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As a generalization of connected domination in a graph G we consider domination by sets having at most k components. The order $\gamma_c^k(G)$ of such a smallest set we relate to $\gamma_c(G)$, the order of a smallest connected dominating set. For a tree T we give bounds on $\gamma_c^k(T)$ in terms of minimum valency and diameter. For trees the inequality $\gamma_c^k(T) \leq n-k-1$ is known to hold, we determine the class of trees, for which equality holds.

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1 Introduction

We consider simple non-oriented graphs. The largest valency in G is denoted by $\Delta(G) = \Delta$, the smallest by $\delta(G) = \delta$. By P_n we denote a path on n vertices and C_n denotes a circuit on n vertices. In a graph a **leaf** or **pendant vertex** is a vertex of valency one and a **stem** is a vertex adjacent to at least one leaf. In K_2 each vertex is both a leaf and a stem. The set of leaves in a graph G is denoted by $\Omega(G)$. The set of neighbours to a vertex x is denoted $N(x)$. By $K_{1,k}$ we denote a star with one central vertex joined to k other vertices. A **subdivided star** is a star with a subdivision vertex on each edge. By the **corona graph** on H we understand the graph $G = H \circ K_1$ obtained from the graph H by adding for each vertex x in H one new vertex x' and one new edge xx'. In a corona graph each vertex is either a leaf or a stem adjacent to exactly one leaf. In particular, if H is a tree, we obtain a **corona tree** $T = H \circ K_1$.

The **eccentricity** $e(x)$ of a vertex x is defined by $e(x) = \max\{d(x,y)|y \in V(G)\}$. The **diameter** of G is diam(G)=$\max\{e(x)|x \in V(G)\}$. Let $D \subseteq V(G)$, then $N(D)$ is the set of vertices which have a neighbour in D and $N[D]$ is the set of vertices which are in D or have a neighbour in D, $N[D] = D \cup N(D)$. A set $D \subseteq V(G)$ **dominates** G if $V(G) \subseteq N[D]$, i.e. each vertex not in D is adjacent to a vertex in D. The **domination number** $\gamma(G)$ is the cardinality of a smallest dominating set in G.

For a given graph G it is NP-hard to determine its domination number $\gamma(G)$, but we can search for for upper bounds as O. Ore started doing about fifty years ago. Also it may be more tractable to restrict the

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minimum dominating set problem to consider only such dominating sets which induce a connected subset of G, this problem is called **the minimum connected dominating problem** and it is still NP-complete; In network design theory it is called the **maximum leaf spanning tree problem** [4], the name will be clear from Section 2 below. We shall study a concept intermediate to the classical and the connected domination, namely by demanding the dominating set to induce at most a given number k of components, we aim at presenting upper bounds for its order γ_c^k. Quite likely there is a corresponding problem in network design theory, although we are aware of no reference.

A comprehensive introduction to domination theory is given in [7, 14] and variations are discussed in [5, 13, 15].

Ore [10] proved the inequality below while C. Payan and N. H. Xuong [11], Fink, Jacobsen, Kinch and Roberts [3] determined its extremal graphs.

Proposition 1 *Let G be a connected graph with n vertices, $n \geq 2$. Then $\gamma(G) \leq \dfrac{n}{2}$ and equality holds if and only if G is either a corona graph or a 4-circuit.*

If a tree T has $\gamma(T) = \dfrac{n}{2}$, then n is even and Proposition 1 implies that T is a corona tree.

Definition For a positive integer k and a graph G with at most k components we define

$$\gamma_c^k(G) = \min\left\{|D| | D \subseteq V(G), D \text{ has at most } k \text{ components and } D \text{ dominates } G\right\}.$$

A set D attaining the minimum above is called a γ_c^k-set for G.

Example

$$\gamma_c^k(P_n) = \gamma_c^k(C_n) = \begin{cases} n - 2k & \text{for } n \geq 3k \\ \lceil \frac{n}{3} \rceil & \text{for } 1 \leq n \leq 3k \end{cases}$$

For $k = 1$ we have that γ_c^1 is the usual connected domination number, $\gamma_c^1(G) = \gamma_c(G)$.
There exists for every graph G a k such that $\gamma_c^k(G) = \gamma(G)$, e.g. $k = |G|$.
For G connected and $k \geq 1$, obviously, $\gamma(G) \leq \gamma_c^k(G) \leq \gamma_c(G)$.

2 General graphs

Let G be a connected graph with n vertices and k a positive integer. Let $\epsilon_F(G)$ be the maximum number of leaves among all spanning forests of G, and $\epsilon_T(G)$ be the maximum number of leaves among all spanning trees of G. With this notation Niemen [9] proved statement (i) below about γ and Hedetniemi and Laskar [8] generalized it to statement (ii) about γ_c.

(i) $\gamma(G) = n - \epsilon_F(G)$,

(ii) $\gamma_c(G) = n - \epsilon_T(G)$.

In the next two theorems we extend these results to γ_c^k.

Theorem 1 *Let G be a connected graph with n vertices and k a positive integer. Let $\epsilon_{F_k}(G)$ be the maximum number of leaves among all spanning forests of G with at most k trees. Then*
$\gamma_c^k(G) = n - \epsilon_{F_k}(G)$.

Proof: In any spanning forest F with at most k trees the leaves will be dominated by their stems, so $\gamma_c^k(G) \leq n - |\Omega(F)|$ and hence $\gamma_c^k(G) \leq n - \epsilon_{F_k}(G)$.

Conversely, let $D = D_1 \cup D_2 \cup \cdots \cup D_t,\ 1 \leq t \leq k$, be a γ_c^k-set for G. Choose for each D_i a spanning tree $T_i, 1 \leq i \leq t$. For each vertex in $V(G) \setminus D$ choose one edge which is incident with a vertex in D. We have constructed a spanning forest F with t components and at least $n - |D| = n - \gamma_c^k(G)$ leaves. Therefore $\epsilon_{F_k}(G) \geq n - \gamma_c^k(G)$ and Theorem 1 is proved. □

Theorem 2 *Let k be a positive integer and G a connected graph. Then*

$$\begin{aligned}\gamma_c^k(G) &= \min\left\{\gamma_c^k(F_k) | F_k \textit{ is a spanning forest of } G \textit{ with at most } k \textit{ trees} \right\} \\ &= \min\left\{\gamma_c^k(T) | T \textit{ is a spanning tree of } G \right\}\end{aligned}$$

Proof: Let F_k be a spanning forest of G with at most k trees. Certainly $\gamma_c^k(G) \leq \gamma_c^k(F_k)$ since a set which dominates F_k also dominates G. Conversely, we can in G find a spanning forest F_k with at most k components such that $\gamma_c^k(G) = \gamma_c^k(F_k)$: As was originally also done in the proofs for (i) and (ii) above we construct F_k from a γ_c^k-set $D = D_1 \cup D_2 \cup \cdots \cup D_t,\ 1 \leq t \leq k$, by choosing a spanning tree T_i in each connected subgraph D_i and joining each vertex in $V(G) \setminus D$ to precisely one vertex in D. Obviously, $\gamma_c^k(F_k) \leq |D| = \gamma_c^k(G)$. This proves the first equality. For the second equality we observe that the first minimum is chosen among a larger set, so that $\min \gamma_c^k(F_k) \leq \min \gamma_c^k(T)$, and also that any F_k by addition of edges can produce a tree T with $\gamma_c^k(T) \leq \gamma_c^k(F_k)$. □

Hartnell and Vestergaard [6] proved the following result.

Proposition 2 *For $k \geq 1$ and G connected*

$$\gamma_c(G) - 2(k-1) \leq \gamma_c^k(G) \leq \gamma_c(G).$$

From Proposition 2 we can easily derive the following corollary which is a classical result proven by Duchet and Meyniel. [2]

Corollary 3 *For any connected graph G, $\gamma_c(G) \leq 3\gamma(G) - 2$.*

Proof: Let G be a connected graph with domination number $\gamma(G)$. Choose $k = \gamma(G)$, then $\gamma_c^k(G) = \gamma(G)$. Substituting into Proposition 2 above we obtain $\gamma_c(G) - 2(k-1) \leq \gamma(G)$ and that proves the corollary. □

2.1 Other bounds on γ_c^k

Theorem 4 *For a positive integer k and a connected graph G with maximum valency Δ we have*

(A) $\gamma_c(G) \leq n - \Delta$ *and for trees T equality holds if and only if T has at most one vertex of valency ≥ 3.*

(B) $\gamma_c^k(G) \leq n - \dfrac{(r-1)(\delta-2)}{3} - 2k$ *if G has diameter $r \geq 3k-1$ and the minimum valency $\delta = \delta(G)$ is at least 3.*

(C) If G is a connected graph with two vertices of valency Δ at distance d apart, $d \geq 3$, then

$$\gamma_c^k(G) \leq n - 2(\Delta - 1) - 2\min\{k-1, \frac{d-2}{3}\}. \tag{1}$$

(D) Let $x \in V(G)$ have valency $d(x)$ and eccentricity $e(x)$. Then

$$\gamma_c^k(G) \leq n - d(x) - 2\min\{k-1, \frac{e(x)-2}{3}\}. \tag{2}$$

Proof:

(A) Let T be a spanning tree of G with $\Delta(T) = \Delta(G) = \Delta$, then T has at least Δ leaves, and hence $\gamma_c(G) \leq \gamma_c(T) \leq n - \Delta$.

If T has two vertices of valency ≥ 3, the number of leaves in T will be larger than Δ, and we get strict inequality in (A). Clearly, a tree T with exactly one vertex of valency $\Delta \geq 3$ has equality in (A) and for $\Delta = 2$, we obtain a path P_n with $\gamma_c(P_n) = n - 2$.

(B) Let $P = v_1 v_2 v_3 \dots v_{3t+u}, \ \ k \leq t, 0 \leq u \leq 2$, be a diametrical path in G. The diameter of T equals the length of P, which is $r = 3t + u - 1$. For $i = 1, \dots, t$ let v_{3i-1} have neighbours v_{3i-2}, v_{3i} on P and a_{ij} off P, $j = 1, \dots, s_i \ \ s_i \geq \delta - 2 \geq 1$. In $G - \{v_{3i}v_{3i+1} | 1 \leq i \leq k-1\}$ consider the $k-1$ disjoint stars with center v_{3i-1} and neighbours $N(v_{3i-1}), \ \ 1 \leq i \leq k-1$, and the remaining tree to the right consisting of the path $v_{3k-2}v_{3k-1}v_{3k} \dots v_{3t+u}$ and leaves $v_{3i-1}a_{3i-1}, \ \ j = 1, \dots, s_i, \ \ s_i \geq \delta - 2 \geq 1$ adjacent to vertices $v_{3i-1}, \ \ k \leq i \leq t$.

Extend this forest of k trees to a spanning forest F with k trees in $G - \{v_{3i}v_{3i+1} | 1 \leq i \leq k-1\}$. The number of leaves in F is at least $t(\delta - 2) + 2k$ and hence $\gamma_c^k(G) \leq n - t(\delta-2) - 2k$. From $t = \dfrac{r+1-u}{3} \geq \dfrac{r-1}{3}$ we obtain the desired result $\gamma_c^k(G) \leq n - \dfrac{(r-1)(\delta-2)}{3} - 2k$.

C Let v_1, v_s be two vertices in G with maximum valency, $d(v_1) = d(v_s) = \Delta$, and let $P = v_1 v_2 \dots v_s$ be a shortest $v_1 v_s$-path, $s = 3t + 1 + u, t \geq 1, 0 \leq u \leq 2$.

Case 1, $t \geq k - 1$**:** In $G - \{v_{3i-1}v_{3i} | 1 \leq i \leq k-2\}$ we extend the k trees listed below to a spanning forest F of G,

1. The star consisting of v_1 joined to all its neighbours,
2. the $k-2$ paths of length two $v_{3i}v_{3i+1}v_{3i+2}, 1 \leq i \leq k-2$,
3. the path $v_{3k-3}v_{3k-2} \dots v_s$ together with all $\Delta - 1$ neighbours of v_s outside of P.

F will have at least $2(\Delta - 1) + 2(k-1)$ leaves.

Case 2, $t \leq k - 2$**:** $s = 3t + 1 + u, d = d(v_1, v_s) = s - 1 = 3t + u, t - 1 = \dfrac{d-u}{3} - 1 \geq \dfrac{d-2}{3} - 1$. As before, we can find a spanning forest F of G whose number of leaves is at least $2\Delta + 2(t-1) \geq 2(\Delta - 1) + 2\dfrac{d-2}{3}$ and consequently $\gamma_c^k(G) \leq n - 2(\Delta - 1) - 2\dfrac{d-2}{3}$.

The proof of D is similar.

□

3 Trees

For trees Hartnell and Vestergaard [6] found

Proposition 3 *Let k be a positive integer and T a tree with $|V(T)| = n, n \geq 2k+1$. Then $\gamma_c^k(T) \leq n-k-1$.*

This inequality is best possible. For $k = 1$ the extremal trees are paths P_n and for $k \geq 2$ extremal trees will be described in the following Theorem 5.

A tree T is of type A if it contains a vertex x_0 such that $T - x_0$ is a forest of trees $T_1, T_2, \ldots, T_\alpha, \alpha \geq 1$, such that each tree T_i is a corona tree and x_0 is joined to a stem in each of the trees T_i, $1 \leq i \leq \alpha$. We note that a subdivision of a star is a tree of type A.

A tree T is of type B if it contains a path uvw such that $T - \{u, v, w\}$ is a forest of corona trees $T_1, T_2, \ldots, T_s,\ T_{s+1,}, \ldots, T_\alpha,\ \alpha \geq 2, 1 \leq s < \alpha$ and u is joined to a stem in each of the trees $T_1, T_2, \ldots, T_s$, while w is joined to a stem in each of the trees $T_{s+1,}, \ldots, T_\alpha$.

Proposition 4 below was proven by Randerath and Volkmann [12], Baogen, Cockayne, Haynes, Hedetniemi and Shangchao [1].

Proposition 4 *If T is a tree with n vertices, n odd, and $\gamma(T) = \lfloor \frac{n}{2} \rfloor$ then T is a tree of type A or B.*

We shall now determine the trees extremal for Proposition 3.

Theorem 5 *Let $k \geq 2$ be a positive integer and T a tree with n vertices, $n \geq 2k+1$. Then $\gamma_c^k(T) = n-k-1$ if and only if one of cases (i)-(iii) below occur.*

(i) $k = \frac{n-1}{2}$, $\gamma_c^k(T) = \gamma(T) = \frac{n-1}{2}$ *and T is of type A or B.*

(ii) $k = \frac{n-2}{2}$, $\gamma_c^k(T) = \gamma(T) = \frac{n}{2}$ *and T is a corona tree.*

(iii) $k = \frac{n-3}{2}$, $\gamma_c^k(T) = \frac{n+1}{2}$, $\gamma(T) = \frac{n-1}{2}$ *and T is a star $K_{1,k+1}$ with a subdvision vertex on each edge.*

Proof: First, let $k \geq 2$ and a tree T of order n be given such that $n \geq 2k+1$ and $\gamma_c^k(T) = n-k-1$. We shall prove that T is as described in one of the three cases (i)-(iii).

We note in passing that

Remark 1 *$\gamma(T) \leq k$ implies $\gamma_c^k(T) = \gamma(T)$, and that likewise $\gamma_c^k(T) \leq k$ implies $\gamma_c^k(T) = \gamma(T)$.*

If $n = 2k+1$, or equivalently $k = \frac{n-1}{2}$, we have by assumption $\gamma_c^k(T) = n-k-1 = k$ and, as just observed above, that implies that also $\gamma(T) = k$. Since $k = \lfloor \frac{n}{2} \rfloor$ we obtain from Proposition 4 that T is a tree of type A or B, so Case (i) occurs.

If $n = 2k+2$, or equivalently $k = \frac{n-2}{2}$ we have by assumption $\gamma_c^k(T) = n-k-1 = k+1$. Certainly $\gamma(T) \leq \gamma_c^k(T)$, but if $\gamma(T) \leq k$ then we should have that $\gamma_c^k(T) = \gamma(T) \leq k$ in contradiction to $\gamma_c^k(T) = k+1$, therefore $\gamma(T) = k+1 = \dfrac{n}{2}$. From Proposition 1 we obtain that T is a corona tree, i.e. Case (ii) occurs.

We may now assume $n \geq 2k+3$, and we shall prove that, in fact, n equals $2k+3$ and that Case (iii) occurs.

Let $v_1 v_2 \ldots v_\alpha$ be a longest path in T. Since $\gamma_c^k(T) = n-k-1 \geq k+2 \geq 4$, T is neither a star nor a bistar and therefore $\alpha \geq 5$. We must have $d_T(v_2) = 2$, because otherwise $d_T(v_2) \geq 3$ and we could from T delete three leaves adjacent to v_2, if $d_T(v_2) \geq 4$, and in case $d_T(v_2) = 3$ we could delete v_2 and its two adjacent leaves. In both cases we would obtain a tree T' of order $n-3 \geq 2(k-1)+1$ which by Proposition 3 has $\gamma_c^{k-1}(T') \leq (n-3)-(k-1)-1 \leq n-k-3$. Adding v_2 to a $\gamma_c^{k-1}(T')$-set we would obtain $\gamma_c^k(T) \leq n-k-2$, a contradiction. Therefore $d_T(v_2) = 2$.

The vertex v_3 cannot be adjacent to two leaves c and d, say, because, then the tree $T' = T - \{v_1, v_2, c, d\}$ would have order $n-4 \geq 2(k-1)+1$. Thus Proposition 3 gives that $\gamma_c^{k-1}(T') \leq (n-4)-(k-1)-1$ $\leq n-k-4$ and adding v_2, v_3 to a $\gamma_c^{k-1}(T')$-set we would obtain $\gamma_c^k(T) \leq n-k-2$, a contradiction. So v_3 can be adjacent to at most one leaf. The case $d_T(v_3) = 3$ and v_3 adjacent to one leaf c can similarly be seen to be impossible by considering $T' = T \setminus \{v_1, v_2, v_3, c\}$.

On the other hand $d_T(v_3) \geq 3$, for assume $d_T(v_3) = 2$, then $T' = T \setminus \{v_1, v_2, v_3\}$ has $\gamma_c^{k-1}(T') \leq n-k-3$ and addition of v_2 to a $\gamma_c^{k-1}(T')$-set would give $\gamma_c^k(T) \leq n-k-2$, a contradiction.

Assume therefore that v_3 besides v_2 and v_4 is adjacent to precisely one leaf c and to at least one further vertex a, where a has valency two and is adjacent to the leaf b. Then $T' = T \setminus \{v_1, v_2, a, b\}$ has order $n-4 \geq 2(k-1)+1$ and Proposition 3 gives that (3) $\gamma_c^{k-1}(T') \leq (n-4)-(k-1)-1 \leq n-k-4$. In T' the vertex c is a leaf and as any γ_c^{k-1}-set for T' must contain one of $\{v_3, c\}$, we may assume it contains v_3. Addition of $\{v_2, a\}$ to a $\gamma_c^{k-1}(T')$-set now gives the contradiction $\gamma_c^k(T) \leq n-k-2$.

Assume finally that v_3 has no leaf but besides v_2 and v_4 is adjacent to $a_1, a_2, \ldots, a_t$, $t \geq 1$, where each a_i has valency two and is adjacent to the leaf b_i, $1 \leq i \leq t$.

We have $k-t \geq 1$ because $V(T) \setminus \{v_1, b_1, b_2, \ldots, b_t, v_\alpha\}$ is a connected subgraph with $n-t-2$ vertices which dominate T, so that $n-k-1 = \gamma_c^k(T) \leq n-t-2$ giving $k-t \geq 1$. Consider the tree $T' = T \setminus \{v_1, v_2, a_1, a_2, \ldots, b_1, b_2, \ldots, b_t, v_3\}$ of order $n-2t-3$.

If $n-2t-3 \geq 2(k-t)+1$ we obtain by Proposition 3 that $\gamma_c^{k-t}(T') \leq (n-2t-3)-(k-t)-1 \leq n-k-t-4$, and by addition of the $t+2$ vertices $\{v_2, v_3, a_1, a_2, \ldots, a_t\}$, (which span a connected subgraph of T), to a $\gamma_c^{k-t}(T')$-set we obtain $\gamma_c^k(T) \leq n-k-2$, a contradiction. So we have $n-2t-3 \leq 2(k-t)$ and now $|V(T')| = n-2t-3 \leq 2(k-t)$ implies $\gamma(T') \leq \dfrac{|V(T')|}{2} \leq k-t$ which by remark 1 gives that $\gamma_c^{k-t}(T') = \gamma(T')$ and hence addition of the $t+2$ vertices $\{v_2, v_3, a_1, a_2, \ldots, a_t\}$ to a $\gamma_c^{k-t}(T')$-set (having at most $k-t$ vertices) gives $\gamma_c^{k-t+1}(T) \leq k+2$. We now have $n-k-1 = \gamma_c^k(T) \leq \gamma_c^{k-t+1}(T) \leq k+2$ giving $n \leq 2k+3$, so the assumption $n \geq 2k+3$ implies $n = 2k+3$. By hypothesis $\gamma_c^k(T) = k+2$ and we have $\gamma(T) \leq k+1$ by Proposition 1.

Thus $\gamma(T) = k+1$, (because otherwise $\gamma_c^k(T) = \gamma(T) < k+2$), and any $\gamma(T)$-set must consist of $k+1$ isolated vertices. As $\gamma(T) = \lfloor \dfrac{n}{2} \rfloor$ the tree T by Proposition 4 is of type A or B. But T cannot be of type B, for assume T is of type B. Then T consists of a 3-path , uvw, with each of its ends joined to stems of corona trees, and since we have just seen that $v_3, v_{\alpha-2}$ are neither stems nor leaves, they must play the role of u, w, so $\alpha = 7$ and T consists of two subdivided stars centered respectively at $u = v_3$ and $w = v_5$ and a vertex $v = v_4$ joined to u and w. Among its γ-sets this tree T has one with two adjacent vertices, namely v_2 and v_3, a contradiction, so T is of type A .

Using, in analogy to v_2, v_3, that $d_T(v_{\alpha-1}) = 2$ and that $v_{\alpha-2}$ is not a stem, we get that $\alpha = 5$ and T is a subdivided star so that (iii) occurs.

Conversely, it is easy to see that if (i), (ii) or (iii) holds then $\gamma_c^k(T) = \gamma(T) = n - k + 1$. This proves Theorem 5. □

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