HAL
archives-ouvertes.fr

Queue Layouts of Graph Products and Powers

David R. Wood

▶ To cite this version:

David R. Wood. Queue Layouts of Graph Products and Powers. Discrete Mathematics and Theoretical Computer Science, DMTCS, 2005, 7, pp.255-268. <hal-00959041>

HAL Id: hal-00959041
https://hal.inria.fr/hal-00959041

Submitted on 13 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Discrete Mathematics and Theoretical Computer Science DMTCS vol. **7**, 2005, 255–268

Queue Layouts of Graph Products and Powers[†]

David R. Wood

Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain.
`david.wood@upc.edu`

received May 26, 2005, *revised Nov 3, 2005*, *accepted Nov 7, 2005*.

A *k-queue layout* of a graph G consists of a linear order σ of $V(G)$, and a partition of $E(G)$ into k sets, each of which contains no two edges that are nested in σ. This paper studies queue layouts of graph products and powers.

Keywords: graph, queue layout, cartesian product, d-dimensional grid graph, d-dimensional toroidal grid graph, Hamming graph.

2000 MSC classification: 05C62 (graph representations)

1 Introduction

Let G be a graph. (All graphs considered are finite, simple and undirected.) The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *density* of G is $\eta(G) := |E(G)|/|V(G)|$.

A *vertex ordering* of G is a bijection $\sigma : V(G) \to \{1, 2, \dots, |V(G)|\}$. In a vertex ordering σ of G, let $L_\sigma(e)$ and $R_\sigma(e)$ denote the endpoints of each edge $e \in E(G)$ such that $\sigma(L_\sigma(e)) < \sigma(R_\sigma(e))$. Where the vertex ordering σ is clear from the context, we will abbreviate $L_\sigma(e)$ and $R_\sigma(e)$ by L_e and R_e, respectively. For edges e and f of G with no endpoint in common, there are the following three possible relations with respect to σ, as illustrated in Figure 1:

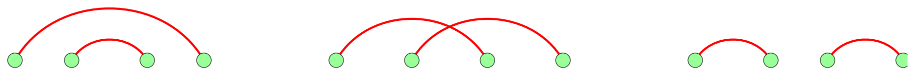
(a) e and f *nest* if $\sigma(L_e) < \sigma(L_f) < \sigma(R_f) < \sigma(R_e)$,

(b) e and f *cross* if $\sigma(L_e) < \sigma(L_f) < \sigma(R_e) < \sigma(R_f)$,

(c) e and f are *disjoint* if $\sigma(L_e) < \sigma(R_e) < \sigma(L_f) < \sigma(R_f)$.

[†]Supported by the Government of Spain grant MEC SB2003-0270, and by the projects MCYT-FEDER BFM2003-00368 and Gen. Cat 2001SGR00224.

1365–8050 © 2005 Discrete Mathematics and Theoretical Computer Science (DMTCS), Nancy, France



(a) nested (b) crossing (c) disjoint

Fig. 1: Relationships between pairs of edges with no common endpoint in a vertex ordering.

A *queue* in σ is a set of edges $Q \subseteq E(G)$ such that no two edges in Q are nested. Observe that when traversing σ from left to right, the left and right endpoints of the edges in a queue are reached in first-in-first-out order—hence the name 'queue'. Observe that $Q \subseteq E(G)$ is a queue if and only if for all edges $e, f \in Q$,

$$\begin{aligned} &\sigma(L_e) \le \sigma(L_f) \text{ and } \sigma(R_e) \le \sigma(R_f) \ , \\ \text{or } &\sigma(L_f) \le \sigma(L_e) \text{ and } \sigma(R_f) \le \sigma(R_e) \ . \end{aligned} \tag{1}$$

A k*-queue layout* of G is a pair

$$(\sigma, \{Q_1, Q_2, \dots, Q_k\})$$

where σ is a vertex ordering of G, and $\{Q_1, Q_2, \dots, Q_k\}$ is a partition of $E(G)$, such that each Q_i is a queue in σ. The *queue-number* of a graph G, denoted by $\mathsf{qn}(G)$, is the minimum k such that there is a k-queue layout of G.

Queue layouts were introduced by Heath et al. [15, 19]. Applications of queue layouts include sorting permutations [12, 20, 22, 24, 27], parallel process scheduling [3], matrix computations [23], and graph drawing [4, 6]. Other aspects of queue layouts have been studied in the literature [7, 9, 10, 13, 25, 26]. Queue layouts of directed graphs [5, 11, 17, 18] and posets [16] have also been investigated.

Table 1 describes the best known upper bounds on the queue-number of various classes of graphs. Planar graphs are an interesting class of graphs for which it is not known whether the queue-number is bounded (see [6, 23]).

This paper studies queue layouts of graph products and graph powers. To prove optimality we use the following lower bound by Heath and Rosenberg [19]. See Pemmaraju [23] and Dujmović and Wood [9] for slightly more exact lower bounds.

Lemma 1 ([19]) *Every graph G has queue-number $\mathsf{qn}(G) > \eta(G)/2$.*

This paper is organised as follows. In Section 2 we introduce the concepts of strict queue layout and strict queue-number. Many of the upper bounds on the queue-number that are presented in later sections will be expressed as functions of the strict queue-number. In Section 3 we prove bounds on the queue-number of the power of a graph in terms of the queue-number of the underlying graph. In Section 4 we define the graph products that will be studied in later sections. In Section 5 we study the queue-number of the cartesian product of graphs. Finally in Section 6 we study the queue-number of the direct and strong products of graphs.

‡ Dujmović and Wood [8] gave a simple proof of this result.

Tab. 1: Upper bounds on the queue-number.

graph family	queue-number	reference
n vertices	$\lfloor \frac{n}{2} \rfloor$	Heath and Rosenberg [19]
m edges	$\boldsymbol{e}\sqrt{m}$	Dujmović and Wood [9]
tree-width w	$3^w \cdot 6^{(4^w - 3w - 1)/9} - 1$	Dujmović *et al.* [6]
tree-width w, max. degree Δ	$36\Delta w$	Wood [29]
path-width p	p	Dujmović *et al.* [6]
band-width b	$\lceil \frac{b}{2} \rceil$	Heath and Rosenberg [19]
track-number t	$t - 1$	Dujmović *et al.* [6]
2-trees	3	Rengarajan and Veni Madhavan [25]‡
k-ary butterfly	$\lfloor \frac{k}{2} \rfloor$+1	Hasunuma [14]
d-ary de Bruijn	d	Hasunuma [14]
Halin	3	Ganley [13]
X-trees	2	Heath and Rosenberg [19]
outerplanar	2	Heath *et al.* [15]
arched levelled planar	1	Heath *et al.* [15]
trees	1	Heath and Rosenberg [19]

2 Strict Queue Layouts

Let σ be a vertex ordering of a graph G. We say an edge e is *inside* a distinct edge f, and e and f *overlap*, if

$$\sigma(L_f) \leq \sigma(L_e) < \sigma(R_e) \leq \sigma(R_f) \ .$$

A set of edges $Q \subseteq E(G)$ is a *strict queue* in σ if no edge in Q is inside another edge in Q. Alternatively, Q is a *strict queue* in σ if

$$\begin{aligned} &\sigma(L_e) < \sigma(L_f) \ \text{ and } \ \sigma(R_e) < \sigma(R_f) \ , \\ \text{or } &\sigma(L_f) < \sigma(L_e) \ \text{ and } \ \sigma(R_f) < \sigma(R_e) \ . \end{aligned} \tag{2}$$

Note that Equation (2) is obtained from Equation (1) by replacing "$\leq$" by "$<$".

Hence a strict queue is a set of edges, no two of which are nested or overlapping, as illustrated in Figure 2. Note that edges forming a 'butterfly' can be in a single strict queue.

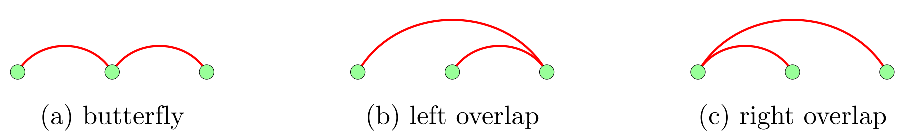
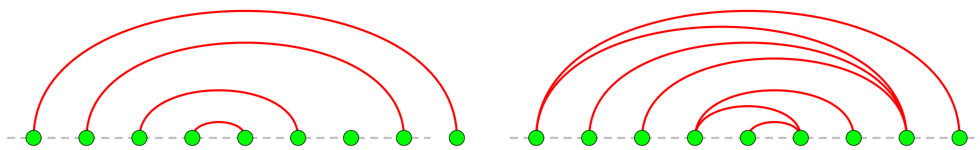


Fig. 2: Relationships between pairs of edges with a common endpoint in a vertex ordering.

A *strict k-queue layout* of G is a pair $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$ where σ is a vertex ordering of G, and $\{Q_1, Q_2, \dots, Q_k\}$ is a partition of $E(G)$, such that each Q_i is a strict queue in σ. We sometimes write

$\mathsf{queue}(e) = i$ for each edge $e \in Q_i$. The *strict-queue-number* of a graph G, denoted by $\mathsf{sqn}(G)$, is the minimum k such that there is a strict k-queue layout of G.

Heath and Rosenberg [19] proved that a fixed vertex ordering of a graph G admits a k-queue layout of G if and only if it has no $(k+1)$-edge rainbow, where a *rainbow* is a set of pairwise nested edges, as illustrated in Figure 3(a). Consider the analogous problem for strict queues: assign the edges of a graph G to the minimum number of strict queues given a fixed vertex ordering σ of G. As illustrated in Figure 3(b), a *weak rainbow* in σ is a set of edges R such that for every pair of edges $e, f \in R$, e is inside f or f is inside e.



(a) (b)

Fig. 3: (a) rainbow, (b) weak rainbow

Lemma 2 *A vertex ordering of a graph G admits a strict k-queue layout of G if and only if it has no $(k+1)$-edge weak rainbow.*

Proof: A strict k-queue layout has no $(k+1)$-edge weak rainbow since each edge of a weak rainbow must be in a distinct strict queue. Conversely, suppose we have a vertex ordering with no $(k+1)$-edge weak rainbow. For every edge $e \in E(G)$, let $\mathsf{queue}(e)$ be one plus the maximum number of edges in a weak rainbow consisting of edges that are inside e. If e is inside f then $\mathsf{queue}(e) < \mathsf{queue}(f)$. Hence we have a valid strict queue assignment. The number of strict queues is at most k. □

A *linear forest* is a graph in which every component is a path. The *linear arboricity* of a graph G, denoted by $\mathsf{la}(G)$, is the minimum integer k such that $E(G)$ can be partitioned in k linear forests; see [1, 2, 30, 31]. We have the following lower bounds on $\mathsf{sqn}(G)$.

Lemma 3 *The strict queue-number of every graph G satisfies:*

(a) $\mathsf{sqn}(G) \geq \mathsf{la}(G) > \eta(G)$,

(b) $\mathsf{sqn}(G) \geq \mathsf{la}(G) \geq \Delta(G)/2$, *and*

(c) $\mathsf{sqn}(G) \geq \delta(G)$.

Proof: Say Q is a strict queue in a vertex ordering σ of G. Every 2-edge path (u, v, w) in Q has $\sigma(u) < \sigma(v) < \sigma(w)$ (or $\sigma(w) < \sigma(v) < \sigma(u)$). Thus no vertex is incident to three edges in Q, and Q induces a linear forest. Hence $\mathsf{la}(G) \leq \mathsf{sqn}(G)$.

Since a linear forest in G has at most $|V(G)| - 1$ edges, $\mathsf{la}(G) \geq |E(G)|/(|V(G)| - 1) > \eta(G)$. This proves (a). At most two edges incident to each vertex are a linear forest. Thus $\mathsf{la}(G) \geq \Delta(G)/2$. This proves (b).

In every vertex ordering of G, every edge incident to the first vertex is in a distinct strict queue. Hence $\mathsf{sqn}(G) \geq \delta(G)$. This proves (c). $\square$

Obviously a proper edge $(\Delta(G)+1)$-colouring [28] can be combined with a $\mathsf{qn}(G)$-queue layout to obtain a strict queue layout.

Lemma 4 *Every graph G has strict queue-number* $\mathsf{sqn}(G) \leq (\Delta(G)+1)\cdot\mathsf{qn}(G)$. $\square$

3 Graph Powers

Let G be a graph, and let $d \in \mathbb{Z}^+$. The d-th *power* of G, denoted by G^d, is the graph with vertex set $V(G^d) = V(G)$, where $vw \in E(G^d)$ if and only if the distance between v and w in G is at most d. The following general result is similar to a theorem of Dujmović and Wood [10].

Theorem 1 *For every graph G and $d \in \mathbb{Z}^+$,*

$$\mathsf{qn}(G^d) \leq \frac{(2\,\mathsf{sqn}(G))^{d+1}-1}{2\,\mathsf{sqn}(G)-1} - \mathsf{sqn}(G) - 1 \ .$$

Proof: Let σ be the vertex ordering in a strict $\mathsf{sqn}(G)$-queue layout of G. Consider σ to be a vertex ordering of G^d. For every pair of vertices $v, w \in V(G)$ with $\sigma(v) < \sigma(w)$ and at distance $\ell \leq d$, fix a path $P(vw)$ from v to w in G with exactly ℓ edges. Suppose $P(vw) = (x_0, x_1, \dots, x_\ell)$, where $v = x_0$ and $w = x_\ell$. For each $1 \leq i \leq \ell$, let $\mathsf{dir}(x_{i-1}x_i)$ be '$+$' if $\sigma(x_{i-1}) < \sigma(x_i)$, and '$-$' otherwise. Let $f(vw)$ be the vector

$$f(vw) = \Big[\big(\mathsf{queue}(x_{i-1}x_i), \mathsf{dir}(x_{i-1}x_i)\big) : 1 \leq i \leq \ell\Big] \ .$$

Consider two edges $vw, pq \in E(G^d)$ with $f(vw) = f(pq)$. Then $|P(vw)| = |P(pq)|$. Let $P(vw) = (x_0, x_1, \dots, x_\ell)$ and $P(pq) = (y_0, y_1, \dots, y_\ell)$. We have $\mathsf{dir}(x_0x_1) = \mathsf{dir}(y_0y_1)$ and $\mathsf{queue}(x_0x_1) = \mathsf{queue}(y_0y_1)$. Thus $x_0 \neq y_0$. Without loss of generality $\sigma(x_0) < \sigma(y_0)$. By Equation (2), $\sigma(x_1) < \sigma(y_1)$. In general, $\sigma(x_{i-1}) < \sigma(y_{i-1})$ implies $\sigma(x_i) < \sigma(y_i)$, since $\mathsf{queue}(x_{i-1}x_i) = \mathsf{queue}(y_{i-1}y_i)$ and $\mathsf{dir}(x_{i-1}x_i) = \mathsf{dir}(y_{i-1}y_i)$. By induction, $\sigma(x_i) < \sigma(y_i)$ for all $0 \leq i \leq \ell$. In particular, $\sigma(w) < \sigma(q)$. Thus vw and pq can be in the same strict queue. If we partition the edges of G^d by the value of f we obtain a strict queue layout of G^d. The number of queues is

$$\sum_{\ell=1}^{d} (2\,\mathsf{sqn}(G))^\ell \ = \ \frac{(2\,\mathsf{sqn}(G))^{d+1}-1}{2\,\mathsf{sqn}(G)-1} - 1 \ .$$

Observe that for the edges of G we have counted $2\,\mathsf{sqn}(G)$ queues. Of course we need only $\mathsf{sqn}(G)$ queues. Thus the total number of queues is as claimed. $\square$

3.1 Powers of Paths and Cycles

In a vertex ordering σ of a graph G, the *width* of an edge e is $\sigma(R_e) - \sigma(L_e)$. The *bandwidth* of σ is the maximum width of an edge of G. The *bandwidth* of G, denoted by $\mathsf{bw}(G)$, is the minimum bandwidth of a vertex ordering of G. Alternatively, $\mathsf{bw}(G) = \min\{k : G \subseteq P_n^k\}$ for every n-vertex graph G.

Heath and Rosenberg [19] observed that edges whose widths differ by at most one are not nested. Thus $\mathsf{qn}(G) \leq \lceil \mathsf{bw}(G)/2 \rceil$, as mentioned in Table 1. In a vertex ordering, edges with the same width are not nested or overlapping, and thus form a strict queue. The next lemma follows.

Lemma 5 *Every graph G has strict queue-number $\mathsf{sqn}(G) \leq \mathsf{bw}(G)$.* □

We have the following results that give more precise bounds on the queue-number and strict-queue-number of powers of paths and cycles than Theorem 1.

Lemma 6 *The k-th power of a path P_n ($n \geq k+1$) has queue-number $\mathsf{qn}(P_n^k) = \lceil k/2 \rceil$ and strict queue-number $\mathsf{sqn}(P_n^k) = k$*

Proof: The bandwidth of a graph G can be thought of as the minimum integer k such that $G \subseteq P_n^k$. Thus the upper bound is nothing more than the result $\mathsf{qn}(G) \leq \lceil \mathsf{bw}(G)/2 \rceil$ of Heath and Rosenberg [19]. The lower bound follows since P_n^k contains a $(k+1)$-clique, which contains $\lceil k/2 \rceil$ pairwise nested edges in any vertex ordering, all of which must be assigned to distinct queues.

The natural vertex-ordering of P_n^k has no $(k+1)$-edge weak rainbow. Thus $\mathsf{sqn}(P_n^k) \leq k$ by Lemma 2. The lower bound follows since P_n^k contains a $(k+1)$-clique, which contains a k-edge weak rainbow in any vertex ordering. □

A graph is *unicyclic* if every connected component has at most one cycle. Heath and Rosenberg [19] proved that any unicyclic graph has a 1-queue layout. In particular, every cycle has a 1-queue layout. More generally,

Lemma 7 *The k-th power of a cycle C_n ($n \geq 2k$) has queue-number $\frac{k}{2} < \mathsf{qn}(C_n^k) \leq k$, and strict queue-number $\mathsf{sqn}(C_n^k) = 2k$.*

Proof: Observe that $\delta(C_n^k) = \Delta(C_n^k) = 2k$ and $\eta(C_n^k) = k$. Thus the claimed lower bounds follow from Lemmata 1 and 3. For the upper bounds, say $C_n = (v_1, v_2, \dots, v_n)$. By considering the vertex ordering

$$(v_1, v_n; v_2, v_{n-1}; \dots; v_i, v_{n-i+1}; \dots; v_{\lfloor n/2 \rfloor}, v_{\lceil n/2 \rceil}) \ , \tag{3}$$

we see that $C_n^k \subset P_n^{2k}$. The result follows from Lemma 6. □

4 Graph Products

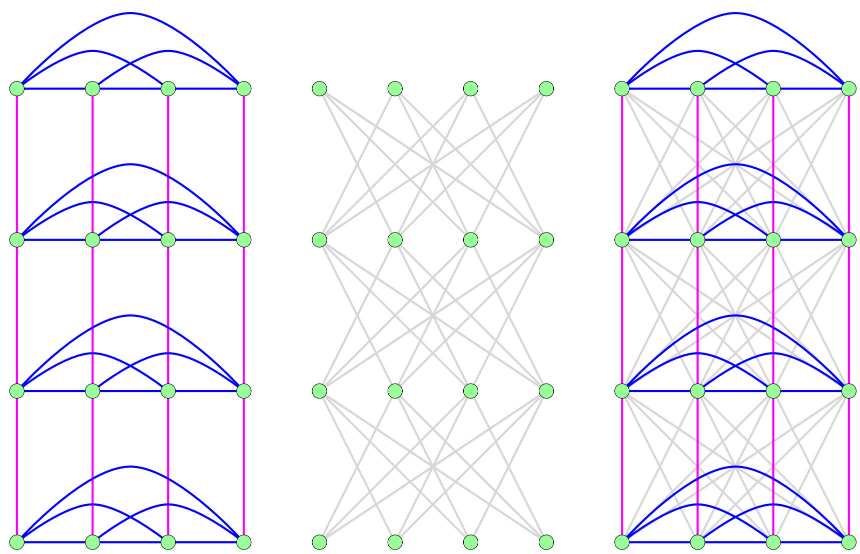
Let G_1 and G_2 be graphs. Below we define a number of graph products whose vertex set is

$$V(G_1) \times V(G_2) = \{(a,v) : a \in V(G_1), v \in V(G_2)\}) \ .$$

We classify a potential edge $(a,v)(b,w)$ as follows:

- G_1-*edge*: $ab \in E(G_1)$ and $v = w$.
- G_2-*edge*: $a = b$ and $vw \in E(G_2)$.
- *direct edge*: $ab \in E(G_1)$ and $vw \in E(G_2)$.

The *cartesian product* $G_1 \square G_2$ consists of the G_1-edges and the G_2-edges. The *direct product* $G_1 \times G_2$ consists of the direct edges. The *strong product* $G_1 \boxtimes G_2$ consists of the G_1-edges, the G_2-edges, and the direct edges. That is, $G_1 \boxtimes G_2 = (G_1 \square G_2) \cup (G_1 \times G_2)$. Note that other names abound for these graph products. Our notation is taken from the survey by Klavžar [21]. Assuming isomorphic graphs are equal, each of the above three products are associative, and for instance, $G_1 \square G_2 \square \cdots \square G_d$ is well-defined. Figure 4 illustrates these three types of graphs products.



(a) $P_4 \square K_4$ (b) $P_4 \times K_4$ (c) $P_4 \boxtimes K_4$

Fig. 4: Examples of graph products: (a) cartesian, (b) direct, (c) strong.

The following lemma is well-known and easily proved.

Lemma 8 *For all graphs G_1 and G_2, the density satisfies*

(a) $\eta(G_1 \square G_2) = \eta(G_1) + \eta(G_2)$,

(b) $\eta(G_1 \times G_2) = 2\eta(G_1) \cdot \eta(G_2)$,

(c) $\eta(G_1 \boxtimes G_2) = 2\eta(G_1) \cdot \eta(G_2) + \eta(G_1) + \eta(G_2)$.

5 The Cartesian Product

We have the following bounds on the queue-number of a cartesian product. In a vertex ordering σ of a graph product, we abbreviate $\sigma((v, a))$ by $\sigma(v, a)$.

Theorem 2 *For all graphs G and H,*

$$\mathsf{qn}(G \square H) \leq \mathsf{sqn}(G) + \mathsf{qn}(H) \ .$$

Furthermore, if for some constant c we have $\mathsf{sqn}(G) \leq c \cdot \eta(G)$ *and* $\mathsf{qn}(H) \leq c \cdot \eta(H)$*, then*

$$\mathsf{qn}(G \,\square\, H) \;\geq\; \tfrac{1}{2c}\left(\mathsf{sqn}(G) + \mathsf{qn}(H)\right) \;.$$

Proof: First we prove the upper bound. Let σ be the vertex ordering in a strict $\mathsf{sqn}(G)$-queue layout of G. Let π be the vertex ordering in a $\mathsf{qn}(H)$-queue layout of H. Let ϕ be the vertex ordering of $G \,\square\, H$ in which $\phi(v, a) < \phi(w, b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$.

For all edges e of G and for all vertices a of H, we have $\phi(L_e, a) < \phi(R_e, a)$. Similarly, for all edges e of H and for all vertices v of G, we have $\phi(v, L_e) < \phi(v, R_e)$.

Consider two G-edges $(L_e, a)(R_e, a)$ and $(L_f, b)(R_f, b)$ of $G \,\square\, H$, for which e and f are in the same strict queue of G. By Equation (2), without loss of generality, $\sigma(L_e) < \sigma(L_f)$ and $\sigma(R_e) < \sigma(R_f)$. Thus $\phi(L_e, a) < \phi(L_f, b)$ and $\phi(R_e, a) < \phi(R_f, b)$. Hence for each strict queue in G, the corresponding G-edges of $G \,\square\, H$ form a strict queue in ϕ.

Consider two H-edges $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ of $G \,\square\, H$, for which e and f are in the same queue of H. By Equation (1), without loss of generality, $\pi(L_e) \leq \pi(L_f)$ and $\pi(R_e) \leq \pi(R_f)$. First suppose that $\sigma(v) \leq \sigma(w)$. Then $\phi(v, L_e) \leq \phi(w, L_f)$ and $\phi(v, R_e) \leq \phi(w, R_f)$. Thus $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ are not nested in ϕ. Now suppose that $\sigma(w) < \sigma(v)$. Then $\phi(w, L_f) < \phi(w, R_f) < \phi(v, L_e) < \phi(v, R_e)$. Thus $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ are disjoint. Thus for each queue in H, the corresponding H-edges of $G \,\square\, H$ form a queue in ϕ. Therefore ϕ admits a $(\mathsf{sqn}(G)+\mathsf{qn}(H))$-queue layout of $G \,\square\, H$.

Now we prove the lower bound. By Lemmata 1 and 8(a), $\mathsf{qn}(G \,\square\, H) > \eta(G \,\square\, H)/2 = (\eta(G) + \eta(H))/2$. The result follows since $\eta(G) \geq \frac{1}{c}\,\mathsf{sqn}(G)$ and $\eta(H) \geq \frac{1}{c}\,\mathsf{qn}(H)$. □

Theorem 2 has the following immediate corollary.

Corollary 1 *For all graphs* $G_1, G_2, \ldots, G_d$*,*

$$\mathsf{qn}(G_1 \,\square\, G_2 \,\square\, \cdots \,\square\, G_d) \;\leq\; \mathsf{qn}(G_1) + \sum_{i=2}^{d} \mathsf{sqn}(G_i) \;.$$

□

5.1 Grids

A *d-dimensional grid* is a graph $P_{n_1} \,\square\, P_{n_2} \,\square\, \cdots \,\square\, P_{n_d}$, for all $n_i \geq 1$. Heath and Rosenberg [19] determined the queue-number of every 2-dimensional grid.

Lemma 9 ([19]) *Every* 2*-dimensional grid has queue-number one.* □

A *generalised d-dimensional grid* is a graph $G = P^k_{n_1} \,\square\, P^k_{n_2} \,\square\, \cdots \,\square\, P^k_{n_d}$, for all $k \geq 1$ and $n_i \geq k+1$. Now P^k_n has $kn - k(k+1)/2$ edges. Thus $\eta(P^k_n) = k - \frac{k(k+1)}{2n}$. By Lemma 8(a),

$$\eta(G) \;=\; \sum_{i=1}^{d}(k - \frac{k(k+1)}{2n_i}) \;=\; dk - \frac{1}{2}\,k(k+1)\sum_{i=1}^{d}\frac{1}{n_i} \;. \tag{4}$$

Lemma 9 generalises as follows.

Theorem 3 *For all $d \geq 2$, the queue-number of a d-dimensional grid $G = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_d}$ satisfies:*

$$\frac{d}{4} \leq \frac{1}{2}\left(d - \sum_{i=1}^{d} \frac{1}{n_i}\right) < \mathsf{qn}(G) \leq d-1 .$$

Proof: The lower bound follows from Lemma 1 and Equation (4) with $k = 1$.

For the upper bound, we have $\mathsf{qn}(P_{n_1} \square P_{n_2}) = 1$ by Lemma 9. Obviously $\mathsf{sqn}(P_{n_i}) = 1$ for all $i \geq 3$. Thus $\mathsf{qn}(G) \leq d - 1$ by Corollary 1.

We now give an alternative proof of the upper bound using a different construction. The graph G can be thought of as having vertex set $\{((x_1, x_2, \ldots, x_d) : 1 \leq x_i \leq n_i, 1 \leq i \leq d\}$, where two vertices $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ are adjacent if and only if $|x_i - y_i| = 1$ for some i, and $x_j = y_j$ for all $j \neq i$. We say this edge is in the i-th dimension. For all $s \geq 0$, let V_s be the set of vertices

$$V_s = \{(x_1, x_2, \ldots, x_d) : \sum_{i=1}^{d} x_i = s\} .$$

Order the vertices $(V_0, V_1, \ldots)$, where each V_s is ordered lexicographically. If vw is an edge then v and w differ in exactly one coordinate, and $v \in V_s$ and $w \in V_{s+1}$ for some s. Thus if two edges vw and pq are nested then $v, p \in V_s$ and $w, q \in V_{s+1}$ for some s. Let Q_i be the set of edges in the i-th dimension. Consider two edges e and f in Q_i. Say

$$e = (x_1, x_2, \ldots, x_d)(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_d) ,$$

and

$$f = (y_1, y_2, \ldots, y_d)(y_1, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_d) .$$

Without loss of generality $(x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)$, which implies that

$$(x_1, \ldots, x_{i-1}, x_i + j, x_{i+1}, \ldots, x_d) \prec (y_1, \ldots, y_{i-1}, y_i + j, y_{i+1}, \ldots, y_d) .$$

Thus e and f are not nested, and Q_i is a queue. Hence we have a d-queue layout. (At this point we have in fact proved that the lexicographical order admits a d-queue layout.)

We now prove that $Q_{d-1} \cup Q_d$ is a queue, and thus we obtain the claimed $(d-1)$-queue layout. Suppose two edges $e \in Q_{d-1}$ and $f \in Q_d$ are nested. Say

$$e = (x_1, x_2, \ldots, x_d)(x_1, x_2, \ldots, x_{d-1} + 1, x_d) ,$$

and

$$f = (y_1, y_2, \ldots, y_d)(y_1, y_2, \ldots, y_{d-1}, y_d + 1) .$$

Then for some s, both $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ are in V_s, and both $(x_1, x_2, \ldots, x_{d-1}+1, x_d)$ and $(y_1, y_2, \ldots, y_{d-1}, y_d + 1)$ are in V_{s+1}.

Case 1. $(x_1, x_2, \ldots, x_d) \prec (y_1, y_2, \ldots, y_d)$: Let j be the first dimension for which $x_j < y_j$. If $j \leq d - 2$ then

$$(x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \ldots, y_{d-1}, y_d + 1) ,$$

which implies that e and f are not nested. Observe that $j \neq d$ as $(x_1, x_2, \ldots, x_d)$ and $(y_1, y_2, \ldots, y_d)$ differ in at least two coordinates, since $\sum_i x_i = \sum_i y_i$. Thus $j = d - 1$. That is,

$$x_{d-1} \leq y_{d-1} - 1 \ . \tag{5}$$

Since e and f are nested, we have $(y_1, y_2, \ldots, y_{d-1}, y_d + 1) \prec (x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d)$, which implies that $y_{d-1} \leq x_{d-1} + 1$. By Equation (5), $x_{d-1} = y_{d-1} - 1$. Since $x_{d-1} + x_d = y_{d-1} + y_d$, we have $x_d = y_d + 1$, which implies that

$$(y_1, y_2, \ldots, y_{d-1}, y_d + 1) = (x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \ .$$

That is, the right-hand endpoints of e and f are the same vertex. Hence e and f are not nested.

Case 2. $(y_1, y_2, \ldots, y_d) \prec (x_1, x_2, \ldots, x_d)$: By the same argument employed above, the first coordinate for which $(y_1, y_2, \ldots, y_d)$ and $(x_1, x_2, \ldots, x_d)$ differ is $d - 1$. That is,

$$y_{d-1} < x_{d-1} \ . \tag{6}$$

Since e and f are nested, we have $(x_1, x_2, \ldots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \ldots, y_{d-1}, y_d + 1)$. Thus $x_{d-1} + 1 < y_{d-1}$, which contradicts Equation (6). Hence e and f are not nested.

Therefore $Q_1, Q_2, \ldots, Q_{d-2}, Q_{d-1} \cup Q_d$ is the desired $(d-1)$-queue layout. □

More generally we have the following.

Theorem 4 *The queue-number of a generalised d-dimensional grid $G = P^k_{n_1} \,\square\, P^k_{n_2} \,\square\, \cdots \,\square\, P^k_{n_d}$ (where $n_i \geq k + 1$) satisfies:*

$$\frac{dk}{4} \ \leq\ \frac{dk}{2} - \frac{k(k+1)}{4} \sum_{i=1}^{d} \frac{1}{n_i} \ <\ \mathsf{qn}(G) \ \leq\ \left\lceil (d - \tfrac{1}{2})k \right\rceil \ .$$

Proof: By Lemma 6, $\mathsf{qn}(P^k_n) = \lceil \frac{k}{2} \rceil$ and $\mathsf{sqn}(P^k_n) \leq k$. Thus, the upper bound follows from Corollary 1. Thus the lower bound follows from Lemma 1 and Equation (4). □

By Theorem 4 with $k = n - 1$ we have the following.

Corollary 2 *The queue-number of the d-dimensional Hamming graph $G = K_n \,\square\, K_n \,\square\, \cdots \,\square\, K_n$ satisfies:*

$$\frac{d(n-1)}{4} \ <\ \mathsf{qn}(G) \ \leq\ \left\lceil (d - \tfrac{1}{2})(n-1) \right\rceil \ .$$

A *generalised d-dimensional toroidal grid* is a graph $C^k_{n_1} \,\square\, C^k_{n_2} \,\square\, \cdots \,\square\, C^k_{n_d}$ for all $k \geq 1$ and $n_i \geq 2k + 1$.

Theorem 5 *The queue-number of a generalised toroidal grid $G = C^k_{n_1} \,\square\, C^k_{n_2} \,\square\, \cdots \,\square\, C^k_{n_d}$ (where $n_i \geq 2k + 1$) satisfies:*

$$\frac{kd}{2} < \mathsf{qn}(G) \leq (2d - 1)k \ .$$

Proof: Since $\eta(G) = kd$, we have that $\mathsf{qn}(G) > \frac{kd}{2}$ by Lemma 1. Thus $\mathsf{qn}(G) \geq \lfloor \frac{d}{2} \rfloor + 1$. By Lemma 7, $\mathsf{qn}(C^k_{n_1}) \leq k$ and $\mathsf{sqn}(C^k_{n_1}) \leq 2k$. By Corollary 1, $\mathsf{qn}(G) \leq 2k(d-1) + k = (2d-1)k$ □

6 Direct and Strong Products

We have the following bounds on the queue-number of direct and strong products.

Theorem 6 *For all graphs G and H,*

$$\mathsf{qn}(G \times H) \leq 2\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) \ .$$

Furthermore, if $\mathsf{sqn}(G) \leq c \cdot \eta(G)$ and $\mathsf{qn}(H) \leq c \cdot \eta(H)$, then

$$\mathsf{qn}(G \times H) \ > \ \tfrac{1}{c^2}\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) \ .$$

Proof: First we prove the upper bound. Let $k := \mathsf{sqn}(G)$, and let $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$ be a strict k-queue layout of G. Let $\ell := \mathsf{qn}(H)$, and let $(\pi, \{P_1, P_2, \dots, P_\ell\})$ be an ℓ-queue layout of H. For $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let

$$E'_{i,j} := \{(v,a)(w,b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(a) < \pi(b)\}$$
$$E''_{i,j} := \{(v,a)(w,b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(b) < \pi(a)\}$$

Then $\{E'_{i,j}, E''_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ is a partition of $E(G \times H)$ into $2k\ell$ sets. Let ϕ be the vertex ordering of $G \times H$ in which $\phi(v,a) < \phi(w,b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$. We claim that each set $E'_{i,j}$ and $E''_{i,j}$ is a queue in ϕ.

Suppose that two edges $(v,a)(w,b), (x,c)(y,d) \in E'_{i,j}$ are nested. Without loss of generality, $\phi(v,a) < \phi(x,c) < \phi(y,d) < \phi(w,b)$. If $v \neq x$ and $y \neq w$, then $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ are nested in σ. If $v \neq x$ and $y = w$, then $\sigma(v) < \sigma(x) < \sigma(y) = \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in σ. If $v = x$ and $y \neq w$, then $\sigma(v) = \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in σ. Each of these outcomes contradict the assumption that Q_i is a strict queue in σ. Otherwise $v = x$ and $y = w$, in which case $\pi(a) < \pi(c) < \pi(d) < \pi(b)$, and ab and cd are nested in π. This contradicts the assumption that P_j is a queue in π. Thus each $E'_{i,j}$ is queue in ϕ. By symmetry, each $E''_{i,j}$ is also a queue in ϕ.

Now we prove the lower bound. Lemmata 1 and 8(b) imply that

$$\mathsf{qn}(G \times H) > \eta(G \times H)/2 = \eta(G) \cdot \eta(H) \geq \tfrac{1}{c}\,\mathsf{sqn}(G) \cdot \tfrac{1}{c}\,\mathsf{qn}(H) \ .$$

□

Theorem 7 *For all graphs G and H,*

$$\mathsf{qn}(G \boxtimes H) \leq 2\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) + \mathsf{sqn}(G) + \mathsf{qn}(H) \ .$$

Furthermore, if $\mathsf{sqn}(G) \leq c \cdot \eta(G)$ and $\mathsf{qn}(H) \leq c \cdot \eta(H)$, then

$$\mathsf{qn}(G \boxtimes H) \ > \ \tfrac{1}{c^2}\,\mathsf{sqn}(G) \cdot \mathsf{qn}(H) + \tfrac{1}{2c}\big(\mathsf{sqn}(G) + \mathsf{qn}(H)\big) \ .$$

Proof: To prove the upper bound, observe that the vertex ordering ϕ defined in Theorems 2 and 6 is the same. By Theorem 2, ϕ admits a $\mathsf{sqn}(G) + \mathsf{qn}(H)$-queue layout of $G \,\square\, H$. By Theorem 6, ϕ admits a

$2\,\mathsf{sqn}(G)\cdot\mathsf{qn}(H)$-queue layout of $G \times H$. Since $G \boxtimes H = (G \,\square\, H) \cup (G \times H)$, ϕ admits the claimed queue layout of $G \boxtimes H$.

For the lower bound, Lemmata 1 and 8(c) imply that

$$\mathsf{qn}(G \boxtimes H) > \tfrac{1}{2}\eta(G \boxtimes H) = \eta(G)\cdot\eta(H) + \tfrac{1}{2}(\eta(G)+\eta(H)) \geq \frac{1}{c}\,\mathsf{sqn}(G)\cdot\frac{1}{c}\,\mathsf{qn}(H) \enspace .$$

$\square$

Acknowledgements

Many thanks to Toru Hasunuma for invaluable comments on a preliminary draft of this paper. Thanks to the anonymous reviewers and the editor, Therese Biedl, for helpful suggestions.

References

[1] N. ALON. The linear arboricity of graphs. *Israel J. Math.*, 62(3):311–325, 1988.

[2] NOGA ALON, V. J. TEAGUE, AND N. C. WORMALD. Linear arboricity and linear k-arboricity of regular graphs. *Graphs Combin.*, 17(1):11–16, 2001.

[3] SANDEEP N. BHATT, FAN R. K. CHUNG, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. Scheduling tree-dags using FIFO queues: A control-memory trade-off. *J. Parallel Distrib. Comput.*, 33:55–68, 1996.

[4] EMILIO DI GIACOMO, GIUSEPPE LIOTTA, AND HENK MEIJER. Computing straight-line 3D grid drawings of graphs in linear volume. *Comput. Geom.*, 32(1):26–58, 2005.

[5] EMILIO DI GIACOMO, GIUSEPPE LIOTTA, HENK MEIJER, AND STEPHEN K. WISMATH. Volume requirements of 3D upward drawings. In PATRICK HEALY, ed., *Proc. 13th International Symp. on Graph Drawing* (GD '05), Lecture Notes in Comput. Sci. Springer, to appear.

[6] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, 34(3):553–579, 2005.

[7] VIDA DUJMOVIĆ, ATTILA PÓR, AND DAVID R. WOOD. Track layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):497–522, 2004.

[8] VIDA DUJMOVIĆ AND DAVID R. WOOD. Tree-partitions of k-trees with applications in graph layout. Tech. Rep. TR-2002-03, School of Computer Science, Carleton University, Ottawa, Canada, 2002.

[9] VIDA DUJMOVIĆ AND DAVID R. WOOD. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):339–358, 2004.

[10] VIDA DUJMOVIĆ AND DAVID R. WOOD. Stacks, queues and tracks: Layouts of graph subdivisions. *Discrete Math. Theor. Comput. Sci.*, 7:155–202, 2005.

[11] VIDA DUJMOVIĆ AND DAVID R. WOOD. Three-dimensional upward grid drawings of graphs. 2005.

[12] SHIMON EVEN AND A. ITAI. Queues, stacks, and graphs. In ZVI KOHAVI AND AZARIA PAZ, eds., *Proc. International Symp. on Theory of Machines and Computations*, pp. 71–86. Academic Press, 1971.

[13] JOSEPH L. GANLEY. Stack and queue layouts of Halin graphs, 1995. Manuscript.

[14] TORU HASUNUMA. Laying out iterated line digraphs using queues. In GUISEPPE LIOTTA, ed., *Proc. 11th International Symp. on Graph Drawing* (GD '03), vol. 2912 of *Lecture Notes in Comput. Sci.*, pp. 202–213. Springer, 2004.

[15] LENWOOD S. HEATH, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. Comparing queues and stacks as mechanisms for laying out graphs. *SIAM J. Discrete Math.*, 5(3):398–412, 1992.

[16] LENWOOD S. HEATH AND SRIRAM V. PEMMARAJU. Stack and queue layouts of posets. *SIAM J. Discrete Math.*, 10(4):599–625, 1997.

[17] LENWOOD S. HEATH AND SRIRAM V. PEMMARAJU. Stack and queue layouts of directed acyclic graphs. part II. *SIAM J. Comput.*, 28(5):1588–1626, 1999.

[18] LENWOOD S. HEATH, SRIRAM V. PEMMARAJU, AND ANN N. TRENK. Stack and queue layouts of directed acyclic graphs. part I. *SIAM J. Comput.*, 28(4):1510–1539, 1999.

[19] LENWOOD S. HEATH AND ARNOLD L. ROSENBERG. Laying out graphs using queues. *SIAM J. Comput.*, 21(5):927–958, 1992.

[20] ATSUMI IMAMIYA AND AKIHIRO NOZAKI. Generating and sorting permutations using restricted-deques. *Information Processing in Japan*, 17:80–86, 1977.

[21] SANDI KLAVŽAR. Coloring graph products—a survey. *Discrete Math.*, 155(1-3):135–145, 1996.

[22] EDWARD T. ORDMAN AND WILLIAM SCHMITT. Permutations using stacks and queues. In *Proc. 24th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 96 of *Congr. Numer.*, pp. 57–64. 1993.

[23] SRIRAM V. PEMMARAJU. *Exploring the Powers of Stacks and Queues via Graph Layouts*. Ph.D. thesis, Virginia Polytechnic Institute and State University, U.S.A., 1992.

[24] VAUGHAN R. PRATT. Computing permutations with double-ended queues. Parallel stacks and parallel queues. In *Proc. 5th Annual ACM Symp. on Theory of Computing* (STOC '73), pp. 268–277. ACM, 1973.

[25] S. RENGARAJAN AND C. E. VENI MADHAVAN. Stack and queue number of 2-trees. In DING-ZHU DU AND MING LI, eds., *Proc. 1st Annual International Conf. on Computing and Combinatorics* (COCOON '95), vol. 959 of *Lecture Notes in Comput. Sci.*, pp. 203–212. Springer, 1995.

[26] Farhad Shahrokhi and Weiping Shi. On crossing sets, disjoint sets, and pagenumber. *J. Algorithms*, 34(1):40–53, 2000.

[27] Robert E. Tarjan. Sorting using networks of queues and stacks. *J. Assoc. Comput. Mach.*, 19:341–346, 1972.

[28] Vadim G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, 3:25–30, 1964.

[29] David R. Wood. Queue layouts, tree-width, and three-dimensional graph drawing. In Manindra Agrawal and Anil Seth, eds., *Proc. 22nd Foundations of Software Technology and Theoretical Computer Science* (FST TCS '02), vol. 2556 of *Lecture Notes in Comput. Sci.*, pp. 348–359. Springer, 2002.

[30] Jian-Liang Wu. On the linear arboricity of planar graphs. *J. Graph Theory*, 31(2):129–134, 1999.

[31] Jianliang Wu. The linear arboricity of series-parallel graphs. *Graphs Combin.*, 16(3):367–372, 2000.