Finite-time-observer design for nonlinear impulsive systems with impact perturbation

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Abstract

This paper investigates the observer design problem of nonlinear impulsive systems with impact perturbation. By using the concept of normal form, this paper proposes a full order finite time observer, which guarantees the finite time convergence independent of the impact perturbation. Reduced order observer is additionally developed for the proposed normal form. An illustrative example is given in order to illustrate the capability of the proposed method.

1 Introduction

Impulsive systems are widely studied in chemical, engineering, and biological fields ([Lakshmikantham (1989)]) where instantaneous changes of the state variables are admitted at various time instants. Moreover, from the theoretical point of view, some non smooth dynamical systems ([Brogliato(1999)]) and sampled-data systems ([Sun (1993), Sagfors (1998)]) can be treated from an impulsive systems point of view as well.

Fundamental problems for impulsive systems, such as observability, reachability and controllability, have been widely investigated for different types of impulsive systems ([Lakshmikantham (1989), Yang(2001), Li (2005)]). Particularly, in the case of linear impulsive systems with constant coefficients, the concepts of observability/reachability and controllability proved to be equivalent to those of linear time-invariant systems ([George (2000)]). Controllability and observability results were developed in [Guan (2002)] for linear impulsive systems where the controlled actuation were available for the continuous-time dynamics at impact time instants only and the impulsive effects were limited to scalings of the state. For the same type of linear impulsive system, [Xie and Wang(2005)] used a geometric framework to generalize the results of [Guan (2002)]. For nonlinear systems, finite time observers have been exhaustively discussed in the literature, by using different methods, such as sliding mode technique ([Perruquetti (1998)]), homogeneity ([Perruquetti (2008)]) delay measurement ([Sauvage (2007)]), output injection ([Engel and Kreisselmeier(2002)]) and algebraic methods ([Fliess and Sira-Ramírez(2004), Barbot (2007)]). However, few results are reported on finite time observer design for impulsive systems.

In this paper, we study observer design problem for nonlinear impulsive systems with persistent impact perturbation. This work is motivated by the fact that the perturbation cannot be avoided at each impact instant for some practical applications, for example, for the walking robot, the perturbation always exists when a robot's feet touches the ground. For some advanced controllers relying on the unmeasured states, a finite time observer, which can always estimate system states during a prescribed interval, is required under persistent impact perturbations.

Recently, [Raff and Allgöwer(2007)] proposed to simply couple two Luenberger observers to form a finite time observer for linear systems. This paper adopts this idea to synthesize a finite time observer for nonlinear impulsive systems with persistent impact perturbation, whose convergence duration is independent of the persistent perturbation at each impact instant, and can be fixed a priori.

The concept of a normal form is additionally adopted in order to study the finite time observer design problem for nonlinear impulsive systems. Roughly speaking, such a form represents a class of systems possessing the same properties of the stability, controllability and so on. A normal form used for the observability study and observer design can be found in [Krener and Isidori(1983), Krener and Respondek(1985), Boutat (2009), Zheng (2007)] and in the references, quoted therein. The basic idea is to deduce a diffeomorphism, transforming a system in question into a common and simple normal form for which the observability and observer design problems have been already solved. Then, by inverting the deduced diffeomorphism, the observability and observer design problems are solved for the original system as well.

This paper is organized as follows. Notations and problem statement are given in Section 2. Section 3 presents an appropriate normal form and necessary and sufficient conditions for a transform of a nonlinear impulsive system into such a normal form to exist. The corresponding full order observer and reduced order observer are proposed in Section 4. Section 5 highlights the capability of the proposed method by means of an illustrative example.

2 Notations and problem statement

Consider the following nonlinear impulsive system

$$\left\{\begin{array}{ll} \dot{\xi}(t)=f(\xi(t))+g\left(\xi(t)\right)u(t), & t\in[t_{k-1},t_k) \text{ with } k\in Z^{+} \\ \xi(t^{+})=\tilde{\gamma}(\xi(t^{-}))+\tilde{\Gamma}_p(t^{-}), & t=t_k \\ y(t)=h(\xi(t)) & \\ \xi(t_0^{+})=\xi_0, & t_0=0 \end{array}\right. \tag{1}$$

with an unknown persistent impact perturbation $\tilde{\Gamma}_p$. Hereinafter, $\xi\in\mathbb{R}^n$, $u\in\mathbb{R}^p$ and $y\in\mathbb{R}$ stand for the state, the admissible control input and the output, respectively; $f(\xi), g(\xi), h(\xi)$ and $\tilde{\gamma}$ are known smooth functions of appropriate dimensions; Z^+ represents the set of positive integer, and $t_0 < t_1 < \cdots < t_k < \cdots$ are impact time instants such that

$$D_{\min} = \inf_{k \in Z^+} \{t_k - t_{k-1}\} > 0 \tag{2}$$

and $D_{max} = \sup_{k \in Z^+} \{t_k - t_{k-1}\} < \infty$ and $\lim_{k \to \infty} t_k = \infty$.

The left and right values at impulse time t_k are defined as usual

$$\xi(t_k^+) = \lim_{h \to 0} \xi(t_k + h), \xi(t_k^-) = \lim_{h \to 0} \xi(t_k - h).$$

with $h > 0$. For certainty, it is assumed that $\xi(t_k^-) = \xi(t_k)$, thereby yielding a solution of (1) to be left continuous at t_k. It is also assumed throughout that (1) is observable ([Hermann and Krener(1977)]) in the impact-free case.

Since in what follows, the underlying system (1) is periodically affected by unknown impulsive disturbances, many conventional concepts, such as that of asymptotic observer, is to be re-worked for impulsive systems with impact uncertainties.

Given $t_{k-1} \leq t < t_k$ with $k \in Z^+$, denote $x(t) = x(t, t_{k-1}^+, x(t_{k-1}^+))$ a solution of (1) initialized at t_{k-1}^+ with $x(t_{k-1}^+)$.

Definition 1 *The following impact system*

$$\begin{cases} \dot{z} = f_1(z, u, y), & t \in [t_{k-1}, T_k) \cup (T_K, t_k) \\ z(t^+) = g_1(z(t^-)), & t = t_{k-1} + T_k \\ z(t^+) = g_2(z(t^-)), & t = t_k \end{cases}$$

with $z \in \mathbb{R}^q$ where $q \geq n$ and some user chosen functions f_1, g_1 and g_2, is an exponential observer of (1) with the output $\eta : \mathbb{R}^q \to \mathbb{R}^n$ if the following inequality

$$||\eta\left(z(t)\right) - x(t)|| \leq a e^{-b(t-t_{k-1})} ||\eta\left(z(t_{k-1})\right) - x(t_{k-1})||, \text{ for } t_{k-1} \leq t < t_k$$

is satisfied for each interval $[t_{k-1}, t_k),\ k \in Z^+$ and some positive constants a and b.

It is a finite time observer of (1) with the output $\eta : \mathbb{R}^q \to \mathbb{R}^n$ if there exist corresponding $\sigma_k < t_k - t_{k-1},\ k \in Z^+$ such that

$$||\eta\left(z(t)\right) - x(t)|| = 0, \text{ when } t \in [t_{k-1} + \sigma_k, t_k),\ k \in Z^+.$$

This system is said to be a uniform finite time observer of (1) iff $\sigma = \sup_{k \in Z^+}\{\sigma_k\} \in (0, D_{\min})$ where $D_{\min}$ is specified in (2). Such a σ is further referred to as a settling time estimate of the observer.

To motivate the above definition it should be noted that although one can design an asymptotical observer for (1) on each interval $[t_{k-1}, t_k),\ k \in Z^+$, however, the convergence of such an observer on a specific interval $[t_{k-1}, t_k)$ would depend on the perturbation $\widetilde{\Gamma}_p(t_{k-1}^-)$, thereby possibly resulting in inappropriate state estimate due to accumulating an estimation error.

3 Transformation to normal form

By using the concept of normal form, this section is devoted to presenting a normal form for which an observer is proposed, and then deducing necessary and sufficient conditions under which (1) can be transformed into this normal form.

Throughout this article, $L_f^k h$ is the k^{th} Lie derivative of h along f and $\theta_i = dL_f^{i-1}h$ stands for the associated differentiation for $1 \leq i \leq n$ with $dL_f^0 h = dh$. Since once not affected by impulses, system (1) is assumed to be locally observable, the 1-forms θ_i are thus linearly independent, i.e. $rank\{\theta_i, 1 \leq i \leq n\} = n$. Then, one can construct the well-known Krener & Isidori ([Krener and Isidori(1983)]) frame

$$\tau = (\tau_1, ..., \tau_n) \tag{3}$$

with the first vector field τ_1 given by the following algebraic equations

$$\begin{cases} \theta_i\left(\tau_1\right) = 0 \text{ for } 1 \leq i \leq n-1 \\ \theta_n\left(\tau_1\right) = 1 \end{cases}$$

whereas the rest of vector fields is obtained by iterating on i:

$$\tau_{i+1} = [\tau_i, f] \text{ for } 1 \leq i \leq n-1$$

where $[,]$ denotes the conventional Lie bracket.

Setting

$$\theta = \left(\theta_1, \cdots, \theta_n\right)^T, \tag{4}$$

let us introduce

$$\Xi = \theta\tau = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & l_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & l_{n-1,n-1} & l_{n-1,n} \\ 1 & l_{n,2} & \cdots & l_{n,n-1} & l_{n,n} \end{pmatrix} \tag{5}$$

with $l_{i,j} = \theta_i \tau_j$ for $2 \leq i \leq n$ and $n-i+2 \leq j \leq n$. It is clear that Ξ is invertible. With this in mind, we denote

$$\omega = \Xi^{-1}\theta. \tag{6}$$

Thus, we arrive at the following result which can be viewed as a natural extension of [Krener and Isidori(1983)] to nonlinear systems with impacts.

Theorem 1 *There exists a diffeomorphism $x = \phi(\xi)$ which transforms nonlinear impulsive system (1) into*

$$\begin{cases} \dot{x}(t) = Ax(t) + \beta\left(y(t)\right) + \rho\left(y\left(t\right), u(t)\right), & t \in [t_{k-1}, t_k) \textit{ with } k \in Z^+ \\ x(t^+) = \gamma(x(t^-)) + \Gamma_p(t^-), & t = t_k \\ y(t) = Cx(t) & \\ x(t_0^+) = x_0 = \phi(\xi_0), & t_0 = 0 \end{cases} \tag{7}$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, C = (0, \cdots, 0, 1)$$

$$\gamma(x) = \phi\left(\tilde{\gamma}(\xi(t^-))\right)_{|_{\xi=\phi^{-1}(x)}}$$

$$\Gamma_p(t^-) = \left[\phi(\tilde{\gamma}(\xi(t^-)) + \tilde{\Gamma}_p(t^-)) - \phi(\tilde{\gamma}(\xi(t^-)))\right]_{|_{\xi=\phi^{-1}(x)}}$$

if and only if one of the following conditions holds:

1. $[\tau_i, \tau_j] = 0$ *for* $1 \le i \le n$, $1 \le j \le n$ *and* $[g, \tau_l] = 0$ *for* $1 \le l \le n-1$;
2. $d\omega = 0$ *and* $[g, \tau_l] = 0$ *for* $1 \le l \le n-1$.

Proof 1 *This theorem is based on the result in [Krener and Isidori(1983)] where nonlinear systems without impact perturbations were considered. The first part of this proof is based on [Zheng (2007)], but in order to keep the paper self contained, we still detail here the proof of the first part of this theorem.*

To begin with, let us show the equivalence between the first condition and the second condition of the theorem. For this purpose, compute the differential $d\omega$ *of* ω *on two vector fields* $X = \tau_i$ *and* $Y = \tau_j$*:*

$$d\omega(X, Y) = L_X\omega(Y) - L_Y\omega(X) - \omega([X, Y]),$$

where L_X *is the Lie derivative in* X *direction. Since* $\omega(\tau_i) = e_i$ *and* $\omega(\tau_j) = e_j$ *are constant, then one has* $d\omega(\tau_i, \tau_j) = \omega([\tau_i, \tau_j])$*. Taking into account that* ω *is an isomorphism, then one obtains the following equivalence*

$$d\omega = 0 \Leftrightarrow [\tau_i, \tau_j] = 0$$

which results in the equivalence between the afore-mentioned conditions of the theorem.

Necessity*: Next let us proof that once (1) can be transformed into (7) through a diffeomorphism* $x = \phi(\xi)$*, Condition 1 of the theorem is satisfied. Indeed, just in case, one has* $\phi_*(\tau_i) = \frac{\partial}{\partial x_i}$ *for* $1 \le i \le n$ *and* $\phi_*(g) = \rho(y, u)\frac{\partial}{\partial x_i}$ *with* $y = x_n$*, thereby ensuring that* $[\phi_*(\tau_i), \phi_*(\tau_j)] = \phi_*([\tau_i, \tau_j]) = 0$ *for* $1 \le i \le n$ *and* $1 \le j \le n$ *and* $[\phi_*(\tau_l), \phi_*(g)] = \phi_*([\tau_l, g]) = 0$ *for* $1 \le l \le n-1$*. Moreover, since* $\xi = \phi(x)$ *is a diffeomorphism, one has* $[\tau_i, \tau_j] = 0$ *for* $1 \le i \le n$ *and* $1 \le j \le n$ *and* $[\tau_l, g] = 0$ *for* $1 \le l \le n-1$*.*

Sufficiency*: In order to prove the sufficiency suppose that* $[\tau_i, \tau_j] = 0$ *for* $1 \le i \le n$ *and* $1 \le j \le n$*. Then according to Poincaré theorem, there exists a local diffeomorphism* $x = \phi(\xi)$ *such that* $\omega = d\phi = \phi_*$, $\phi_*(\tau_i) = \frac{\partial}{\partial x_i}$ *for* $1 \le i \le n$*. Thus one has*

$$\frac{\partial \phi_*(f)}{\partial x_i} = \phi_*([\tau_i, f]) = \phi_*(\tau_{i+1}) = \frac{\partial}{\partial x_{i+1}}$$

for $1 \leq i \leq n-1$,*and by integration one arrives at*

$$\phi_*(f) = Ax + \beta(y).$$

Moreover, one has

$$\frac{\partial \phi_*(g)}{\partial x_i} = \phi_*([\tau_i, g]) = 0$$

for $1 \leq l \leq n-1$, *and therefore, one obtains*

$$\begin{aligned} \dot{x} &= \phi_*(f) + \phi_*(g) \\ &= Ax + \beta(y) + \rho(y, u). \end{aligned} \tag{8}$$

By definition (6) of ω, *one has*

$$\omega = \Xi^{-1}\theta = \begin{pmatrix} * & * & \cdots & * & 1 \\ * & * & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} dh \\ dL_f h \\ \vdots \\ dL_f^{n-2} h \\ dL_f^{n-1} h \end{pmatrix} = \begin{pmatrix} * \\ * \\ \vdots \\ * \\ dh \end{pmatrix}$$

which ensures that $x_n = \phi_n(\xi) = h(\xi)$, *thereby yielding* $y = Cx$ *with* $C = (0, \cdots, 0, 1)$.

Finally, with the deduced diffeomorphism $x = \phi(\xi)$, *it is straightforward to check that*

$$x(t^+) = \gamma(x(t^-)) + \Gamma_p(t^-)$$

at $t = t_k$ *where* γ *and* Γ_p *are specified in the statement of the theorem. Hence, system (1) can be transformed into (7) through the diffeomorphism* $x = \phi(\xi)$ *provided that Condition 1 of the theorem is satisfied. The proof of the theorem is thus completed.*

Theorem 1 gives necessary and sufficient conditions which guarantee the equivalence, via a diffeomorphism $x = \phi(\xi)$, between the nonlinear impulsive systems (1) and the proposed normal form (7). Due to this equivalence, one can design an observer for the transformed form (7) to estimate x, and then by applying the inverse of the diffeomorphism one can obtain the estimation of ξ, instead of designing observers for (1) to estimate ξ directly. It should also be noted that even if the observer is global for the transformed form (7), only a local state estimation of the original system (1) is available with the present approach that can become singular, dependent of the properties of the chosen diffeomorphism ([Ciccarella (1993)]). Thus motivated, the subsequent synthesis of full order and reduced order finite time observers is confined to the transformed system (7).

4 Observer design and observation error analysis

4.1 Uniform finite time observer

Provided that either condition of Theorem 1 is satisfied, the nonlinear system (1) can be represented, via a diffeomorphism, in the normal form (7) where the pair (A, C) is observable. Hence one can always find two different vectors K_1 and K_2, such that

$$P_i = (A - K_i C) \tag{9}$$

is Hurwitz for $1 \leq i \leq 2$.

For a prescribed small positive constant $\sigma \in (0, D_{\min})$ where $D_{\min}$ is defined in (2), it is obvious that one can always find two different Hurwitz matrices P_1 and P_2 in (9), such that the matrix $(I_{n\times n} - e^{P_2\sigma}e^{-P_1\sigma})$ is well-posed, thus is invertible. Hence the following matrix

$$\Lambda = (\Lambda_1, \Lambda_2) = (I_{n\times n} - e^{P_2\sigma}e^{-P_1\sigma})^{-1}(-e^{P_2\sigma}e^{-P_1\sigma}, I_{n\times n}) \in \mathbb{R}^{n\times 2n} \tag{10}$$

is well-posed. Then the following result is in order.

Theorem 2 *Let either Condition 1 or Condition 2 of Theorem 1 be satisfied. Given a prescribed positive constant $\sigma \in (0, D_{\min})$, let K_1 and K_2 be two different vectors such that $P_i,\ i = 1, 2$ in (9) are Hurwitz, and Λ in (10) is well-posed. Then the following coupled impulsive dynamics*

$$\left\{\begin{array}{ll} \dot{z}_i = Az_i + \beta(y) + \rho(y, u) + K_i(y - Cz_i), & t \neq t_{k-1} \text{ and } t \neq t_{k-1} + \sigma \\ z_i(t^+) = \Lambda \left[\begin{array}{c} z_1(t^-) \\ z_2(t^-) \end{array}\right], & t = t_{k-1} + \sigma \\ z_i(t^+) = \gamma\left(z(t^-)\right), & t = t_k \\ z_i(t_0^+) = z_0, & t_0 = 0 \end{array}\right. \tag{11}$$

with $i = 1, 2$ and the output $\eta = z_1$ (alternatively, with the output $\eta = z_2$) form a uniform finite time observer for (7) with the prescribed settling time estimate σ.

Proof 2 *Denote $\epsilon_i = z_i - x$ for $i = 1, 2$. For a prescribed $\sigma \in (0, D_{\min})$, one has $t_{k-1} + \sigma < t_k$. When $k = 1$, (7) and (11) yield*

$$\dot{\epsilon}_i(t) = (A - K_iC)\epsilon_i(t) = P_i\epsilon_i(t), \text{ for } t \in [t_0, t_0 + \sigma)$$

which ensures that

$$\begin{array}{l} z_1((t_0+\sigma)^-) = x(t_0+\sigma) + e^{P_1\sigma}(z_0 - x_0) \\ z_2((t_0+\sigma)^-) = x(t_0+\sigma) + e^{P_2\sigma}(z_0 - x_0) \end{array} \tag{12}$$

Multiplying (12) by Λ specified by (10) results in

$$x(t_0 + \sigma) = \Lambda \left[\begin{array}{c} z_1((t_0+\sigma)^-) \\ z_2((t_0+\sigma)^-) \end{array}\right]$$

The above equality, combined with (11), yields

$$z_i((t_0+\sigma)^+) = x(t_0+\sigma),$$

thereby ensuring the exact estimation of $x(t)$ when $t_0+\sigma \le t \le t_1^-$.

Since $z_i((t_0+\sigma)^+) = x(t_0+\sigma)$ for $i = 1, 2$, thus z_i of system (11) has the same dynamics as x defined in (7) when $t \in (t_0+\sigma, t_1^-]$, and it implies

$$z_i(t_1^-) = x(t_1^-), \text{ for } i = 1, 2$$

and

$$z_i(t_1^+) = \gamma(z_i(t_1^-)), \text{ for } i = 1, 2$$

which can be seen again that (11) has the same initial condition $z_i(t_1^+)$ for the two coupled systems. By induction, for any $k \in Z^+$ and the corresponding time interval $[t_{k-1}, t_k)$, one has

$$\begin{array}{l} z_1(t_{k-1}^+) = z_2(t_{k-1}^+) \\ z_i((t_{k-1}+\sigma)^+) = x(t_{k-1}+\sigma) \\ z_i(t) = x(t), \text{ for } t_{k-1}+\sigma \le t \le t_k^- \end{array}$$

that completes the proof of Theorem 2.

Remark 1 *The validity of Theorem 2 relies on the well-posedness of the matrix $(I_{n\times n} - e^{P_2\sigma}e^{-P_1\sigma})$. Thus the prescribed finite time σ cannot be too close to zero, otherwise, this matrix becomes ill-posed. In this regard, given a fixed σ, the higher negative real parts of eigenvalues of both P_1 and P_2 are chosen as well as the larger distances between the real parts of eigenvalues of P_1 and P_2 are the better for inverting the matrix $(I_{n\times n} - e^{P_2\sigma}e^{-P_1\sigma})$ numerically.*

4.2 Observation error analysis

For the proposed observer, it is proved that it estimates the state of the original system in finite time for each interval $[t_{k-1}, t_k)$. Because of the existence of periodically acting impulse perturbation, it is important to globally characterize the influence of those impulse perturbations.

Theorem 3 *The observation error of the proposed finite time observer (11) is bounded, and the following L_2-gain inequality is satisfied:*

$$\int_{t_0^+}^{\infty} \epsilon_i(t)^T \epsilon_i(t) dt \le \rho_i \sum_{k=1}^{\infty} ||\Gamma_p(t_{k-1})||_2^2 \tag{13}$$

with the attenuation level

$$\rho_i = \frac{e^{2\sigma\lambda_i^{max}} - 1}{2\lambda_i^{max}} \tag{14}$$

where $\Gamma_p(t_{k-1}) = \Gamma_p(t_{k-1}^-)$ for $1 \le k$ with $\Gamma(t_0) = z_0 - x_0$ being the observation error of initial conditions, and λ_i^{max} is the maximum real part of the eigenvalues of $\frac{P_i+P_i^T}{2}$ defined in (9).

Proof 3 *As shown in the proof of Theorem 2, for any $k \in N$ and for a fixed interval $[t_{k-1}, t_k)$, one always has*

$$\dot{\epsilon}_i(t) = P_i \epsilon(t), \text{ when } t_{k-1} \le t < t_{k-1} + \sigma$$

where $\epsilon(t_{k-1}) = \Gamma_p(t_{k-1})$ with $\Gamma_p(t_{k-1}) = z_0 - x_0$, and

$$\epsilon_i(t) = 0, \text{ when } t_{k-1} + \sigma \le t < t_k$$

Then one has

$$\epsilon_i(t) = e^{P_i(t-t_{k-1})}\Gamma_p(t_{k-1}), \text{ when } t_{k-1} \le t < t_{k-1} + \sigma$$

which yields

$$\begin{aligned} ||\epsilon_i(t)||_2^2 &= ||e^{P_i(t-t_{k-1})}\Gamma_p(t_{k-1})||_2^2 \\ &\le e^{2\lambda_i^{max}(t-t_{k-1})}||\Gamma_p(t_{k-1})||_2^2 \end{aligned}$$

thus arriving at

$$\begin{aligned} \int_{t_{k-1}^+}^{t_k} \epsilon_i(t)^T \epsilon_i(t) dt &= \int_{t_{k-1}^+}^{t_{k-1}+\sigma} ||e^{P_i(t-t_{k-1})}\Gamma_p(t_{k-1})||_2^2 dt \\ &\le ||\Gamma_p(t_{k-1})||_2^2 \int_{t_{k-1}^+}^{t_{k-1}+\sigma} e^{2\lambda_i^{max}(t-t_{k-1})} dt \\ &= \frac{e^{2\lambda_i^{max}\sigma} - 1}{2\lambda_i^{max}} ||\Gamma_p(t_{k-1})||_2^2 \\ &= \rho_i ||\Gamma_p(t_{k-1})||_2^2 \end{aligned}$$

It follows

$$\begin{aligned} \int_{t_0^+}^{\infty} \epsilon_i(t)^T \epsilon_i(t) dt &= \sum_{k=1}^{\infty} \int_{t_{k-1}^+}^{t_k} \epsilon_i(t)^T \epsilon_i(t) dt \\ &\le \rho_i \sum_{k=1}^{\infty} ||\Gamma_p(t_{k-1})||_2^2. \end{aligned}$$

The proof is completed.

Remark 2 *The attenuation level ρ_i given by (14) depends only on the maximum real part of the eigenvalues of P_i and the prescribed finite time σ, thus it is straightforward to obtain that*

$$\lim_{\lambda_i^{max} \to 0^-} \rho_i = \sigma, \text{ and } \lim_{\lambda_i^{max} \to -\infty} \rho_i = 0$$

which yields the following bounds on the observation error:

$$0 \le \int_{t_0^+}^{\infty} \epsilon_i(t)^T \epsilon_i(t) dt \le \sigma \sum_{k=1}^{\infty} ||\Gamma_p(t_{k-1})||_2^2$$

Since the pair (A, C) is observable the values of λ_i^{max}, $i = 1, 2$ can be prespecified to have an arbitrarily large magnitude if vectors K_1 and K_2 are properly chosen, so that ρ_i can be tuned to be small enough in order to perfectly attenuate the influence of the disturbance to an arbitrarily small attenuation level, given a priori. The role of the map $\Gamma_p(t_{k-1})$ represents the initial error introduced by the impact disturbance at each t_{k-1}.

4.3 Robustness with noisy measurement

It is well known that the measurements are usually corrupted by noises, thus this section discusses the robustness issue for the proposed observer (11) with respect to noisy measurements. Under this situation, without loss of generalities, let us suppose that the output of the transformed normal form (7) is corrupted by a bounded noise $\mu(t)$ as follows:

$$y(t) = Cx(t) + \mu(t) \tag{15}$$

then one can state the following theorem.

Theorem 4 *For the transformed canonical form* (7) *with the noisy output* (15)*, the observation error for the proposed observer* (11) *at the prescribed time* $t_k+\sigma$ *is bounded, independent of the unknown persistent impact perturbation* $\Gamma_p(t_k^-)$*, and satisfies the following inequality:*

$$||\epsilon_1((t_k+\sigma)^+)|| = ||\epsilon_2((t_k+\sigma)^+)|| \le \kappa\mu_0 \tag{16}$$

where $\mu_0 = \sup_{t\ge 0}\{\mu(t)\}$ *and*

$$\kappa = \tilde{\rho}_1||K_1|| + \frac{\tilde{\rho}_1||K_1|| + \tilde{\rho}_2||K_2||}{|1-||e^{(P_2-P_1)\sigma}||_2|}$$

with $\tilde{\rho}_1 = \frac{e^{\lambda_1^{max}\sigma}-1}{\lambda_1^{max}}$ *and* $\tilde{\rho}_2 = \frac{e^{\lambda_2^{max}\sigma}-1}{\lambda_2^{max}}$*,* σ *is the prescribed small positive constant,* K_i *and* P_i *for* $1\le i\le 2$ *are given in* (9)*.*

Proof 4 *Firstly, let us prove the boundedness of* Λ *defined in* (10)*. Since*

$$\begin{aligned} ||I_{n\times n} - e^{P_2\sigma}e^{-P_1\sigma}||_2 &\ge |||I_{n\times n}||_2 - ||e^{P_2\sigma}e^{-P_1\sigma}||_2| \\ &= |1-||e^{P_2\sigma}e^{-P_1\sigma}||_2| \end{aligned}$$

thus

$$||\Lambda_2||_2 \;=\; ||(I_{n\times n} - e^{P_2\sigma}e^{-P_1\sigma})^{-1}||_2 \le \frac{1}{|1-||e^{P_2\sigma}e^{-P_1\sigma}||_2|}$$

and

$$||\Lambda_1||_2 \;=\; ||I_{n\times n} - \Lambda_2||_2 \le 1 + ||\Lambda_2||_2 \le 1 + \frac{1}{|1-||e^{P_2\sigma}e^{-P_1\sigma}||_2|}$$

Secondly, let us prove the boundedness of the observation error at the prescribed time σ*. For the sake of simplicities, consider firstly* $t\in(t_0,t_1)$*. With the noisy output* (15) *and following the proof of Theorem 2, for* $1\le i\le 2$ *one has*

$$\dot{\epsilon}_i(t) = (A-K_iC)\epsilon_i(t) + K_i\mu(t) = P_i\epsilon_i(t) + K_i\mu(t), \;\textit{for } t\in[t_0,t_0+\sigma)$$

After the multiplication with Λ defined in (10), this gives the following observation error when $t = t_0 + \sigma$:

$$\epsilon_i((t_0+\sigma)^+) = \Lambda_1 \int_{t_0}^{t_0+\sigma} e^{P_1(t_0+\sigma-\tau)} K_1 \mu(\tau)\, d\tau + \Lambda_2 \int_{t_0}^{t_0+\sigma} e^{P_2(t_0+\sigma-\tau)} K_2 \mu(\tau)\, d\tau$$

which implies that

$$\begin{aligned}
||\epsilon_i((t_0+\sigma)^+)|| &\leq \mu_0 ||K_1|| ||\Lambda_1||_2 \int_{t_0}^{t_0+\sigma} ||e^{P_1(t_0+\sigma-\tau)}|| d\tau + \mu_0 ||K_2|| ||\Lambda_2||_2 \int_{t_0}^{t_0+\sigma} ||e^{P_2(t_0+\sigma-\tau)}|| d\tau \\
&\leq \mu_0 ||K_1|| ||\Lambda_1||_2 \int_{t_0}^{t_0+\sigma} e^{\lambda_1^{max}(t_0+\sigma-\tau)} d\tau + \mu_0 ||K_2|| ||\Lambda_2||_2 \int_{t_0}^{t_0+\sigma} e^{\lambda_2^{max}(t_0+\sigma-\tau)} d\tau \\
&\leq \mu_0 ||K_1|| ||\Lambda_1||_2 \frac{e^{\lambda_1^{max}\sigma} - 1}{\lambda_1^{max}} + \mu_0 ||K_2|| ||\Lambda_2||_2 \frac{e^{\lambda_2^{max}\sigma} - 1}{\lambda_2^{max}} \\
&\leq \mu_0 ||K_1|| ||\Lambda_1||_2 \tilde{\rho}_1 + \mu_0 ||K_2|| ||\Lambda_2||_2 \tilde{\rho}_2 = \kappa \mu_0
\end{aligned}$$

Since at each time $t = t_k$, the observer states are set as $z_1(t^+) = z_2(t^+) = \gamma(z(t^-))$, thus the above argument can be applied as well for any interval $[t_k, t_{k+1})$. Therefore, by induction, one can deduce that at each time $t = t_k + \sigma$, the observation errors are always bounded as (16). Moreover, since κ in (16) does not depend on the unknown persistent impact perturbation $\Gamma_p(t_k^-)$, one can conclude that the observation error for the proposed observer (11) at the prescribed time $t_k + \sigma$ is bounded, and does not depend on the unknown persistent impact perturbation.

Remark 3 *Due to the existence of noise in the measurement, one can only conclude the boundedness of $\epsilon_i((t_k+\sigma)^+)$, which is however equal to 0 for the noise free case. Thus, for $t_k + \sigma \leq t \leq t_{k+1}^-$, the observation errors are influenced only by the measurement noise, but not by the unknown persistent impact perturbation, and this error is governed by the following dynamics:*

$$\dot{\epsilon}_i(t) = P_i \epsilon_i(t) + K_i \mu(t)$$

therefore it can be easily seen that the ϵ_i is ISS ([Sontag(1995)]), since the initial condition $\epsilon_i((t_k+\sigma)^+)$ and the measurement noise $\mu(t)$ are both bounded.

4.4 Reduced order uniform finite time observer

The uniform finite time observer given in (11) is of dimension $2n$, since two coupled Luenberger observers are involved. It is well known that, for linear system, the existence of full order observer implies the existence of reduced order observer, which is used to only estimate the states except the output. Hence we can also design a reduced order observer for the systems with impacts. For this, let us reconsider the proposed normal form (7) which can be decomposed into the following form

$$\begin{cases} \dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + \beta_1(y) + \gamma_1(y,u), & t \in [t_{k-1}, t_k) \\ \dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + \beta_2(y) + \gamma_2(y,u), & t \in [t_{k-1}, t_k) \\ x(t^+) = \gamma(x(t^-)) + \Gamma_p(t^-), & t = t_k \\ y(t) = x_2(t) = Cx(t) \\ x(t_0^+) = x_0, & t_0 = 0 \end{cases} \tag{17}$$

where $x_1 \in \mathbb{R}^{n-1}$ and $x_2 \in \mathbb{R}$ are the decomposed state vectors; $A_{11} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}$, $A_{21} = [0, \cdots, 0, 1] \in \mathbb{R}^{1\times(n-1)}$, $A_{12} = 0_{(n-1)\times 1} \in \mathbb{R}^{(n-1)\times 1}$ and $A_{22} = 0_{1\times 1} \in \mathbb{R}$ are the corresponding decomposed matrices.

It is worth noticing that the observability of (A, C) in (7) ensures that (A_{11}, A_{21}) in (17) is observable as well. Thus, similarly to full order observer design for (7), one can also find two different vectors K_1 and K_2, such that

$$Q_i = (A_{11} - K_i A_{21}) \tag{18}$$

is Hurwitz for $i = 1, 2$.

Following the same argument as that for full order observer design, for a prescribed small positive constant $\sigma \in (0, D_{\min})$ where $D_{\min}$ is defined in (2), one can also find two different Hurwitz matrices Q_1 and Q_2 defined in (18), such that the matrix $(I_{(n-1)\times(n-1)} - e^{Q_2\sigma}e^{-Q_1\sigma})$ is well-posed.

Denote

$$\Omega = (I_{(n-1)\times(n-1)} - e^{Q_2\sigma}e^{-Q_1\sigma})^{-1}(-e^{Q_2\sigma}e^{-Q_1\sigma}, I_{(n-1)\times(n-1)}) \in \mathbb{R}^{(n-1)\times 2(n-1)} \tag{19}$$

then one has the following result.

Theorem 5 *For a prescribed positive constant $\sigma \in (0, D_{\min})$, let K_i for $i = 1, 2$ be two different vectors such that Q_i in (18) is Hurwitz, and Ω in (19) is well-defined. Then the following coupled impulsive dynamics*

$$\begin{cases} \dot{\zeta}_i = (A_{11} - K_i A_{21})\zeta_i + \beta_1(y) + \gamma_1(y,u) + (A_{11}K_i + A_{12} - K_i A_{22} - K_i A_{21} K_i)y \\ \qquad -K_i\left(\beta_2(y) + \gamma_2(y,u)\right), \quad t \neq t_{k-1} \text{ and } t \neq t_{k-1} + \sigma \\ \zeta_i(t^+) = \Omega \begin{pmatrix} \zeta_1(t^-) + K_1 y \\ \zeta_2(t^-) + K_2 y \end{pmatrix} - K_i y, \quad t = t_{k-1} + \sigma \\ \zeta_i(t^+) = \gamma(\zeta(t^-) + k_i y(t^-)) - k_i y(t^-), \quad t = t_k \\ z_i = \zeta_i + K_i y \end{cases} \tag{20}$$

with initial conditions satisfying

$$\zeta_2(t_0^+) = \zeta_1(t_0^+) + (K_1 - K_2)y(t_0)$$

and the output $\eta = [z_1, y]^T$ (alternatively, with the output $\eta = [z_2, y]^T$) form a uniform finite time observer for (7) with the prescribed settling time estimate σ.

Proof 5 *Similarly to the proof of Theorem 2, denote $\epsilon_i = z_i - x_1$ for $i = 1, 2$. According to (20), for $t \neq t_{k-1} + \sigma$ where $\sigma \in (0, D_{\min})$ is a prescribed constant, one obtains*

$$\begin{aligned} \dot{z}_i &= \dot{\zeta}_i + K_i \dot{y} \\ &= Q_i z_i + A_{12} y + K_1 A_{21} x_1 + \beta_1(y) + \rho_1(y, u) \end{aligned}$$

and this, combined with (17), yields the following dynamics

$$\dot{\epsilon}_i(t) = Q_i \epsilon_i(t), \text{ for } t \neq t_{k-1} + \sigma$$

When $k = 1$, the above equality implies

$$\begin{aligned} z_1((t_0 + \sigma)^-) = x_1(t_0 + \sigma) + e^{Q_1 \sigma}(z_1(t_0^+) - x_0) \\ z_2((t_0 + \sigma)^-) = x_1(t_0 + \sigma) + e^{Q_2 \sigma}(z_2(t_0^+) - x_0) \end{aligned} \tag{21}$$

Since

$$\zeta_2(t_0^+) = \zeta_1(t_0^+) + (K_1 - K_2) y(t_0)$$

then one has

$$z_1(t_0^+) = z_2(t_0^+)$$

Consequently, multiplying (21) by Ω defined in (19) gives

$$x_1(t_0 + \sigma) = \Omega \begin{pmatrix} z_1((t_0 + \sigma)^-) \\ z_2((t_0 + \sigma)^-) \end{pmatrix}$$

From the impulsive dynamics of ζ_i in (20), one obtains

$$z_i((t_0 + \sigma)^+) = x_1(t_0 + \sigma)$$

and the rest of the proof follows the same line of reasoning as that of the proof of Theorem 2.

5 Illustrative example

In order to highlight the proposed finite time observers, let us study the following system

$$\left\{ \begin{array}{ll} \dot{\xi}_1 = \frac{\xi_2^2 - 2\xi_1^2\xi_2 - 2\xi_1^2\xi_2^3 - 2\xi_1\xi_2^2}{1+\xi_2^2} + \frac{1}{1+\xi_2^2} u, & t \in [t_{k-1}, t_k] \\ \dot{\xi}_2 = \xi_1 + \xi_1 \xi_2^2 + \xi_2 & \\ \xi(t_k^+) = \left[\begin{array}{l} \gamma(\xi_1(t_k^-) + \xi_1(t_k^-)\xi_2^2(t_k^-)) \\ \xi_2^2(t_k^-) \end{array} \right] + \Gamma_p(t_k^-), & t = t_k \\ y = \xi_2 & \\ \xi(t_0^+) = \xi_0, & t_0 = 0 \end{array} \right. \tag{22}$$

One has

$$\theta_1 = d\xi_2, \theta_2 = (1+\xi_2^2)d\xi_1 + (1+2\xi_2)d\xi_2$$

thus

$$\tau_1 = \tfrac{1}{1+\xi_2^2}\tfrac{\partial}{\partial\xi_1}, \tau_2 = -\tfrac{2\xi_1\xi_2}{1+\xi_2^2}\tfrac{\partial}{\partial\xi_1} + \tfrac{\partial}{\partial\xi_2}$$

It is straightforward to check that $[\tau_1, \tau_2] = 0$ and $[\tau_1, g] = [\tau_2, g] = 0$ with $g = \frac{1}{1+\xi_2^2}\frac{\partial}{\partial\xi_1}$.

Then one has

$$\Xi = \theta\tau = \begin{pmatrix} 0, & 1 \\ 1, & -2\xi_1\xi_2 + 2\xi_2 \end{pmatrix}$$

which yields

$$\omega = \Xi^{-1}\theta = \begin{pmatrix} 1+\xi_2^2, & 2\xi_1\xi_2 \\ 0, & 1 \end{pmatrix} = \begin{pmatrix} d\left(\xi_1(1+\xi_2^2)\right) \\ d\xi_2 \end{pmatrix}$$

This gives the following diffeomorphism:

$$\phi = \left(\xi_1(1+\xi_2^2), \xi_2\right)^T$$

through which the studied system proves to be equivalent to the following form:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} y^2 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, & t \in [t_{k-1}, t_k) \\ x(t_k^+) = \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} x(t_k^-) + \Gamma_p(t_k^-), & t = t_k \\ y(t) = [0, 1]\, x(t) & \\ x(t_0^+) = x_0, & t_0 = 0 \end{cases} \tag{23}$$

where Γ_p represents the unknown impact uncertainties.

By choosing $K_1 = [4, 4]^T$ and $K_2 = [100, 20]^T$, one arrives at Hurwitz matrices P_1 and P_2. Setting $\sigma \in \{0.2s, 0.8s\}$, the states of (23) are estimated for each $[t_{k-1}, t_k)$ at the finite time $\sigma = 0.2s$ or $0.8s$. Thus, one can design a full order finite time observer for (23) of the form (11). In the simulation, we set $t_k - t_{k-1} = 5s$ for $k \in Z^+$, $\gamma = -0.8$ and the unknown impact uncertainty $\Gamma_p(t_k^-)$ belongs to the interval $(-0.4, 0.4)$; $u = \ddot{y}_r - 100(x_1 - \dot{y}_r) - 20(x_2 - y_r)$ with $y_r = |\sin(\pi/5t)|$ for $i \in \{1, 2\}$. In the simulation, the Dini derivative ([Garg(1998)]) is considered for the non differential reference y_r at time $t = 5k$ for $k \in Z^+$. The simulation results, supporting the theory, are depicted in Fig. 1 and 2. Good performance of the proposed observer design is concluded from these figures.

Figure 1: Simulation of the full order finite time observer with $\sigma = 0.8s$.

Figure 2: Simulation of the full order finite time observer with $\sigma = 0.2s$.

6 Conclusion

Due to the existence of impact perturbation in nonlinear impulsive systems, the convergence of traditional observer highly depends on the initial condition and periodic impact perturbation, thus resulting in an inconsistent performance. This paper conceptually develops the finite time observer design of impact systems. First a normal form is specified for nonlinear impulsive systems and then based on such a form, two types of finite time observers of full and reduced orders are justified to estimate the state of the underlying system in a prescribed small interval in spite of persisten impact perturbations.

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