incidence coloring. Therefore, $\chi_i(G) = \Delta(G) + 1$.*

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