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# On the maximum average degree and the incidence chromatic number of a graph

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We prove that the incidence chromatic number of every 3-degenerated graph  $G$  is at most  $\Delta(G) + 4$ . It is known that the incidence chromatic number of every graph  $G$  with maximum average degree  $mad(G) < 3$  is at most  $\Delta(G) + 3$ . We show that when  $\Delta(G) \geq 5$ , this bound may be decreased to  $\Delta(G) + 2$ . Moreover, we show that for every graph  $G$  with  $mad(G) < 22/9$  (resp. with  $mad(G) < 16/7$  and  $\Delta(G) \geq 4$ ), this bound may be decreased to  $\Delta(G) + 2$  (resp. to  $\Delta(G) + 1$ ).

**Keywords:** incidence coloring,  $k$ -degenerated graph, planar graph, maximum average degree

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## 1 Introduction

The concept of incidence coloring was introduced by Brualdi and Massey (3) in 1993.

Let  $G = (V(G), E(G))$  be a graph. An *incidence* in  $G$  is a pair  $(v, e)$  with  $v \in V(G)$ ,  $e \in E(G)$ , such that  $v$  and  $e$  are incident. We denote by  $I(G)$  the set of all incidences in  $G$ . For every vertex  $v$ , we denote by  $I_v$  the set of incidences of the form  $(v, vw)$  and by  $A_v$  the set of incidences of the form  $(w, wv)$ . Two incidences  $(v, e)$  and  $(w, f)$  are *adjacent* if one of the following holds: (i)  $v = w$ , (ii)  $e = f$  or (iii) the edge  $vw$  equals  $e$  or  $f$ .

A  *$k$ -incidence coloring* of a graph  $G$  is a mapping  $\sigma$  of  $I(G)$  to a set  $C$  of  $k$  colors such that adjacent incidences are assigned distinct colors. The *incidence chromatic number*  $\chi_i(G)$  of  $G$  is the smallest  $k$  such that  $G$  admits a  $k$ -incidence coloring.

For a graph  $G$ , let  $\Delta(G)$ ,  $\delta(G)$  denote the maximum and minimum degree of  $G$  respectively. It is easy to observe that for every graph  $G$  we have  $\chi_i(G) \geq \Delta(G) + 1$  (for a vertex  $v$  of degree  $\Delta(G)$  we must use  $\Delta(G)$  colors for coloring  $I_v$  and at least one additional color for coloring  $A_v$ ). Brualdi and Massey proved the following upper bound:

**Theorem 1** (3) For every graph  $G$ ,  $\chi_i(G) \leq 2\Delta(G)$ .

Guiduli (4) showed that the concept of incidence coloring is a particular case of directed star arboricity, introduced by Algor and Alon (1). Following an example from (1), Guiduli proved that there exist graphs  $G$  with  $\chi_i(G) \geq \Delta(G) + \Omega(\log \Delta(G))$ . He also proved that For every graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + O(\log \Delta(G))$ .

Concerning the incidence chromatic number of special classes of graphs, the following is known:

- For every  $n \geq 2$ ,  $\chi_i(K_n) = n = \Delta(K_n) + 1$  (3).
- For every  $m \geq n \geq 2$ ,  $\chi_i(K_{m,n}) = m + 2 = \Delta(K_{m,n}) + 2$  (3).
- For every tree  $T$  of order  $n \geq 2$ ,  $\chi_i(T) = \Delta(T) + 1$  (3).
- For every Halin graph  $G$  with  $\Delta(G) \geq 5$ ,  $\chi_i(G) = \Delta(G) + 1$  (8).
- For every  $k$ -degenerated graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + 2k - 1$  (5).
- For every  $K_4$ -minor free graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + 2$  and this bound is tight (5).
- For every cubic graph  $G$ ,  $\chi_i(G) \leq 5$  and this bound is tight (6).
- For every planar graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + 7$  (5).

The *maximum average degree* of a graph  $G$ , denoted by  $mad(G)$ , is defined as the maximum of the average degrees  $ad(H) = 2 \cdot |E(H)|/|V(H)|$  taken over all the subgraphs  $H$  of  $G$ .

In this paper we consider the class of 3-degenerated graphs (recall that a graph  $G$  is  $k$ -degenerated if  $\delta(H) \leq k$  for every subgraph  $H$  of  $G$ ), which includes for instance the class of triangle-free planar graphs and the class of graphs with maximum average degree at most 3. More precisely, we shall prove the following:

1. If  $G$  is a 3-degenerated graph, then  $\chi_i(G) \leq \Delta(G) + 4$  (Theorem 2).
2. If  $G$  is a graph with  $mad(G) < 3$ , then  $\chi_i(G) \leq \Delta(G) + 3$  (Corollary 5).
3. If  $G$  a graph with  $mad(G) < 3$  and  $\Delta(G) \geq 5$ , then  $\chi_i(G) \leq \Delta(G) + 2$  (Theorem 8).
4. If  $G$  is a graph with  $mad(G) < 22/9$ , then  $\chi_i(G) \leq \Delta(G) + 2$  (Theorem 11).
5. If  $G$  is a graph with  $mad(G) < 16/7$  and  $\Delta(G) \geq 4$ , then  $\chi_i(G) = \Delta(G) + 1$  (Theorem 13).

In fact we shall prove something stronger, namely that one can construct for these classes of graphs incidence colorings such that for every vertex  $v$ , the number of colors that are used on the incidences of the form  $(w, vw)$  is bounded by some fixed constant not depending on the maximum degree of the graph.

More precisely, we define a  $(k, \ell)$ -incidence coloring of a graph  $G$  as a  $k$ -incidence coloring  $\sigma$  of  $G$  such that for every vertex  $v \in V(G)$ ,  $|\sigma(A_v)| \leq \ell$ .

We end this section by introducing some notation that we shall use in the rest of the paper.

Let  $G$  be a graph. If  $v$  is a vertex in  $G$  and  $vw$  is an edge in  $G$ , we denote by  $N_G(v)$  the set of neighbors of  $v$ , by  $d_G(v) = |N_G(v)|$  the degree of  $v$ , by  $G \setminus v$  the graph obtained from  $G$  by deleting the vertex  $v$  and by  $G \setminus vw$  the graph obtained from  $G$  by deleting the edge  $vw$ .

Let  $G$  be a graph and  $\sigma'$  a *partial* incidence coloring of  $G$ , that is an incidence coloring only defined on some subset  $I$  of  $I(G)$ . For every uncolored incidence  $(v, vw) \in I(G) \setminus I$ , we denote by  $F_G^{\sigma'}(v, vw)$  the set of *forbidden colors* of  $(v, vw)$ , that is:

$$F_G^{\sigma'}(v, vw) = \sigma'(A_v) \cup \sigma'(I_v) \cup \sigma'(I_w).$$

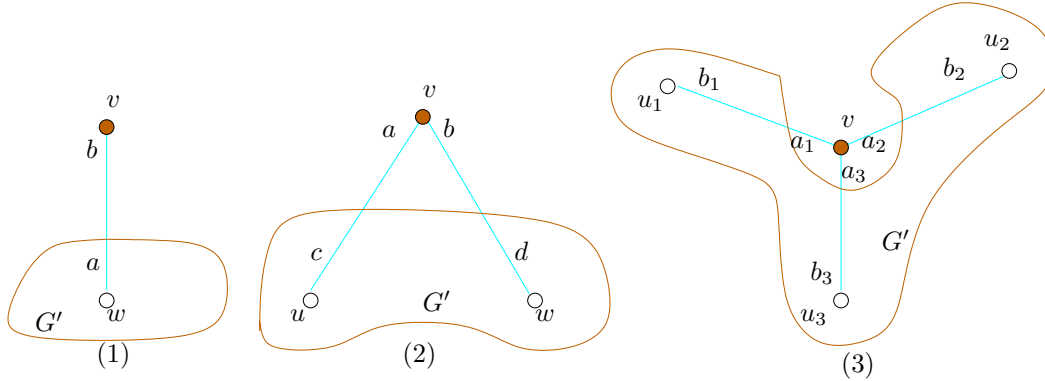


Fig. 1: Configurations for the proof of Theorem 2

We shall often say that we extend such a partial incidence coloring  $\sigma'$  to some incidence coloring  $\sigma$  of  $G$ . In that case, it should be understood that we set  $\sigma(v, vw) = \sigma'(v, vw)$  for every incidence  $(v, vw) \in I$ .

We shall make extensive use of the fact that every  $(k, \ell)$ -incidence coloring may be viewed as a  $(k', \ell)$ -incidence coloring for any  $k' > k$ .

**Drawing convention.** In a figure representing a forbidden configuration, all the neighbors of “black” or “grey” vertices are drawn, whereas “white” vertices may have other neighbors in the graph.

## 2 3-degenerated graphs

In this section, we prove the following:

**Theorem 2** *Every 3-degenerated graph  $G$  admits a  $(\Delta(G)+4, 3)$ -incidence coloring. Therefore,  $\chi_i(G) \leq \Delta(G) + 4$ .*

**Proof:** Let  $G$  be a 3-degenerated graph. Observe first that if  $\Delta(G) \leq 3$  then, by Theorem 1,  $\chi_i(G) \leq 2\Delta(G) < \Delta(G) + 4 \leq 7$  and every  $(\Delta(G) + 4)$ -incidence coloring of  $G$  is obviously a  $(\Delta(G) + 4, 3)$ -incidence coloring.

Therefore, we assume  $\Delta(G) \geq 4$  and we prove the theorem by induction on the number of vertices of  $G$ . If  $G$  has at most 5 vertices then  $G \subseteq K_5$ . Since for every  $k > 0$ ,  $\chi_i(K_n) = n$ , we obtain  $\chi_i(G) \leq \chi_i(K_5) = \Delta(K_5) + 1 = 5$ , and every 5-incidence coloring of  $G$  is obviously a  $(\Delta(G) + 4, 3)$ -incidence coloring. We assume now that  $G$  has  $n + 1$  vertices,  $n \geq 5$ , and that the theorem is true for all 3-degenerated graphs with at most  $n$  vertices.

Let  $v$  be a vertex of  $G$  with minimum degree. Since  $G$  is 3-degenerated, we have  $d_G(v) \leq 3$ . We consider three cases according to  $d_G(v)$ .

$d_G(v) = 1$ :

Let  $w$  denote the unique neighbor of  $v$  in  $G$  (see Figure 1.(1)). Due to the induction hypothesis, the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring of  $G$ . Since  $|F_G^{\sigma'}(w, vw)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G) - 1 + 3 = \Delta(G) + 2$ ,

there is a color  $a$  such that  $a \notin F_G^{\sigma'}(w, vw)$ . We then set  $\sigma(w, vw) = a$  and  $\sigma(v, vw) = b$ , for any color  $b$  in  $\sigma'(A_w)$ .

$d_G(v) = 2$ :

Let  $u, w$  be the two neighbors of  $v$  in  $G$  (see Figure 1.(2)). Due to the induction hypothesis, the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  as follows. We first set  $\sigma(v, vu) = a$  for a color  $a \in \sigma'(A_u)$  (if  $d_G(u) = 1$ , we have the case 1). Now, if  $|\sigma'(A_w)| \geq 2$ , there is a color  $b \in \sigma'(A_w) \setminus \{a\}$  and if  $|\sigma'(A_w)| = 1$ , since  $|F_G^{\sigma'}(v, vw)| = |\sigma'(I_w) \cup \{a\}| \leq \Delta(G) - 1 + 1 = \Delta(G)$ , there is a color  $b$  distinct from  $a$  such that  $b \notin F_G^{\sigma'}(v, vw)$ . We set  $\sigma(v, vw) = b$ .

We still have to color the two incidences  $(u, uv)$  and  $(w, wv)$ . Since  $a \in \sigma'(A_u)$ , we have  $|F_G^{\sigma'}(u, uv)| = |\sigma'(I_u) \cup \sigma'(A_u) \cup \{a, b\}| \leq \Delta(G) - 1 + 3 + 2 - 1 = \Delta(G) + 3$ . Therefore, there is a color  $c$  such that  $c \notin F_G^{\sigma'}(u, uv)$ . Similarly, since  $b \in \sigma'(A_w)$ , we have  $|F_G^{\sigma'}(w, wv)| \leq \Delta(G) + 3$  and there exists a color  $d$  such that  $d \notin F_G^{\sigma'}(w, wv)$ . We can extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  by setting  $\sigma(u, uv) = c$  and  $\sigma(w, wv) = d$ .

$d_G(v) = 3$ :

Let  $u_1, u_2$  and  $u_3$  be the three neighbors of  $v$  in  $G$  (see Figure 1.(3)). Due to the induction hypothesis, the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma'$ .

Observe first that for every  $i, 1 \leq i \leq 3$ , since  $|F_G^{\sigma'}(v, vu_i)| \leq \Delta(G) - 1$  and since we have  $\Delta(G) + 4$  colors, we have at least five colors which are not in  $F_G^{\sigma'}(v, vu_i)$ . Moreover, if  $|A_{u_i}| < 3$  then any of these five colors may be assigned to the incidence  $(v, vu_i)$  whereas we have only three possible choices (among these five) if  $|A_{u_i}| = 3$ . In the following, we shall see that having only three available colors is enough, and therefore assume that  $|\sigma'(A_{u_i})| = 3$  for every  $i, 1 \leq i \leq 3$ .

We define the sets  $B$  and  $B_{i,j}$  as follows:

$$\begin{aligned} & - \forall i, j, 1 \leq i, j \leq 3, i \neq j, B_{i,j} := (\sigma'(I_{u_i}) \cup \sigma'(A_{u_i})) \cap \sigma'(A_{u_j}) \\ & - B := \bigcup_{1 \leq i, j \leq 3} B_{i,j}, i \neq j. \end{aligned}$$

We consider now four subcases according to the degrees of  $u_1, u_2$  and  $u_3$ :

1.  $\forall i, 1 \leq i \leq 3, d_G(u_i) < \Delta(G)$ .

In this case, since we have 3 colors for the incidence  $(v, vu_i)$  for every  $i, 1 \leq i \leq 3$ , we can find 3 distinct colors  $a_1, a_2, a_3$  such that  $a_i \notin F_G^{\sigma'}(v, vu_i)$ . We set  $\sigma(v, vu_i) = a_i$  for every  $i, 1 \leq i \leq 3$ .

We still have to color the three incidences  $(u_i, u_i v), 1 \leq i \leq 3$ . Since  $a_i \in \sigma'(A_{u_i})$ , we have  $|F_G^{\sigma'}(u_i, u_i v)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}| \leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3$  for every  $i, 1 \leq i \leq 3$ . So, there exist three colors  $b_1, b_2, b_3$  such that  $b_i \notin F_G^{\sigma'}(u_i, u_i v), 1 \leq i \leq 3$ . We can extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  by setting  $\sigma(u_i, u_i v) = b_i$  for every  $i, 1 \leq i \leq 3$ .

2. Only one of the vertices  $u_i$  is of degree  $\Delta(G)$ .

We can suppose without loss of generality that  $d_G(u_1), d_G(u_2) < \Delta(G)$  and  $d_G(u_3) = \Delta(G)$ .

Since  $|\sigma'(I_{u_3}) \cup \sigma'(A_{u_3})| = \Delta(G) - 1 + 3 = \Delta(G) + 2$  and  $|\sigma'(A_{u_1})| = 3$ , we have  $B_{3,1} \neq \emptyset$ . Let  $a_1 \in B_{3,1}$ . Since  $|\sigma'(A_{u_i})| = 3$  for every  $i, 1 \leq i \leq 3$ , there exist two distinct colors  $a_2$  and  $a_3$  distinct from  $a_1$  such that  $a_2 \in \sigma'(A_{u_2})$  and  $a_3 \in \sigma'(A_{u_3})$ . We set  $\sigma(v, vu_i) = a_i$  for every  $i, 1 \leq i \leq 3$ .

We still have to color the three incidences of form  $(u_i, u_iv)$ . Since  $a_1 \in B_{3,1}$  and  $a_3 \in \sigma'(A_{u_3})$  we have:

$$\begin{aligned} |F_G^\sigma(u_3, u_3v)| &= |\sigma'(I_{u_3}) \cup \sigma(A_{u_3}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta + 3 \end{aligned}$$

and since  $a_i \in \sigma'(A_{u_i})$  for every  $i = 1, 2$  we have:

$$\begin{aligned} |F_G^{\sigma'}(u_i, u_iv)| &= |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta + 3. \end{aligned}$$

Therefore, there exist three colors  $b_1, b_2, b_3$  such that  $b_i \notin F_G^\sigma(u_i, u_iv) \cup \{a_1, a_2, a_3\}, 1 \leq i \leq 3$ . We can extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  by setting  $\sigma(u_i, u_iv) = b_i$  for every  $i, 1 \leq i \leq 3$ .

3. Only one vertex among the  $u_i$ 's is of degree less than  $\Delta(G)$ .

We can suppose without loss of generality that  $d_G(u_1) < \Delta(G)$  and  $d_G(u_2) = d_G(u_3) = \Delta(G)$ .

Similarly to the previous case, we have  $B_{2,1} \neq \emptyset$  and  $B_{3,2} \neq \emptyset$ . We consider two cases:

$B_{2,1} \neq B_{3,2}$

Let  $a_1 \in B_{2,1}, a_2 \in B_{3,2} \setminus \{a_1\}$  and  $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$ . We set  $\sigma(v, vu_i) = a_i$  for every  $i, 1 \leq i \leq 3$ .

We still have to color the three incidences  $(u_i, u_iv), 1 \leq i \leq 3$ . Since  $a_1 \in \sigma'(A_{u_1})$  we have:

$$\begin{aligned} |F_G^\sigma(u_1, u_1v)| &= |\sigma'(I_{u_1}) \cup \sigma(A_{u_1}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3 \end{aligned}$$

and since  $a_i \in B_{i+1,i}$  for  $i = 1, 2$  and  $a_j \in \sigma'(A_{u_j})$  for  $j = 2, 3$ , we have:

$$\begin{aligned} |F_G^\sigma(u_i, u_iv)| &= |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3. \end{aligned}$$

Therefore, there exist three colors  $b_1, b_2, b_3$  such that  $b_i \notin F_G^\sigma(u_i, u_iv), 1 \leq i \leq 3$ . We can extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  by setting  $\sigma(u_i, u_iv) = b_i$  for every  $i, 1 \leq i \leq 3$ .

$B_{2,1} = B_{3,2}$

Let  $a_1 \in B_{2,1} = B_{3,2}, a_2 \in \sigma'(A_{u_2}) \setminus \{a_1\}$  and  $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$ . We set  $\sigma(v, vu_i) = a_i$  for every  $i, 1 \leq i \leq 3$ .

We still have to color the three incidences  $(u_i, u_iv)$ ,  $1 \leq i \leq 3$ . Since  $a_1 \in \sigma'(A_{u_1})$  we have:

$$\begin{aligned} |F_G^\sigma(u_1, u_1v)| &= |\sigma'(I_{u_1}) \cup \sigma(A_{u_1}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 2 + 3 + 3 - 1 = \Delta(G) + 3 \end{aligned}$$

and since  $a_1 \in B_{2,1} = B_{3,2}$  and  $a_j \in \sigma'(A_{u_j})$  for  $j = 2, 3$ , we have:

$$\begin{aligned} |F_G^\sigma(u_i, u_iv)| &= |\sigma'(I_{u_i}) \cup \sigma(A_{u_i}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3. \end{aligned}$$

Therefore, there exist three colors  $b_1, b_2, b_3$  such that  $b_i \notin F_G^\sigma(u_i, u_iv)$ ,  $1 \leq i \leq 3$ . We can extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  by setting  $\sigma(u_i, u_iv) = b_i$  for every  $i$ ,  $1 \leq i \leq 3$ .

4.  $d_G(u_1) = d_G(u_2) = d_G(u_3) = \Delta(G)$ .

Similarly to the case (b) we have  $B_{i,j} \neq \emptyset$  for every  $i$  and  $j$ ,  $1 \leq i, j \leq 3$  and thus  $|B| \geq 1$ . We prove first that in this case  $|B| \geq 2$ . Suppose that  $|B| = |\{x\}| = 1$ ; in other words,  $(\sigma'(I_{u_i}) \cup A'_{u_i}) \cap A'_{u_j} = \{x\}$  for every  $i$  and  $j$ ,  $1 \leq i, j \leq 3$ . Thus we have:

$$\begin{aligned} |\sigma'(A_{u_1}) \cup \sigma'(I_{u_1}) \cup \sigma'(A_{u_2}) \cup \sigma'(A_{u_3})| &= \Delta(G) - 1 + 3 + 3 + 3 - 1 - 1 \\ &= \Delta(G) + 6. \end{aligned} \tag{1}$$

But the relation (1) is in contradiction with the fact that  $\sigma'$  is a  $(\Delta(G) + 4, 3)$ -incidence coloring and we then get  $|B| \geq 2$ .

Let  $a_1$  and  $a_2$  be two distinct colors in  $B$ . We can suppose without loss of generality that  $a_1 \in B_{2,1}$  and  $a_2 \in B_{3,2}$ .

We consider the two following subcases:

$$B_{1,3} \setminus \{a_1, a_2\} \neq \emptyset$$

Let  $a_3$  be a color in  $B_{1,3} \setminus \{a_1, a_2\}$ . We set  $\sigma(v, vu_i) = a_i$  for every  $i$ ,  $1 \leq i \leq 3$ .

Since  $a_i \in B_{j,i} = (\sigma'(I_{u_j}) \cup \sigma'(A_{u_j})) \cap \sigma'(A_{u_i})$ ,  $j = i + 1 \pmod{3}$ , and  $a_i \in \sigma'(A_{u_i})$  for every  $i$ ,  $1 \leq i \leq 3$ , we have:

$$\begin{aligned} |F_G^\sigma(u_i, u_iv)| &= |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3. \end{aligned}$$

Therefore, there exist three colors  $b_1, b_2, b_3$  such that  $b_i \notin F_G^\sigma(u_i, u_iv)$ ,  $1 \leq i \leq 3$ . We can extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  by setting  $\sigma(u_i, u_iv) = b_i$  for every  $i$ ,  $1 \leq i \leq 3$ .

$$B_{1,3} \setminus \{a_1, a_2\} = \emptyset$$

Since  $B_{1,3} \neq \emptyset$  we can suppose without loss of generality that  $a_2 \in B_{1,3}$ . Let  $a_3 \in \sigma'(A_{u_3}) \setminus \{a_1, a_2\}$ . We set  $\sigma(v, vu_i) = a_i$  for every  $i$ ,  $1 \leq i \leq 3$ .

Since  $a_i \in B_{j,i} = (\sigma'(I_{u_j}) \cup \sigma'(A_{u_j})) \cap \sigma'(A_{u_i})$ ,  $j = i + 1 \pmod{3}$ , and  $a_i \in \sigma'(A_{u_i})$  for  $i = 1, 2$ , we have:

$$|F_G^\sigma(u_i, u_iv)| = |\sigma'(I_{u_i}) \cup \sigma'(A_{u_i}) \cup \{a_1, a_2, a_3\}|$$

$$\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3$$

and since  $a_2 \in \sigma'(I_{u_1}) \cup \sigma'(A_{u_1})$  and  $a_1 \in \sigma'(A - u_1)$  we have:

$$\begin{aligned} |F_G^\sigma(u_1, u_1v)| &= |\sigma'(I_{u_1}) \cup \sigma'(A_{u_1}) \cup \{a_1, a_2, a_3\}| \\ &\leq \Delta(G) - 1 + 3 + 3 - 1 - 1 = \Delta(G) + 3. \end{aligned}$$

Therefore, there exist three colors  $b_1, b_2, b_3$  such that  $b_i \notin F_G^\sigma(u_i, u_iv)$ ,  $1 \leq i \leq 3$ . We can extend  $\sigma'$  to a  $(\Delta(G) + 4, 3)$ -incidence coloring  $\sigma$  of  $G$  by setting  $\sigma(u_i, u_iv) = b_i$  for every  $i$ ,  $1 \leq i \leq 3$ .

It is easy to check that in all cases we obtain a  $(\Delta(G) + 4, 3)$ -incidence coloring of  $G$  and the theorem is proved.  $\square$

Since every triangle free planar graph is 3-degenerated, we have:

**Corollary 3** For every triangle free planar graph  $G$ ,  $\chi_i(G) \leq \Delta(G) + 4$ .

### 3 Graphs with bounded maximum average degree

In this section we study the incidence chromatic number of graphs with bounded maximum average degree. The following result has been proved in (5).

**Theorem 4** Every  $k$ -degenerated graph  $G$  admits a  $(\Delta(G) + 2k - 1, k)$ -incidence coloring.

Since every graph  $G$  with  $mad(G) < 3$  is 2-degenerated, we get the following:

**Corollary 5** Every graph  $G$  with  $mad(G) < 3$  admits a  $(\Delta(G) + 3, 2)$ -incidence coloring. Therefore,  $\chi_i(G) \leq \Delta(G) + 3$ .

Concerning planar graphs, we have the following:

**Observation 6** (2) For every planar graph  $G$  with girth at least  $g$ ,  $mad(G) < 2g/(g - 2)$ .

Hence, we obtain:

**Corollary 7** Every planar graph  $G$  with girth  $g \geq 6$  admits a  $(\Delta(G) + 3, 2)$ -incidence coloring. Therefore,  $\chi_i(G) \leq \Delta(G) + 3$ .

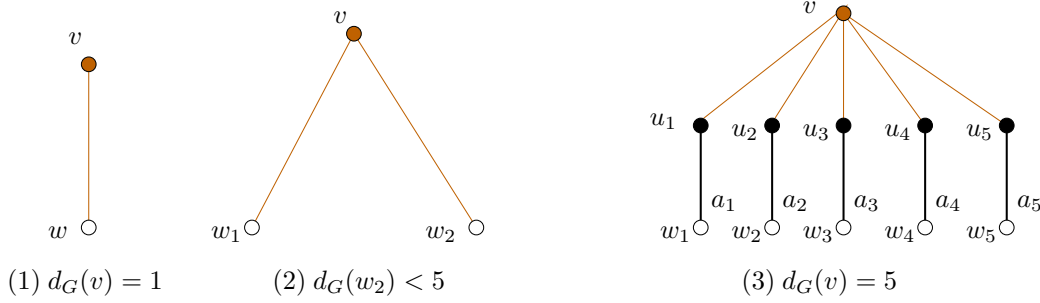
**Proof:** By Observation 6 we have  $mad(G) < 2g/(g - 2) \leq (2 \times 6)/(6 - 2) = 3$  and we get the result from Corollary 5.  $\square$

If the graph has maximum degree at least 5, the previous result can be improved:

**Theorem 8** Every graph  $G$  with  $mad(G) < 3$  and  $\Delta(G) \geq 5$  admits a  $(\Delta(G) + 2, 2)$ -incidence coloring. Therefore,  $\chi_i(G) \leq \Delta(G) + 2$ .

**Proof:** Suppose that the theorem is false and let  $G$  be a minimal counter-example (with respect to the number of vertices). We first show that  $G$  must avoid all the configurations depicted in Fig. 2.





**Fig. 2:** Forbidden configurations for the proof of Theorem 8

- (1) Let  $w$  denote the unique neighbor of  $v$  in  $G$ . Due to the minimality of  $G$ , the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma$  of  $G$ . Since  $|F_G^\sigma(w, wv)| = |\sigma'(I_w) \cup \sigma'(A_w)| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$ , there is a color  $a$  such that  $a \notin F_G^\sigma(w, wv)$ . We set  $\sigma(w, wv) = a$  and  $\sigma(v, vw) = b$ , for any color  $b$  in  $\sigma'(A_w)$ .
- (2) Let  $w_1, w_2$  denote the two neighbors of  $v$  in  $G$ . Due to the minimality of  $G$ , the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma$  of  $G$ .

Since  $|F_G^\sigma(w_1, w_1v)| = |\sigma'(I_{w_1}) \cup \sigma'(A_{w_1})| \leq \Delta(G) - 1 + 2 = \Delta(G) + 1$  and since we have  $\Delta(G) + 2$  possible colors, there is a color  $a$  such that  $a \notin F_G^\sigma(w_1, w_1v)$ . We set  $\sigma(w_1, w_1v) = a$ . If  $|\sigma'(A_{w_2}) \setminus \{a\}| \geq 1$  then there is a color  $b \in \sigma'(A_{w_2}) \setminus \{a\}$  and if  $\sigma'(A_{w_2}) = \{a\}$ , since  $|F_G^\sigma(v, vw_2)| = |\sigma'(I_{w_2}) \cup \{a\}| \leq 3 + 1 = 4 \leq \Delta(G) - 1$ , there is a color  $b$  such that  $b \notin F_G^\sigma(v, vw_2)$ . We set  $\sigma(v, vw_2) = b$ .

Now, if  $|\sigma'(A_{w_1}) \setminus \{b\}| \geq 1$  then there is a color  $c \in \sigma'(A_{w_1}) \setminus \{b\}$  and if  $\sigma'(A_{w_1}) = \{b\}$ , since  $|F_G^\sigma(v, vw_1)| = |\sigma(I_{w_1}) \cup \{b\}| \leq \Delta(G) + 1$ , there is a color  $c$  such that  $c \notin F_G^\sigma(v, vw_1)$ . We set  $\sigma(v, vw_1) = c$ .

Since  $|F_G^\sigma(w_2, w_2v)| = |\sigma'(I_{w_2}) \cup \sigma(A_{w_2}) \cup \{c\}| \leq 3 + 2 + 1 = 6 \leq \Delta(G) + 1$ , there is a color  $d$  such that  $d \notin F_G^\sigma(w_2, w_2v)$  and we set  $\sigma(w_2, w_2v) = d$ .

- (3) Let  $u_i, 1 \leq i \leq 5$ , denote the five neighbors of  $v$  and  $w_i$  denote the other neighbor of  $u_i$  in  $G$  (see Figure 2.(3)). Due to the minimality of  $G$ , the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma$  of  $G$ .

Let  $a_i = \sigma'(w_i, w_iu_i), 1 \leq i \leq 5$ . Since we have  $\Delta(G) + 2 \geq 7$  colors, there is a color  $x$  distinct from  $a_i$  for every  $i, 1 \leq i \leq 5$ .

Since  $|F_G^{\sigma'}(u_i, u_iw_i)| = |\sigma'(I_{w_i})| \leq \Delta(G)$  we have two possible colors for the incidence  $(u_i, u_iw_i)$  for every  $i, 1 \leq i \leq 5$ . So, we can suppose that  $\sigma'(u_i, u_iw_i) \neq x$  for every  $i, 1 \leq i \leq 5$ . We set  $\sigma(u_i, u_iv) = x$  for every  $i, 1 \leq i \leq 5$ .

Since  $F_G^\sigma(v, vu_i) = \{x, \sigma'(u_i, u_iw_i)\}$  for every  $i, 1 \leq i \leq 5$ , and since we have at least 7 colors, there is 5 distinct colors  $c_1, c_2, \dots, c_5$  such that  $c_i \notin \{x, \sigma'(u_i, u_iw_i)\}, 1 \leq i \leq 5$ , and we set  $\sigma(v, vu_i) = c_i$  for every  $i, 1 \leq i \leq 5$ .

It is easy to check that in every case we have obtained a  $(\Delta(G) + 2, 2)$ -incidence coloring of  $G$ , which contradicts our assumption.

We now associate with each vertex  $v$  of  $G$  an initial charge  $d(v) = d_G(v)$ , and we use the following discharging procedure: each vertex of degree at least 5 gives  $1/2$  to each of its 2-neighbors.

We shall prove that the modernized degree  $d^*$  of each vertex of  $G$  is at least 3 which contradicts the assumption  $mad(G) < 3$  (since  $\sum_{u \in G} d^*(u) = \sum_{u \in G} d(u)$ ). Let  $v$  be a vertex of  $G$ ; we consider the possible cases for old degree  $d_G(v)$  of  $v$  (since  $G$  does not contain the configuration 2(1), we have  $d_G(v) \geq 2$ ):

$$d_G(v) = 2.$$

Since  $G$  does not contain the configuration 2(2) the two neighbors of  $v$  are of degree at least 5. Therefore,  $v$  receives  $1/2$  from each of its neighbors so that  $d^*(v) = 2 + 1/2 + 1/2 = 3$ .

$$3 \leq d_G(v) \leq 4.$$

In this case we have  $d^*(v) = d_G(v) \geq 3$ .

$$d_G(v) = 5.$$

Since  $G$  does not contain the configuration 2(3) at least one of the neighbors of  $v$  is of degree at least 3 and  $v$  gives at most  $4 \times 1/2 = 2$ . We obtain  $d^*(v) \geq 5 - 2 = 3$ .

$$d_G(v) = k \geq 6.$$

In this case  $v$  gives at most  $k \times (1/2)$  so that  $d^*(v) \geq k - k/2 = k/2 \geq 6/2 = 3$ .

Therefore, every vertex in  $G$  gets a modernized degree of at least 3 and the theorem is proved.  $\square$

**Remark 9** The previous result also holds for graphs with maximum degree 2 and for graphs with maximum degree 3 (by the result from (6)) but the question remains open for graphs with maximal degree 4.

As previously, for planar graphs we obtain:

**Corollary 10** Every planar graph  $G$  of girth  $g \geq 6$  with  $\Delta(G) \geq 5$  admits a  $(\Delta(G) + 2, 2)$ -incidence coloring. Therefore,  $\chi_i(G) \leq \Delta(G) + 2$ .

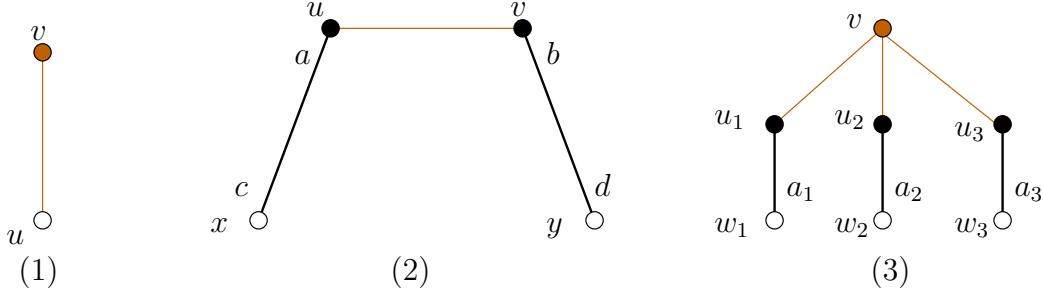
For graphs with maximum average degree less than  $22/9$ , we have:

**Theorem 11** Every graph  $G$  with  $mad(G) < 22/9$  admits a  $(\Delta(G)+2, 2)$ -incidence coloring. Therefore,  $\chi_i(G) \leq \Delta(G) + 2$ .

**Proof:** It is enough to consider the case of graphs with maximum degree at most 4, since for graphs with maximum degree at least 5 the theorem follows from Theorem 8. Suppose that the theorem is false and let  $G$  be a minimal counter-example (with respect to the number of vertices and edges). Observe first that we have  $\Delta(G) \geq 3$  since otherwise we obtain by Theorem 1 that  $\chi_i(G) \leq 2\Delta(G) \leq \Delta(G) + 2$  and every  $(\Delta(G) + 2)$ -incidence coloring of  $G$  is obviously a  $(\Delta(G) + 2, 2)$ -incidence coloring.

We first show that  $G$  cannot contain any of the configurations depicted in Figure 3.

(1) This case is similar to case 1 of Theorem 8.



**Fig. 3:** Forbidden configurations for the proof of Theorem 11

- (2) Let  $x$  (resp.  $y$ ) denote the other neighbor of  $u$  (resp.  $v$ ) in  $G$ . Due to the minimality of  $G$ , the graph  $G' = G \setminus uv$  admits a  $(\Delta(G)+2, 2)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G)+2, 2)$ -incidence coloring  $\sigma$  of  $G$ .

Suppose  $\sigma'(u, ux) = a$ ,  $\sigma'(v, vy) = b$ ,  $\sigma'(x, xu) = c$  and  $\sigma'(y, yv) = d$ .

Suppose first that  $|\{a, b, c, d\}| = 4$ . In that case, we set  $\sigma(u, uv) = d$  and  $\sigma(v, vu) = c$ .

Now, if  $|\{a, b, c, d\}| \leq 3$ , we set  $\sigma(u, uv) = e$  and  $\sigma(v, vu) = f$  for any  $e, f \notin \{a, b, c, d\}$ .

- (3) Let  $u_1, u_2$  and  $u_3$  denote the three neighbors of  $v$  and  $w_i$  denotes the other neighbor of  $u_i$ ,  $1 \leq i \leq 3$ , in  $G$ . Due to the minimality of  $G$ , the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 2, 2)$ -incidence coloring  $\sigma$  of  $G$ .

Suppose that  $a_i = \sigma'(w_i, w_i u_i)$ ,  $1 \leq i \leq 3$ . Since we have  $\Delta(G) + 2 \geq 5$  colors, there is a color  $x$  distinct from  $a_i$  for every  $i$ ,  $1 \leq i \leq 3$ .

Since  $|F_G^{\sigma'}(u_i, u_i w_i)| = |\sigma'(I_{w_i})| \leq \Delta(G)$  we have at least two colors for the incidence  $(u_i, u_i w_i)$  for every  $i$ ,  $1 \leq i \leq 3$ . Thus, we can suppose  $\sigma'(u_i, u_i w_i) \neq x$  for every  $i$ ,  $1 \leq i \leq 3$ . We then set  $\sigma(u_i, u_i v) = x$  for every  $i$ ,  $1 \leq i \leq 3$ .

Since  $F_G^\sigma(v, v u_i) = \{x, \sigma'(u_i, u_i w_i)\}$  for every  $i$ ,  $1 \leq i \leq 3$ , and since we have at least 5 colors, there are 3 distinct colors  $c_1, c_2$  et  $c_3$  such that  $c_i \notin \{x, \sigma'(u_i, u_i w_i)\}$ ,  $1 \leq i \leq 3$ . We then set  $\sigma(v, v u_i) = c_i$  for every  $i$ ,  $1 \leq i \leq 3$ .

Therefore, in all cases we obtain a  $(\Delta(G) + 2, 2)$ -incidence coloring of  $G$ , which contradicts our assumption.

We now associate with each vertex  $v$  of  $G$  an initial charge  $d(v) = d_G(v)$ , and we use the following discharging procedure: each vertex of degree at least 3 gives  $2/9$  to each of its 2-neighbors.

We shall prove that the modernized degree  $d^*$  of each vertex of  $G$  is at least  $22/9$  which contradicts the assumption  $mad(G) < 22/9$ . Let  $v$  be a vertex of  $G$ ; we consider the possible cases for old degree  $d_G(v)$  of  $v$  (since  $G$  does not contain the configuration 3(1), we have  $d_G(v) \geq 2$ ):

$d_G(v) = 2$ .

Since  $G$  does not contain the configuration 3(2) the two neighbors of  $v$  are of degree at least 3. Therefore,  $v$  receives then  $2/9$  from each of its neighbors so that  $d^*(v) = 2 + 2/9 + 2/9 = 22/9$ .

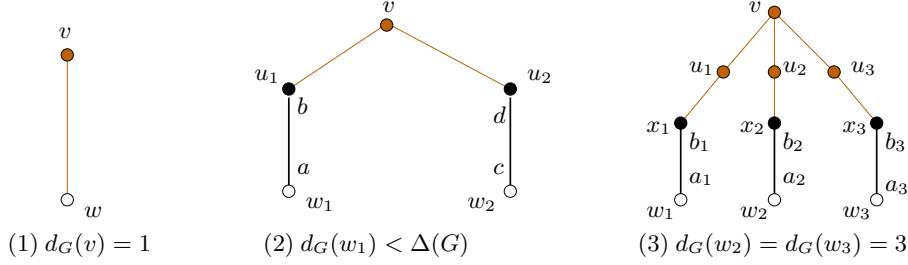


Fig. 4: Forbidden configurations for the proof of Theorem 13

$d_G(v) = 3$ .

Since  $G$  does not contain the configuration 3(3),  $v$  is adjacent to at most two 2-vertices and  $v$  gives at most  $2 \times 2/9 = 4/9$ . We obtain  $d^*(v) \geq 3 - 4/9 = 23/9 \geq 22/9$ .

$d_G(v) = 4$ .

In this case,  $v$  gives at most  $4 \times 2/9 = 8/9$  so that  $d^*(v) \geq 4 - 8/9 = 28/9 \geq 22/9$ .

Therefore, every vertex in  $G$  gets a modernized degree of at least 3 and the theorem is proved.  $\square$

By considering cycles of length  $\ell \not\equiv 0 \pmod{3}$ , we get that the upper bound of Theorem 11 is tight. As previously, for planar graphs we obtain:

**Corollary 12** *Every planar graph  $G$  of girth  $g \geq 11$  admits a  $(\Delta(G) + 2, 2)$ -incidence coloring. Therefore,  $\chi_i(G) \leq \Delta(G) + 2$ .*

Finally, for graphs with maximum average degree less than  $16/7$ , we have:

**Theorem 13** *Every graph  $G$  with  $mad(G) < 16/7$  and  $\Delta(G) \geq 4$  admits a  $(\Delta(G) + 1, 1)$ -incidence coloring. Therefore,  $\chi_i(G) = \Delta(G) + 1$ .*

**Proof:** Since for every graph  $G$ ,  $\chi_i(G) \geq \Delta(G) + 1$ , it is enough to prove that  $G$  admits a  $(\Delta(G) + 1, 1)$ -incidence coloring. Suppose that the theorem is false and let  $G$  be a minimal counter-example (with respect to the number of vertices). We first show that  $G$  cannot contain any of the configurations depicted in Figure 4.

- (1) This case is similar to case 1 of Theorem 8.
- (2) Let  $u_i, i = 1, 2$ , be the two neighbors of  $v$  and  $w_i$  denote the other neighbor of  $u_i$  in  $G$ . Due to the minimality of  $G$ , the graph  $G' = G \setminus v$  admits a  $(\Delta(G) + 1, 1)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 1, 1)$ -incidence coloring  $\sigma$  of  $G$ .

Suppose that  $\sigma'(w_1, w_1u_1) = a$ ,  $\sigma'(u_1, u_1w_1) = b$ ,  $\sigma'(w_2, w_2u_2) = c$  and  $\sigma'(u_2, u_2w_2) = d$ . Since  $|F_{G'}^{\sigma'}(w_1, w_1u_1) \cup \{c\}| = |\sigma'(I_{w_1}) \setminus \{a\} \cup \sigma'(A_{w_1}) \cup \{c\}| \leq \Delta(G) - 2 + 1 + 1 = \Delta(G)$ , we can suppose that  $a \neq c$ . We then set  $\sigma(v, vw_1) = a$  and  $\sigma(v, vw_2) = c$ .

Now, since  $F_G^\sigma(u_1, u_1v) \cup F_G^\sigma(u_2, u_2v) = \{a, b, c, d\}$  and since we have at least  $\Delta(G) + 1 \geq 5$  colors, there is a color  $x$  such that  $x \notin \{a, b, c, d\}$ . We then set  $\sigma(u_1, u_1v) = \sigma(u_2, u_2v) = x$ .

- (3) Let  $u_i$ ,  $1 \leq i \leq 3$  be the three neighbors of  $v$ ,  $x_i$  denote the other neighbor of  $u_i$  and  $w_i$  denote the other neighbor of  $x_i$  in  $G$ . Due to the minimality of  $G$ , the graph  $G' = G \setminus \{v, u_1, u_2, u_3\}$  admits a  $(\Delta(G) + 1, 1)$ -incidence coloring  $\sigma'$ . We extend  $\sigma'$  to a  $(\Delta(G) + 1, 1)$ -incidence coloring  $\sigma$  of  $G$ .

Suppose that  $\sigma'(w_i, w_i x_i) = a_i$  and  $\sigma'(x_i, x_i w_i) = b_i$  for every  $i$ ,  $1 \leq i \leq 3$ . Since  $|F_G^{\sigma'}(w_i, w_i x_i) \cup \{b_1\}| = |\sigma'(I_{w_i}) \setminus \{a_i\} \cup \{b_i, b_1\}| \leq 2 + 2 = 4$  for  $i = 2, 3$ , and since we have  $\Delta(G) + 1 \geq 5$  colors, we can suppose that  $a_2 \neq b_1 \neq a_3$ . We then set  $\sigma(u_i, u_i x_i) = a_i$  and  $\sigma(u_i, u_i v) = b_1$  for every  $i$ ,  $1 \leq i \leq 3$ .

Since  $F_G^\sigma(v, v u_j) \cup F_G^\sigma(x_j, x_j u_j) = \{b_1, b_j, a_j\}$  for  $j = 2, 3$ , there are two distinct colors  $c_2$  and  $c_3$  such that  $c_j \notin \{b_1, b_j, a_j\}$ ,  $j = 2, 3$ . We set  $\sigma(v, v u_j) = \sigma(x_j, x_j u_j) = c_j$ ,  $j = 2, 3$ .

Now, since  $F_G^{\sigma'}(v, v u_1) \cup F_G^{\sigma'}(x_1, x_1 u_1) = \{a_1, b_1, c_2, c_3\}$  and since we have at least 5 colors, there is a color  $c_1$  such that  $c_1 \notin \{a_1, b_1, c_2, c_3\}$ . We then set  $\sigma(v, v u_1) = \sigma(x_1, x_1 u_1) = c_1$ .

Therefore, in all cases we obtain a  $(\Delta(G) + 1, 1)$ -incidence coloring of  $G$ , which contradicts our assumption.

We now associate with each vertex  $v$  of  $G$  an initial charge  $d(v) = d_G(v)$ , and we use the following discharging procedure:

- (R1) each vertex of degree 3 gives  $2/7$  to each of its 2-neighbors which has a 2-neighbor adjacent to a 3-vertex and gives  $1/7$  to its other 2-neighbors.
- (R2) each vertex of degree at least 4 gives  $2/7$  to each of its 2-neighbors and gives  $1/7$  to each 2-vertex which is adjacent to one of its 2-neighbors.

We shall prove that the modernized degree  $d^*$  of each vertex of  $G$  is at least  $16/7$  which contradicts the assumption  $mad(G) < 16/7$ . Let  $v$  be a vertex of  $G$ , we consider the possible cases for old degree  $d_G(v)$  of  $v$  (since  $G$  does not contain the configuration 4(1), we have  $d_G(v) \geq 2$ ):

$d_G(v) = 2$ . In this case we consider five subcases:

1.  $v$  has two 2-neighbors, say  $z_1$  and  $z_2$ . Let  $y_i$  be the other neighbor of  $z_i$ ,  $i = 1, 2$ , in  $G$ . Since  $G$  does not contain the configuration 4(2),  $y_i$  is of degree  $\Delta(G) \geq 4$  for  $i = 1, 2$ . Each  $y_i$ ,  $i = 1, 2$ , gives  $1/7$  to  $v$  so that  $d^*(v) = 2 + 1/7 + 1/7 = 16/7$ .
2.  $v$  is adjacent to a 3-vertex  $z_1$  and a 2-vertex which is itself adjacent to a 3-vertex. In this case  $v$  receives  $2/7$  from  $z_1$  and we have  $d^*(v) = 2 + 2/7 = 16/7$ .
3.  $v$  is adjacent to a 3-vertex  $z_1$  and a 2-vertex which is itself adjacent to a vertex  $z_2$  of degree at least 4. In this case  $v$  receives  $1/7$  from  $z_1$  and  $1/7$  from  $z_2$  so that  $d^*(v) = 2 + 1/7 + 1/7 = 16/7$ .
4.  $v$  is adjacent to two 3-vertices that both gives  $1/7$  to  $v$  so that  $d^*(v) = 2 + 1/7 + 1/7 = 16/7$ .
5. One of the two neighbors of  $v$  is of degree at least 4. In this case  $v$  receives at least  $2/7$  so that  $d^*(v) \geq 2 + 2/7 = 16/7$ .

$d_G(v) = 3$ .

Let  $u_1, u_2$  and  $u_3$  be the three neighbors of  $v$ . We consider two subcases according to the degrees of  $u_i$ 's.

1. One of the  $u_i$ 's is of degree at least 3, say  $u_1$ . In this case  $v$  gives at most  $2/7$  to  $u_2$  and  $2/7$  to  $u_3$  so that  $d^*(v) \geq 3 - 2/7 - 2/7 = 17/7 \geq 16/7$ .
2. All the  $u_i$ 's are of degree 2. Let  $x_i$  be the other neighbor of  $u_i$  in  $G$ ,  $1 \leq i \leq 3$ .
  - (a) One of the  $x_i$ 's is of degree at least 3, say  $x_1$ . In this case  $v$  gives  $1/7$  to  $u_1$ , at most  $2/7$  to  $u_2$  and at most  $2/7$  to  $u_3$ . We then have  $d^*(v) \geq 3 - 1/7 - 2/7 - 2/7 = 16/7$ .
  - (b) All the  $x_i$ 's are of degree 2. Let  $w_i$  be the other neighbor of  $x_i$  in  $G$ ,  $1 \leq i \leq 3$ . Since  $G$  does not contain the configuration 4(2) we have  $d_G(w_i) \geq 3$  for every  $i$ ,  $1 \leq i \leq 3$ , and since  $G$  does not contain the configuration 4(3), at most one of the  $w_i$ 's,  $1 \leq i \leq 3$ , can be of degree 3. Thus, we can suppose without loss of generality that  $d_G(w_1)$  and  $d_G(w_2) \geq 4$ . In this case,  $v$  gives  $1/7$  to  $w_1$ ,  $1/7$  to  $w_2$  and at most  $2/7$  to  $w_3$ . We then have  $d^*(v) \geq 3 - 1/7 - 1/7 - 2/7 = 17/7 \geq 16/7$ .

$$d_G(v) = k \geq 4.$$

In this case,  $v$  gives at most  $k \times (2/7 + 1/7) = 3k/7$  so that  $d^*(v) \geq k - 3k/7 = 4k/7 \geq 16/7$ .

Therefore, every vertex in  $G$  gets a modernized degree of at least  $16/7$  and the theorem is proved.  $\square$

Considering the lower bound discussed in Section 1, we get that the upper bound of Theorem 13 is tight.

**Remark 14** For every graph  $G$ , the *square* of  $G$ , denoted by  $G^2$ , is the graph obtained from  $G$  by linking any two vertices at distance at most 2. It is easy to observe that providing a  $(k, 1)$ -incidence coloring of  $G$  is the same as providing a proper  $k$ -vertex-colouring of  $G^2$ , for every  $k$  (by identifying for every vertex  $v$  the color of  $A_v$  in  $G$  with the color of  $v$  in  $G^2$ ). By considering the cycle  $C_4$  on 4 four vertices (note that  $C_4^2 = K_4$ ) we get that the previous result cannot be extended to the case  $\Delta = 2$ . Consider now the graph  $H$  obtained from the cycle  $C_5$  on five vertices by adding one pending edge with a new vertex. Since  $H^2$  contains a subgraph isomorphic to  $K_5$ , we similarly get that the previous result cannot be extended to the case  $\Delta = 3$ .

As previously, for planar graphs we obtain:

**Corollary 15** Every planar graph  $G$  of girth  $g \geq 16$  and with  $\Delta(G) \geq 4$  admits a  $(\Delta(G) + 1, 1)$ -incidence coloring. Therefore,  $\chi_i(G) = \Delta(G) + 1$ .

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