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# Generalized connected domination in graphs

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As a generalization of connected domination in a graph  $G$  we consider domination by sets having at most  $k$  components. The order  $\gamma_c^k(G)$  of such a smallest set we relate to  $\gamma_c(G)$ , the order of a smallest connected dominating set. For a tree  $T$  we give bounds on  $\gamma_c^k(T)$  in terms of minimum valency and diameter. For trees the inequality  $\gamma_c^k(T) \leq n - k - 1$  is known to hold, we determine the class of trees, for which equality holds.

**Keywords:** connected domination, domination, tree

**Mathematics Subject Classification:** 05C69

## 1 Introduction

We consider simple non-oriented graphs. The largest valency in  $G$  is denoted by  $\Delta(G) = \Delta$ , the smallest by  $\delta(G) = \delta$ . By  $P_n$  we denote a path on  $n$  vertices and  $C_n$  denotes a circuit on  $n$  vertices. In a graph a **leaf** or **pendant vertex** is a vertex of valency one and a **stem** is a vertex adjacent to at least one leaf. In  $K_2$  each vertex is both a leaf and a stem. The set of leaves in a graph  $G$  is denoted by  $\Omega(G)$ . The set of neighbours to a vertex  $x$  is denoted  $N(x)$ . By  $K_{1,k}$  we denote a star with one central vertex joined to  $k$  other vertices. A **subdivided star** is a star with a subdivision vertex on each edge. By the **corona graph** on  $H$  we understand the graph  $G = H \circ K_1$  obtained from the graph  $H$  by adding for each vertex  $x$  in  $H$  one new vertex  $x'$  and one new edge  $xx'$ . In a corona graph each vertex is either a leaf or a stem adjacent to exactly one leaf. In particular, if  $H$  is a tree, we obtain a **corona tree**  $T = H \circ K_1$ .

The **eccentricity**  $e(x)$  of a vertex  $x$  is defined by  $e(x) = \max\{d(x, y) | y \in V(G)\}$ . The **diameter** of  $G$  is  $\text{diam}(G) = \max\{e(x) | x \in V(G)\}$ . Let  $D \subseteq V(G)$ , then  $N(D)$  is the set of vertices which have a neighbour in  $D$  and  $N[D]$  is the set of vertices which are in  $D$  or have a neighbour in  $D$ ,  $N[D] = D \cup N(D)$ . A set  $D \subseteq V(G)$  **dominates**  $G$  if  $V(G) \subseteq N[D]$ , i.e. each vertex not in  $D$  is adjacent to a vertex in  $D$ . The **domination number**  $\gamma(G)$  is the cardinality of a smallest dominating set in  $G$ .

For a given graph  $G$  it is NP-hard to determine its domination number  $\gamma(G)$ , but we can search for upper bounds as O. Ore started doing about fifty years ago. Also it may be more tractable to restrict the

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minimum dominating set problem to consider only such dominating sets which induce a connected subset of  $G$ , this problem is called **the minimum connected dominating problem** and it is still NP-complete; In network design theory it is called the **maximum leaf spanning tree problem** [4], the name will be clear from Section 2 below. We shall study a concept intermediate to the classical and the connected domination, namely by demanding the dominating set to induce at most a given number  $k$  of components, we aim at presenting upper bounds for its order  $\gamma_c^k$ . Quite likely there is a corresponding problem in network design theory, although we are aware of no reference.

A comprehensive introduction to domination theory is given in [7, 14] and variations are discussed in [5, 13, 15].

Ore [10] proved the inequality below while C. Payan and N. H. Xuong [11], Fink, Jacobsen, Kinch and Roberts [3] determined its extremal graphs.

**Proposition 1** *Let  $G$  be a connected graph with  $n$  vertices,  $n \geq 2$ . Then  $\gamma(G) \leq \frac{n}{2}$  and equality holds if and only if  $G$  is either a corona graph or a 4-circuit.*

If a tree  $T$  has  $\gamma(T) = \frac{n}{2}$ , then  $n$  is even and Proposition 1 implies that  $T$  is a corona tree.

**Definition** For a positive integer  $k$  and a graph  $G$  with at most  $k$  components we define

$$\gamma_c^k(G) = \min \{|D| \mid D \subseteq V(G), D \text{ has at most } k \text{ components and } D \text{ dominates } G\}.$$

A set  $D$  attaining the minimum above is called a  $\gamma_c^k$ -set for  $G$ .

**Example**

$$\gamma_c^k(P_n) = \gamma_c^k(C_n) = \begin{cases} n - 2k & \text{for } n \geq 3k \\ \lceil \frac{n}{3} \rceil & \text{for } 1 \leq n \leq 3k \end{cases}$$

For  $k = 1$  we have that  $\gamma_c^1$  is the usual connected domination number,  $\gamma_c^1(G) = \gamma_c(G)$ .

There exists for every graph  $G$  a  $k$  such that  $\gamma_c^k(G) = \gamma(G)$ , e.g.  $k = |G|$ .

For  $G$  connected and  $k \geq 1$ , obviously,  $\gamma(G) \leq \gamma_c^k(G) \leq \gamma_c(G)$ .

## 2 General graphs

Let  $G$  be a connected graph with  $n$  vertices and  $k$  a positive integer. Let  $\epsilon_F(G)$  be the maximum number of leaves among all spanning forests of  $G$ , and  $\epsilon_T(G)$  be the maximum number of leaves among all spanning trees of  $G$ . With this notation Niemen [9] proved statement (i) below about  $\gamma$  and Hedetniemi and Laskar [8] generalized it to statement (ii) about  $\gamma_c$ .

$$(i) \quad \gamma(G) = n - \epsilon_F(G),$$

$$(ii) \quad \gamma_c(G) = n - \epsilon_T(G).$$

In the next two theorems we extend these results to  $\gamma_c^k$ .

**Theorem 1** *Let  $G$  be a connected graph with  $n$  vertices and  $k$  a positive integer. Let  $\epsilon_{F_k}(G)$  be the maximum number of leaves among all spanning forests of  $G$  with at most  $k$  trees. Then*

$$\gamma_c^k(G) = n - \epsilon_{F_k}(G).$$

**Proof:** In any spanning forest  $F$  with at most  $k$  trees the leaves will be dominated by their stems, so  $\gamma_c^k(G) \leq n - |\Omega(F)|$  and hence  $\gamma_c^k(G) \leq n - \epsilon_{F_k}(G)$ .

Conversely, let  $D = D_1 \cup D_2 \cup \dots \cup D_t$ ,  $1 \leq t \leq k$ , be a  $\gamma_c^k$ -set for  $G$ . Choose for each  $D_i$  a spanning tree  $T_i$ ,  $1 \leq i \leq t$ . For each vertex in  $V(G) \setminus D$  choose one edge which is incident with a vertex in  $D$ . We have constructed a spanning forest  $F$  with  $t$  components and at least  $n - |D| = n - \gamma_c^k(G)$  leaves. Therefore  $\epsilon_{F_k}(G) \geq n - \gamma_c^k(G)$  and Theorem 1 is proved.  $\square$

**Theorem 2** *Let  $k$  be a positive integer and  $G$  a connected graph. Then*

$$\begin{aligned} \gamma_c^k(G) &= \min \{ \gamma_c^k(F_k) \mid F_k \text{ is a spanning forest of } G \text{ with at most } k \text{ trees} \} \\ &= \min \{ \gamma_c^k(T) \mid T \text{ is a spanning tree of } G \} \end{aligned}$$

**Proof:** Let  $F_k$  be a spanning forest of  $G$  with at most  $k$  trees. Certainly  $\gamma_c^k(G) \leq \gamma_c^k(F_k)$  since a set which dominates  $F_k$  also dominates  $G$ . Conversely, we can in  $G$  find a spanning forest  $F_k$  with at most  $k$  components such that  $\gamma_c^k(G) = \gamma_c^k(F_k)$ : As was originally also done in the proofs for (i) and (ii) above we construct  $F_k$  from a  $\gamma_c^k$ -set  $D = D_1 \cup D_2 \cup \dots \cup D_t$ ,  $1 \leq t \leq k$ , by choosing a spanning tree  $T_i$  in each connected subgraph  $D_i$  and joining each vertex in  $V(G) \setminus D$  to precisely one vertex in  $D$ . Obviously,  $\gamma_c^k(F_k) \leq |D| = \gamma_c^k(G)$ . This proves the first equality. For the second equality we observe that the first minimum is chosen among a larger set, so that  $\min \gamma_c^k(F_k) \leq \min \gamma_c^k(T)$ , and also that any  $F_k$  by addition of edges can produce a tree  $T$  with  $\gamma_c^k(T) \leq \gamma_c^k(F_k)$ .  $\square$

Hartnell and Vestergaard [6] proved the following result.

**Proposition 2** *For  $k \geq 1$  and  $G$  connected*

$$\gamma_c(G) - 2(k - 1) \leq \gamma_c^k(G) \leq \gamma_c(G).$$

From Proposition 2 we can easily derive the following corollary which is a classical result proven by Duchet and Meyniel. [2]

**Corollary 3** *For any connected graph  $G$ ,  $\gamma_c(G) \leq 3\gamma(G) - 2$ .*

**Proof:** Let  $G$  be a connected graph with domination number  $\gamma(G)$ . Choose  $k = \gamma(G)$ , then  $\gamma_c^k(G) = \gamma(G)$ . Substituting into Proposition 2 above we obtain  $\gamma_c(G) - 2(k - 1) \leq \gamma(G)$  and that proves the corollary.  $\square$

## 2.1 Other bounds on $\gamma_c^k$

**Theorem 4** *For a positive integer  $k$  and a connected graph  $G$  with maximum valency  $\Delta$  we have*

(A)  $\gamma_c(G) \leq n - \Delta$  and for trees  $T$  equality holds if and only if  $T$  has at most one vertex of valency  $\geq 3$ .

(B)  $\gamma_c^k(G) \leq n - \frac{(r-1)(\delta-2)}{3} - 2k$  if  $G$  has diameter  $r \geq 3k - 1$  and the minimum valency  $\delta = \delta(G)$  is at least 3.

(C) If  $G$  is a connected graph with two vertices of valency  $\Delta$  at distance  $d$  apart,  $d \geq 3$ , then

$$\gamma_c^k(G) \leq n - 2(\Delta - 1) - 2 \min\left\{k - 1, \frac{d - 2}{3}\right\}. \quad (1)$$

(D) Let  $x \in V(G)$  have valency  $d(x)$  and eccentricity  $e(x)$ . Then

$$\gamma_c^k(G) \leq n - d(x) - 2 \min\left\{k - 1, \frac{e(x) - 2}{3}\right\}. \quad (2)$$

**Proof:**

(A) Let  $T$  be a spanning tree of  $G$  with  $\Delta(T) = \Delta(G) = \Delta$ , then  $T$  has at least  $\Delta$  leaves, and hence  $\gamma_c(G) \leq \gamma_c(T) \leq n - \Delta$ .

If  $T$  has two vertices of valency  $\geq 3$ , the number of leaves in  $T$  will be larger than  $\Delta$ , and we get strict inequality in (A). Clearly, a tree  $T$  with exactly one vertex of valency  $\Delta \geq 3$  has equality in (A) and for  $\Delta = 2$ , we obtain a path  $P_n$  with  $\gamma_c(P_n) = n - 2$ .

(B) Let  $P = v_1 v_2 v_3 \dots v_{3t+u}$ ,  $k \leq t, 0 \leq u \leq 2$ , be a diametrical path in  $G$ . The diameter of  $T$  equals the length of  $P$ , which is  $r = 3t + u - 1$ . For  $i = 1, \dots, t$  let  $v_{3i-1}$  have neighbours  $v_{3i-2}, v_{3i}$  on  $P$  and  $a_{ij}$  off  $P$ ,  $j = 1, \dots, s_i$ ,  $s_i \geq \delta - 2 \geq 1$ . In  $G - \{v_{3i} v_{3i+1} | 1 \leq i \leq k - 1\}$  consider the  $k - 1$  disjoint stars with center  $v_{3i-1}$  and neighbours  $N(v_{3i-1})$ ,  $1 \leq i \leq k - 1$ , and the remaining tree to the right consisting of the path  $v_{3k-2} v_{3k-1} v_{3k} \dots v_{3t+u}$  and leaves  $v_{3i-1} a_{3i-1, j}$ ,  $j = 1, \dots, s_i$ ,  $s_i \geq \delta - 2 \geq 1$  adjacent to vertices  $v_{3i-1}$ ,  $k \leq i \leq t$ .

Extend this forest of  $k$  trees to a spanning forest  $F$  with  $k$  trees in  $G - \{v_{3i} v_{3i+1} | 1 \leq i \leq k - 1\}$ . The number of leaves in  $F$  is at least  $t(\delta - 2) + 2k$  and hence  $\gamma_c^k(G) \leq n - t(\delta - 2) - 2k$ . From

$$t = \frac{r + 1 - u}{3} \geq \frac{r - 1}{3} \text{ we obtain the desired result } \gamma_c^k(G) \leq n - \frac{(r - 1)(\delta - 2)}{3} - 2k.$$

C Let  $v_1, v_s$  be two vertices in  $G$  with maximum valency,  $d(v_1) = d(v_s) = \Delta$ , and let  $P = v_1 v_2 \dots v_s$  be a shortest  $v_1 v_s$ -path,  $s = 3t + 1 + u, t \geq 1, 0 \leq u \leq 2$ .

**Case 1,  $t \geq k - 1$ :** In  $G - \{v_{3i-1} v_{3i} | 1 \leq i \leq k - 2\}$  we extend the  $k$  trees listed below to a spanning forest  $F$  of  $G$ ,

1. The star consisting of  $v_1$  joined to all its neighbours,
2. the  $k - 2$  paths of length two  $v_{3i} v_{3i+1} v_{3i+2}$ ,  $1 \leq i \leq k - 2$ ,
3. the path  $v_{3k-3} v_{3k-2} \dots v_s$  together with all  $\Delta - 1$  neighbours of  $v_s$  outside of  $P$ .

$F$  will have at least  $2(\Delta - 1) + 2(k - 1)$  leaves.

**Case 2,  $t \leq k - 2$ :**  $s = 3t + 1 + u, d = d(v_1, v_s) = s - 1 = 3t + u, t - 1 = \frac{d - u}{3} - 1 \geq$

$\frac{d - 2}{3} - 1$ . As before, we can find a spanning forest  $F$  of  $G$  whose number of leaves is at least

$$2\Delta + 2(t - 1) \geq 2(\Delta - 1) + 2\frac{d - 2}{3} \text{ and consequently } \gamma_c^k(G) \leq n - 2(\Delta - 1) - 2\frac{d - 2}{3}.$$

The proof of D is similar.

□

### 3 Trees

For trees Hartnell and Vestergaard [6] found

**Proposition 3** *Let  $k$  be a positive integer and  $T$  a tree with  $|V(T)| = n, n \geq 2k + 1$ . Then  $\gamma_c^k(T) \leq n - k - 1$ .*

This inequality is best possible. For  $k = 1$  the extremal trees are paths  $P_n$  and for  $k \geq 2$  extremal trees will be described in the following Theorem 5.

A tree  $T$  is of type A if it contains a vertex  $x_0$  such that  $T - x_0$  is a forest of trees  $T_1, T_2, \dots, T_\alpha, \alpha \geq 1$ , such that each tree  $T_i$  is a corona tree and  $x_0$  is joined to a stem in each of the trees  $T_i, 1 \leq i \leq \alpha$ . We note that a subdivision of a star is a tree of type A.

A tree  $T$  is of type B if it contains a path  $uvw$  such that  $T - \{u, v, w\}$  is a forest of corona trees  $T_1, T_2, \dots, T_s, T_{s+1}, \dots, T_\alpha, \alpha \geq 2, 1 \leq s < \alpha$  and  $u$  is joined to a stem in each of the trees  $T_1, T_2, \dots, T_s$ , while  $w$  is joined to a stem in each of the trees  $T_{s+1}, \dots, T_\alpha$ .

Proposition 4 below was proven by Randerath and Volkmann [12], Baogen, Cockayne, Haynes, Hedetniemi and Shangchao [1].

**Proposition 4** *If  $T$  is a tree with  $n$  vertices,  $n$  odd, and  $\gamma(T) = \lfloor \frac{n}{2} \rfloor$  then  $T$  is a tree of type A or B.*

We shall now determine the trees extremal for Proposition 3.

**Theorem 5** *Let  $k \geq 2$  be a positive integer and  $T$  a tree with  $n$  vertices,  $n \geq 2k + 1$ . Then  $\gamma_c^k(T) = n - k - 1$  if and only if one of cases (i)-(iii) below occur.*

$$(i) \quad k = \frac{n-1}{2}, \gamma_c^k(T) = \gamma(T) = \frac{n-1}{2} \text{ and } T \text{ is of type A or B.}$$

$$(ii) \quad k = \frac{n-2}{2}, \gamma_c^k(T) = \gamma(T) = \frac{n}{2} \text{ and } T \text{ is a corona tree.}$$

$$(iii) \quad k = \frac{n-3}{2}, \gamma_c^k(T) = \frac{n+1}{2}, \gamma(T) = \frac{n-1}{2} \text{ and } T \text{ is a star } K_{1,k+1} \text{ with a subdivision vertex on each edge.}$$

**Proof:** First, let  $k \geq 2$  and a tree  $T$  of order  $n$  be given such that  $n \geq 2k + 1$  and  $\gamma_c^k(T) = n - k - 1$ . We shall prove that  $T$  is as described in one of the three cases (i)-(iii).

We note in passing that

**Remark 1**  $\gamma(T) \leq k$  implies  $\gamma_c^k(T) = \gamma(T)$ , and that likewise  $\gamma_c^k(T) \leq k$  implies  $\gamma_c^k(T) = \gamma(T)$ .

If  $n = 2k + 1$ , or equivalently  $k = \frac{n-1}{2}$ , we have by assumption  $\gamma_c^k(T) = n - k - 1 = k$  and, as just observed above, that implies that also  $\gamma(T) = k$ . Since  $k = \lfloor \frac{n}{2} \rfloor$  we obtain from Proposition 4 that  $T$  is a tree of type A or B, so Case (i) occurs.

If  $n = 2k + 2$ , or equivalently  $k = \frac{n-2}{2}$  we have by assumption  $\gamma_c^k(T) = n - k - 1 = k + 1$ . Certainly  $\gamma(T) \leq \gamma_c^k(T)$ , but if  $\gamma(T) \leq k$  then we should have that  $\gamma_c^k(T) = \gamma(T) \leq k$  in contradiction

to  $\gamma_c^k(T) = k + 1$ , therefore  $\gamma(T) = k + 1 = \frac{n}{2}$ . From Proposition 1 we obtain that  $T$  is a corona tree, i.e. Case (ii) occurs.

We may now assume  $n \geq 2k + 3$ , and we shall prove that, in fact,  $n$  equals  $2k + 3$  and that Case (iii) occurs.

Let  $v_1 v_2 \dots v_\alpha$  be a longest path in  $T$ . Since  $\gamma_c^k(T) = n - k - 1 \geq k + 2 \geq 4$ ,  $T$  is neither a star nor a bistar and therefore  $\alpha \geq 5$ . We must have  $d_T(v_2) = 2$ , because otherwise  $d_T(v_2) \geq 3$  and we could from  $T$  delete three leaves adjacent to  $v_2$ , if  $d_T(v_2) \geq 4$ , and in case  $d_T(v_2) = 3$  we could delete  $v_2$  and its two adjacent leaves. In both cases we would obtain a tree  $T'$  of order  $n - 3 \geq 2(k - 1) + 1$  which by Proposition 3 has  $\gamma_c^{k-1}(T') \leq (n - 3) - (k - 1) - 1 \leq n - k - 3$ . Adding  $v_2$  to a  $\gamma_c^{k-1}(T')$ -set we would obtain  $\gamma_c^k(T) \leq n - k - 2$ , a contradiction. Therefore  $d_T(v_2) = 2$ .

The vertex  $v_3$  cannot be adjacent to two leaves  $c$  and  $d$ , say, because, then the tree  $T' = T - \{v_1, v_2, c, d\}$  would have order  $n - 4 \geq 2(k - 1) + 1$ . Thus Proposition 3 gives that  $\gamma_c^{k-1}(T') \leq (n - 4) - (k - 1) - 1 \leq n - k - 4$  and adding  $v_2, v_3$  to a  $\gamma_c^{k-1}(T')$ -set we would obtain  $\gamma_c^k(T) \leq n - k - 2$ , a contradiction. So  $v_3$  can be adjacent to at most one leaf. The case  $d_T(v_3) = 3$  and  $v_3$  adjacent to one leaf  $c$  can similarly be seen to be impossible by considering  $T' = T \setminus \{v_1, v_2, v_3, c\}$ .

On the other hand  $d_T(v_3) \geq 3$ , for assume  $d_T(v_3) = 2$ , then  $T' = T \setminus \{v_1, v_2, v_3\}$  has  $\gamma_c^{k-1}(T') \leq n - k - 3$  and addition of  $v_2$  to a  $\gamma_c^{k-1}(T')$ -set would give  $\gamma_c^k(T) \leq n - k - 2$ , a contradiction.

Assume therefore that  $v_3$  besides  $v_2$  and  $v_4$  is adjacent to precisely one leaf  $c$  and to at least one further vertex  $a$ , where  $a$  has valency two and is adjacent to the leaf  $b$ . Then  $T' = T \setminus \{v_1, v_2, a, b\}$  has order  $n - 4 \geq 2(k - 1) + 1$  and Proposition 3 gives that (3)  $\gamma_c^{k-1}(T') \leq (n - 4) - (k - 1) - 1 \leq n - k - 4$ . In  $T'$  the vertex  $c$  is a leaf and as any  $\gamma_c^{k-1}$ -set for  $T'$  must contain one of  $\{v_3, c\}$ , we may assume it contains  $v_3$ . Addition of  $\{v_2, a\}$  to a  $\gamma_c^{k-1}(T')$ -set now gives the contradiction  $\gamma_c^k(T) \leq n - k - 2$ .

Assume finally that  $v_3$  has no leaf but besides  $v_2$  and  $v_4$  is adjacent to  $a_1, a_2, \dots, a_t$ ,  $t \geq 1$ , where each  $a_i$  has valency two and is adjacent to the leaf  $b_i$ ,  $1 \leq i \leq t$ .

We have  $k - t \geq 1$  because  $V(T) \setminus \{v_1, b_1, b_2, \dots, b_t, v_\alpha\}$  is a connected subgraph with  $n - t - 2$  vertices which dominate  $T$ , so that  $n - k - 1 = \gamma_c^k(T) \leq n - t - 2$  giving  $k - t \geq 1$ . Consider the tree  $T' = T \setminus \{v_1, v_2, a_1, a_2, \dots, b_1, b_2, \dots, b_t, v_3\}$  of order  $n - 2t - 3$ .

If  $n - 2t - 3 \geq 2(k - t) + 1$  we obtain by Proposition 3 that  $\gamma_c^{k-t}(T') \leq (n - 2t - 3) - (k - t) - 1 \leq n - k - t - 4$ , and by addition of the  $t + 2$  vertices  $\{v_2, v_3, a_1, a_2, \dots, a_t\}$ , (which span a connected subgraph of  $T$ ), to a  $\gamma_c^{k-t}(T')$ -set we obtain  $\gamma_c^k(T) \leq n - k - 2$ , a contradiction. So we have  $n - 2t - 3 \leq 2(k - t)$  and now  $|V(T')| = n - 2t - 3 \leq 2(k - t)$  implies  $\gamma(T') \leq \frac{|V(T')|}{2} \leq k - t$  which by remark 1 gives that  $\gamma_c^{k-t}(T') = \gamma(T')$  and hence addition of the  $t + 2$  vertices  $\{v_2, v_3, a_1, a_2, \dots, a_t\}$  to a  $\gamma_c^{k-t}(T')$ -set (having at most  $k - t$  vertices) gives  $\gamma_c^{k-t+1}(T) \leq k + 2$ . We now have  $n - k - 1 = \gamma_c^k(T) \leq \gamma_c^{k-t+1}(T) \leq k + 2$  giving  $n \leq 2k + 3$ , so the assumption  $n \geq 2k + 3$  implies  $n = 2k + 3$ . By hypothesis  $\gamma_c^k(T) = k + 2$  and we have  $\gamma(T) \leq k + 1$  by Proposition 1.

Thus  $\gamma(T) = k + 1$ , (because otherwise  $\gamma_c^k(T) = \gamma(T) < k + 2$ ), and any  $\gamma(T)$ -set must consist of  $k + 1$  isolated vertices. As  $\gamma(T) = \lfloor \frac{n}{2} \rfloor$  the tree  $T$  by Proposition 4 is of type A or B. But  $T$  cannot be of type B, for assume  $T$  is of type B. Then  $T$  consists of a 3-path,  $uvw$ , with each of its ends joined to stems of corona trees, and since we have just seen that  $v_3, v_{\alpha-2}$  are neither stems nor leaves, they must play the role of  $u, w$ , so  $\alpha = 7$  and  $T$  consists of two subdivided stars centered respectively at  $u = v_3$  and  $w = v_5$  and a vertex  $v = v_4$  joined to  $u$  and  $w$ . Among its  $\gamma$ -sets this tree  $T$  has one with two adjacent vertices, namely  $v_2$  and  $v_3$ , a contradiction, so  $T$  is of type A.

Using, in analogy to  $v_2, v_3$ , that  $d_T(v_{\alpha-1}) = 2$  and that  $v_{\alpha-2}$  is not a stem, we get that  $\alpha = 5$  and  $T$  is a subdivided star so that (iii) occurs.

Conversely, it is easy to see that if (i), (ii) or (iii) holds then  $\gamma_c^k(T) = \gamma(T) = n - k + 1$ . This proves Theorem 5.  $\square$

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