A framework for digital label images

Loïc Mazo

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Loïc Mazo
ICube, Université de Strasbourg, CNRS

Abstract

Digital label images – partitions of a discrete space – need a specific topological model to take into account not only the topologies of the regions but also the topology of the partition. In this article, we propose a topological framework for label images in which all the regions of the initial partition and of any coarser partition of the space can be explicitly represented accordingly to any classical adjacency relation. Moreover, we define a notion of simple point which enables atomic changes in the partition without breaking any topology. Finally, we discuss about the implementation of the framework.

1 Introduction

In this paper, we study, from a topological viewpoint, *digital label images*, that is, images whose domain is $\mathbb{Z}^n$ and whose codomains are sets on which there generally exists no meaningful order relation (unlike grey-level images for instance). Actually, a digital label image is nothing but a labelled partition of $\mathbb{Z}^n$. Label images need a specific approach in topology. Indeed, a label image is much more than a collection of independent objects and we are also interested in the spatial relations between these objects. Thereby, any topologically sound label image processing must pay attention to each object *and* to the partition itself. Nevertheless, as far as we know, the literature devoted to the topology of label images is not well developed and essentially oriented towards specific applications. The most commonly used approach is to process one label at a time while rejecting temporarily the other labels in the background, coming down to a binary image (*e.g.* [15, 4, 10]). With this method, the topology of the partition is generally ignored. Sometimes another structure, like a region adjacency graph, is adjoined. But, except in 2D, this kind of structure can rarely encompass all the topological information of the partition. Furthermore, most of the time, in the applications (see, *e.g.*, [14]) the value of the image on a picture element changes from the background to a particular label, or *vice versa*, but more seldom it goes from a label to another label. The approach

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proposed in [6] makes it possible to change the label of a point in a 3D cellular space by choosing a new label among several ones with the assurance that the homotopy types of the two labels, the new one and the former one, are preserved. Nevertheless, no attention is paid to the topology of the partition. To take this latter topology in consideration, it is proposed in [7] to monitor the boundaries between the regions. Besides the fact that the article is restricted to the $(6, 18)$-adjacency pair in $\mathbb{Z}^3$ (for voxels and regions), it seems difficult to bring out a theoretical model from this framework which mixes notions of digital topology (geodesic neighborhoods), combinatorial topology (collapses on cubical complexes) and discrete surfaces (sets of surfels). In order to take into account the topology of the partition, it is required in [3] that the unions of two labels (in 2D spaces), or two or three labels (in 3D spaces), are watched in any process as well as single labels. However, a careful examination of the examples provided by the authors shows that these conditions are not sufficient to maintain the topology of the partition. In particular, the unions of three labels should also be watched in a 2D space.

In a previous work [13], we have proposed an extensive theoretical framework to deal with label images. In our model, we have retained the idea to consider unions of objects of the initial partition together with the idea found in [16, 17], to equip the set of labels with a lattice structure. In other words, with this framework we propose to see a label image as an initial partition together with some meaningful coarser partitions whose labels are provided by the lattice. Accordingly, we described in our previous article some kinds of *simple points* for label images, that is points whose labels can be changed while preserving the topologies (actually, the weak homotopy types) of the regions of all these partitions. In order to use classical "continuous topology" (as opposed to "discrete topology"), the label images are defined on a poset (partially ordered set), typically the space of cubical complexes equipped with the inclusion, and we use the Alexandroff-Birkhoff topology [1, 5].

Unfortunately, the model described in our previous work has an important drawback at the first stage: the embedding of the digital space in a poset. If the digital objects are not modeled by closed subsets of $\mathbb{R}^n$ (*i.e.* if the adjacency relation for the objects is not the $(3^n - 1)$-adjacency), the proposed embedding cannot preserve the topology of each label. More precisely, we cannot generally define, for each label of the lattice, isomorphisms between the discrete topology structures and the continuous topology structures as illustrated on Figure 1. To overcome this issue, we have been led to develop another model that we describe in this article.

The remainder of this article is organized as follows. In Section 2, we define the covering images, the kind of abstract image that we propose to model a label digital image. In Section 3, we present a notion of simple point for covering images, that is an atomic change on a label image that preserves the topologies of the regions identified by the labels of the lattice. Section 4 explains how the whole framework can be implemented and Section 5 concludes the paper.

(a) (b)

Figure 1: (Color online) Motivation for a new framework.
(a) A digital image λ in $\mathbb{Z}^2$ with 2 labels r (red), g (green) and a background (not depicted). (b) The embedding of the image λ in $\mathbb{F}^2$, the space of cubical 2-complexes, obtained by applying the following membership rule: the label of a 0-, or 1-, dimensional point is the infimum, in the lattice $(2^{\{r,g\}}, \subseteq)$, of the labels of the surrounding 2-dimensional points. We have proved in [12] that this embedding preserves the connected components and the fundamental groups of the object and its complement when a binary digital image is interpreted with the $(2n, 3^n - 1)$-adjacency pair. But, if we identify the two labels, that is if we consider a coarser partition of the space, the topology is not the same on these two images (we have one component on the left and two components on the right).

2 Covering images

In order to model all the topological relations that can be found in a digital label image λ (defined on $\mathbb{Z}^n$), we propose two steps.

1. We split the image λ in a collection of binary images such that each binary image represents a region of interest, that is a region that has been previously labelled (for instance during a segmentation process) or represents a meaningful union of some labeled regions. The unions are labeled thanks to a lattice structure on labels: the label of an union of regions is the supremum of the labels of the regions. No other labels are needed for our purpose, so the lattice of labels, noted T, is an atomistic finite lattice[1] whose atoms are the initial labels that we call *proto-labels* and there is a one-to-one correspondence between this lattice and the collection of binary images (we write λ_t for the binary image associated to the label t in the collection built from the digital label image λ). Figure 2 exemplifies this first stage (in the sequel, the infimum and supremum operators on T are denoted $\wedge$ and $\vee$ while $\perp$ and $\top$ are the minimum and the maximum of T; furthermore, for any $t \in T$, we write $\mathcal{A}(t)$ for the set of atoms under t).

2. In each binary digital image λ_t created in the previous step, we introduce inter-xels elements (pointels, linels, surfels and so on) to have a topology available. The space in which we embed the digital images is the space of nD cubical complexes, $\mathbb{F}^n$, but it could be another cellular decomposition

[1] A lattice is atomistic if any element, but the minimum, is a supremum of atoms.

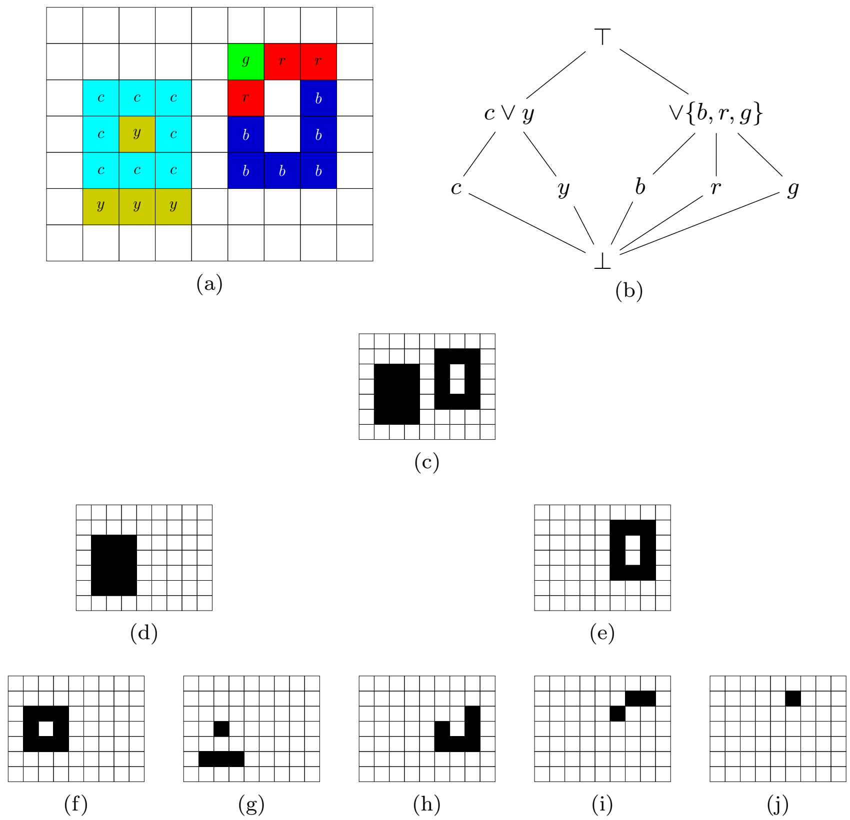
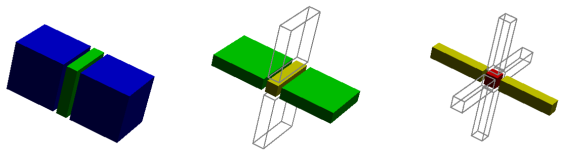


Figure 2: (Color online) A lattice of binary images associated to a label image. (a) A digital label image with five (proto-)labels c, y, b, r, g respectively depicted in cyan, yellow, blue, red and green. (b) A lattice structure T whose atoms are c, y, b, r, g. (c-j) The collection of binary images associated to the lattice T (the binary image associated to $\perp$ is not represented for it is a constant image, with no object).

of the space[2]. The inter-xels elements must be labeled with membership rules with respect to the desired interpretation of the image. Such rules can be found in the literature (see *e.g.* [2]). We have proposed in [12] our own rules which preserve the connected components for the classical adjacency pairs and result in isomorphisms between the digital fundamental groups as defined by Kong [8] and the usual fundamental groups of the regions of $\mathbb{F}^n$. Our rules are defined as follows. If $x, y, z \in \mathbb{F}^n$ are three points such that x is incident to y and z, $\dim(y) = \dim(z) = \dim(x) + 1$ and y and z are not incident to the same xel, we say that y and z are

[2]A formal description of $\mathbb{F}^n$ can be found in [13] but no knowledge of cubical complexes is needed to understand the remainder of the article.

opposite with respect to the point x and we denote by $\mathrm{opp}(x)$ the set of all pairs $\{y,z\}$ for y opposite to z with respect to x (see Figure 3). Finally,



(a) (b) (c)

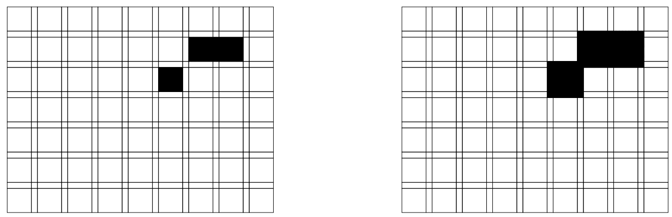
Figure 3: (Color online) Opposite points.
Two opposite points in $\mathbb{F}^3$ with respect to (a) a 2-point, (b) a 1-point, (c) a 0-point. The dashed boxes in (b) and (c) show other pairs of opposite points.

for a sequence ε of n elements in $\{-1,1\}$, we define the *ε-regular (binary) image* μ_t by $\mu_t(x)=\lambda_t(x)$ if x is a xel ($\dim(x)=n$) and, otherwise,

$$\mu_t(x)=\begin{cases}\inf\{\sup\{\mu_t(a),\mu_t(b)\}\mid\{a,b\}\in\mathrm{opp}(x)\} & \text{if } \varepsilon(m+1)=1\\ \sup\{\inf\{\mu_t(a),\mu_t(b)\}\mid\{a,b\}\in\mathrm{opp}(x)\} & \text{if } \varepsilon(m+1)=-1\end{cases}$$

where $m=\dim(x)$ and $\varepsilon(i)$ denotes the i-th element of the sequence ε. For instance, to extend on $\mathbb{F}^2$ a digital image initially defined on $\mathbb{Z}^2$, we set $\varepsilon=(-1,-1)$ if the image has to be interpreted with the $(4,8)$-adjacency pair, or $\varepsilon=(1,1)$ if the image has to be interpreted with the $(8,4)$-adjacency pair (the sequences $(1,-1)$ and $(-1,1)$ correspond respectively to sections of 3D-images equipped with the $(6,18)$-, or the $(18,6)$-, adjacency pair). Note that distinct rules can be applied to distinct labels provided no inconsistency is introduced: for instance, if two voxels are connected in the region R, they cannot be disconnected in a region including R; this leads us to the notion of fiber described below.

After these two steps have been achieved, we get a collection $(\mu_t)_{t\in T}$ (actually a lattice) of binary images, the *sheets*, defined on $\mathbb{F}^n$ (see Figure 4). Now, we can see this collection of binary images as a unique image μ by setting that $\mu(x)$ is equal to the set (we say the *fiber*) whose elements are the labels t such that $\mu_t(x)=1$ (in other words, the labels attached to x, or, equivalently, the regions of interest x belongs to). For instance, let x_0 be the horizontal linel between the label r (red) and the label g (green) on Figure 2(a). If we set $\varepsilon=(-1,-1)$, we find that $\mu_r(x_0)=\{\bigvee\{b,r,g\},\top\}$ while setting $\varepsilon=(1,1)$, we find that $\mu_r(x_0)=\{\{r\},\{g\},\bigvee\{b,r,g\},\top\}$ (provided the same membership rule is applied on all the sheets). When a point in $\mathbb{F}^n$ has not been labeled, for instance, a point in the infinite region surrounding the image, then its fiber is set to $\emptyset$. Since the region obtained by identifying two labels t and u, that is, the region associated to the supremum of the labels t and u, contains all the points that are in the region t or in the region u plus, possibly, some other ponts (like the point x_0 when $\varepsilon=(-1,-1)$ and we take for t the red label and for u the



(a) (b)

Figure 4: Adjacency pairs and ε-regularity.
The sheet μ_r associated to the label r of the digital label image depicted on Figure 2. (a) The sequence ε is equal to $(-1,-1)$ which corresponds to the choice of the $(4,8)$-adjacency pair (the object of the binary image is open, it does not include its boundary). (b) The sequence ε is equal to $(1,1)$ which corresponds to the $(8,4)$-adjacency pair (the object is closed so it includes its boundary).

green one), the fibers are *up-sets* (or *filters*), that is for any fiber S,

$$t \in S \text{ and } t \leq u \Rightarrow u \in S.$$

We write $t^{\uparrow}$ for the upset generated by a label t ($t^{\uparrow} = \{u \in T \mid t \leq u\}$) – such a set is called a principal upset (or principal filter) – and we write $\mathcal{G}_T$ for the family of the up-sets over the lattice T: $\mathcal{G}_T = \left\{\bigcup_{t\in A} t^{\uparrow} \mid A \subseteq T\right\} = \left\{\bigcup_{t\in A} \bigcap_{u\in\mathcal{A}(t)} u^{\uparrow} \mid A \subseteq T\right\}$. Note that $\emptyset \in \mathcal{G}_T$.

Eventually, we can define a covering image.

Definition 1 (Covering image)**.** *Let T be an atomistic finite lattice. A* covering image *μ is a function from $\mathbb{F}^n$ to $\mathcal{G}_T$. For any $t \in T$, the* sheet *μ_t is the binary image defined on $\mathbb{F}^n$ by $\mu_t(x) = 1$ if $t \in \mu(x)$ and $\mu_t(x) = 0$ otherwise. For any $x \in \mathbb{F}^n$, the set $\mu(x)$ is the* fiber *over x. A covering image is* ε-regular *if for any $t \in T$ the sheet μ_t is ε-regular.*

Thanks to the next proposition, a ε-regular covering image μ can be defined without resorting to the sheets μ_t, $t \in T$. This is an important point for the implementation since it enables the encoding of a regular covering image directly from a digital label image (see Section 4).

Proposition 1. *Let T be an atomistic lattice and ε a sequence of n elements in $\{-1, 1\}$. A covering image $\mu : \mathbb{F}^n \to \mathcal{G}_T$ is ε-regular iff, for all $x \in \mathbb{F}^n$ such that* $\dim(x) < n$*:*

$$\mu(x) = \begin{cases} \bigcap_{\{a,b\}\in\mathrm{opp}(x)} \mu(a) \cup \mu(b) & \text{if } \varepsilon(\dim(x)+1) = 1 \\ \bigcup_{\{a,b\}\in\mathrm{opp}(x)} \mu(a) \cap \mu(b) & \text{if } \varepsilon(\dim(x)+1) = -1. \end{cases}$$

Proof. Let us assume that $\dim(x) = m - 1$ and $\varepsilon(m) = 1$. Then, from Definition 1, for all label $t \in T$, one has $\mu_t(x) = \inf\{\sup\{\mu_t(a), \mu_t(b)\} \mid \{a, b\} \in$

opp$(x)\}$. Thereafter,

$$\begin{aligned} t \in \mu(x) &\Leftrightarrow \mu_t(x) = 1 \\ &\Leftrightarrow \forall \{a,b\} \in \operatorname{opp}(x), \mu_t(a) = 1 \text{ or } \mu_t(b) = 1 \\ &\Leftrightarrow \forall \{a,b\} \in \operatorname{opp}(x), t \in \mu(a) \cup \mu(b) \\ &\Leftrightarrow t \in \bigcap_{\{a,b\} \in \operatorname{opp}(x)} \mu(a) \cup \mu(b). \end{aligned}$$

The case $\varepsilon(m) = -1$ is similar. □

Let $t \in T$ be a label. The set $\langle t \rangle_\mu = \mu_t^{-1}(\{1\})$ is the *support* of t in the covering image μ. Thereby, the expressions $x \in \langle t \rangle_\mu$, $\mu_t(x) = 1$ and $t \in \mu(x)$ are synonymous. We write $\langle t \rangle_\mu^c$ for the set $\mathbb{F}^n \setminus \langle t \rangle_\mu$. When there is no ambiguity, we write also $\langle t \rangle$ and $\langle t \rangle^c$ instead of $\langle t \rangle_\mu$ and $\langle t \rangle_\mu^c$. The set $\langle \top \rangle^c = \mu^{-1}(\emptyset)$ is the *background* of μ. The support $\langle \bot \rangle$ contains the points in $\mathbb{F}^n$ that are in the supports of every labels, in particular every proto-labels. Thus $\langle \bot \rangle$ is empty in a covering image obtained from a digital label image (since a digital label image is a partition of $\mathbb{Z}^n$, or a partition of a finite region of $\mathbb{Z}^n$). Nevertheless, thanks to $\langle \bot \rangle$, the dual of a covering image, defined by swapping the foreground and the background in each sheet of the image, is a covering image for the dual order on the lattice T, provided the lattice T is a power set lattice (otherwise, T is not atomistic for its dual order).

3 Simplicity

First, we recall some definitions and properties of the Alexandroff-Birkhoff topology of the cubical complexes. A more detailed presentation of these notions of topology can be found in our previous work on label images [13].

We write $\mathrm{B}(x)$ for the boundary of the face x, $\mathrm{N}(x)$ for the set of faces in $\mathbb{F}^n$ whose boundary contains x. A point $x \in \mathbb{F}^n$ is *unipolar* if $\mathrm{N}(x)$ has a minimum (for the incidence relation) or $\mathrm{B}(x)$ has a maximum. A subset X of $\mathbb{F}^n$ is contractible (it has the homotopy type of a point) iff it can be shrunk to a unique point by a sequential (and greedy) removal of unipolar points.

In a binary image, a point x in the object is *simple* if its removal from the object "preserves topology" [9]. Since a covering image is a collection of binary images (the sheets), we can extend the notion of simple point to covering images: roughly speaking, in a covering image, a point is simple for a fiber S if it is simple in any sheet modified by the assignment $\mu(x) = S$. In our framework, we use β-simple points. A point $x \in \mathbb{F}^n$ is a *β-simple point* for a subset X of $\mathbb{F}^n$ if one of the sets $\mathrm{N}(x) \cap X$ or $\mathrm{B}(x) \cap X$ is contractible. The β-simple points have the advantage to preserve topology twice. Indeed, in the one hand, $\mathbb{F}^n$ can be equipped with the Alexandroff-Birkhoff topology whose open sets are the up-sets of $\mathbb{F}^n$. Then, the deletion of a β-simple point x from a subset X of $\mathbb{F}^n$ is a weak homotopy equivalence, that is, the inclusion $i : X \setminus \{x\} \to X$ induces a one-to-one correspondence between the connected components of both spaces and induces also isomorphisms between the homotopy groups of $X \setminus \{x\}$ and X. Moreover, this is also true for the dual inclusion $i' : \mathbb{F}^n \setminus X \to \mathbb{F}^n \setminus (X \setminus \{x\})$. On the other hand, one can associate to any subset X of $\mathbb{F}^n$ an Euclidean set, denoted $|\mathcal{K}(X)|$, which is the realization of a simplicial complex. The deletion from X of a β-simple point x induces a strong deformation retraction from $|\mathcal{K}(X)|$ to $|\mathcal{K}(X \setminus \{x\})|$. Furthermore, this is also true for the complements (in $\mathbb{R}^n$) of these realizations but, possibly, in a non-monotonic manner.

Eventually, we can give the definition of a simple point in a covering image.

Definition 2 (Simple point)**.** *Let $S \in \mathcal{G}_T$ be a fiber. A point $x \in \mathbb{F}^n$ is* simple for (the fiber) S *if the following two conditions are verified:*

(i) for any label $u \in \mu(x)$ such that $u \notin S$, x is β-simple for the set $\langle u \rangle$ or for the set $\langle u \rangle^c \cup \{x\}$;

(ii) for any label $u \notin \mu(x)$ such that $u \in S$, x is β-simple for the set $\langle u \rangle \cup \{x\}$ or for the set $\langle u \rangle^c$.

The previous definition, and the properties of β-simple points, ensures that, if a point is simple for the fiber S in the image μ, we can set $\mu(x) = S$ while preserving the topology of any region of interest, including the unions pointed out by the choice of the lattice T. Moreover, since the β-simplicity of a point x relies only on the examination of the sets $\mathrm{N}(x)$ and $\mathrm{B}(x)$, modifications of fibers over points in $\mathbb{F}^n$ having the same dimension can be done in parallel, leading to well-balanced algorithms. Figure 5 gives some examples of thinnings, or growings, on a covering image using simple points.

Observe that most of the time the use of simple points to modify a covering image will lose the regularity of the image and will disable the possibility to extract from the covering image a digital label image (defined on $\mathbb{Z}^n$) in a topologically sound manner (Figure 6 illustrates this point). Keeping the possibility to go back to $\mathbb{Z}^n$ is a difficult issue. Some results can be found in our Ph.D. thesis [11], in French, but they concern only three dimensional images equipped with the $(6, 18)$ adjacency pair, or its dual, and the statements are really tedious (fortunately, the implementation has turned out to be less painful than the statements and the proofs).

4 Implementation

In this section, we discuss the implementation of the framework.

We need to encode the lattice T, the fibers and some operations or predicates on fibers (at least union, intersection, is included in). Since the lattice T is atomistic, it is natural to encode the atoms, the proto-labels of the initial digital image, with a single bit in a bit field. Then the other labels in T can be encoded by a bitwise OR on the codes of the atoms of which they are the supremum. Table 1 gives the codes for the labels in the lattice T_1 defined on Figure 5.

In the sequel, we write $[t]$ for the bit field representation of the label t. When the lattice T is the power set of the proto-labels, the supremum, resp. the

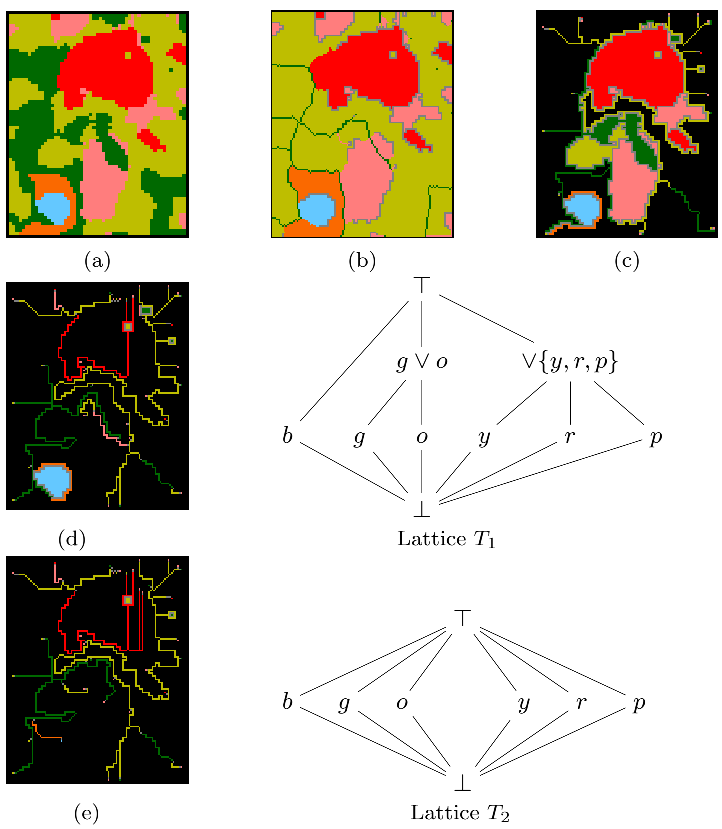
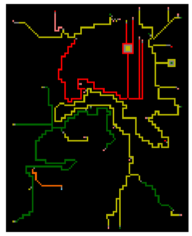
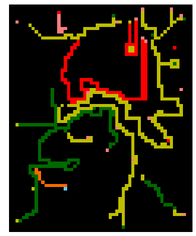


Figure 5: (Color online) Thinning and growing using simple points.
(a) A label image λ defined on $\mathbb{Z}^2$. There are six proto-labels: b (blue), g (green), o (orange), y (yellow), r (red) and p (pink). (b) The image λ has been transformed in a ε-regular covering image μ (with the sequence $\varepsilon = (1, 1)$, which is associated to the $(8, 4)$-adjacency pair, and the proto-label power set lattice) then the support of the green label has been shrunk by use of simple points. More precisely, for each point x whose fiber $\mu(x)$ has a minimal element greater than the green proto-label, we look for a fiber S in its neighborhood that have all its minimal elements not greater than the green label. Then, if x is simple for S, we set $\mu(x) = S$. (c-e) The image λ has been transformed in a ε-regular covering image μ with the sequence $\varepsilon = (-1, -1)$ and the proto-label power set lattice (Figure (c)), the lattice T_1 (Figure (d)), the minimal lattice T_2 (Figure (e)). Then we have grown the background: for each point x that has a non-empty fiber and that is simple for the empty fiber, we set $\mu(x) = \emptyset$.



(a)



(b)

Figure 6: (Color online) From $\mathbb{F}^n$ to $\mathbb{Z}^n$: loss of topological information.
(a) A Covering image μ (same as Figure 5(e)). (b) The image defined on $\mathbb{Z}^n$ extracted from the covering μ. The topologies of the red set (for instance) is not the same on the two images.

Label	$\perp$	b	g	o	y
Code	000000	000001	000010	000100	001000
Label	r	p	$g \vee o$	$\bigvee\{y, r, p\}$	$\top$
Code	0100000	100000	000110	111000	111111

Table 1: Label encoding for the lattice T_1 defined on Figure 5.

infimum, of a pair of labels is obtained by an OR, resp. an AND, operation on the codes of the two labels. Otherwise, we need an ordered list L of the elements of T for the order defined by the number of atoms, then the lexicographic order on the codes. For instance, the ordered list L for the lattice T_1 (Figure 5) is the one induced by Table 1. The code of the supremum of two labels t and u is the smallest element in the list L greater than the bitwise OR of $[t]$ and $[u]$. Alike, the code of the infimum of t and u is the greatest element in the list L smaller than the bitwise AND of $[t]$ and $[u]$.

The fibers are unions of principal upsets $t^{\uparrow}, t \in T$. The upset $t^{\uparrow}$ is encoded as the label t. In other words, an upset is encoded by the code of its minimal element. The other non-empty fibers, which are unions of principal upsets, are encoded as ordered lists of principal upsets (for the same order used for the list L). For a regular image, the length of the list depends on the choice of the adjacency relation and on the number of distinct labels in the neighborhood. For the digital image of Figure 5(a), the mean length of the lists of the ε-regular covering image is 1.13 when $\varepsilon = (1, 1)$ and all the fibers are principal upsets when $\varepsilon = (-1, -1)$. We need also a code for the empty fiber. Observe that in Table 1 the bit field 000000 codes the label $\perp$, or the fiber $T = \perp^{\uparrow}$. Hence, it cannot be used to code the empty set. This is the reason why we adjoin another bit. Then, the bit field $1 \ldots 1$ (all bits set to '1') codes the empty set (the greater the code, the lesser the fiber).

To obtain a regular image from an initial digital label image, we need to compute iteratively unions and/or intersections of fibers (see Definition 1). The unions are performed by concatenating the lists of principal upsets and reordering. Moreover, from the list we delete the principal upsets that are included in an other element of the list: a principal upset $t^{\uparrow}$ is included in the upset $u^{\uparrow}$ iff the bit field ([t] AND [u]) is equal to [u]. The code of the intersection of two principal upsets $t^{\uparrow}$ and $u^{\uparrow}$ is the code of the supremum of the labels t and u (for $t^{\uparrow} \cap u^{\uparrow} = (t \vee u)^{\uparrow}$). To intersect two fibers, we compute all the intersections of their principal upsets, delete the unnecessary elements and sort the resulting list.

The extraction of a sheet μ_t ($t \in T$) is done by testing for each $x \in \mathbb{F}^n$ whether $t^{\uparrow}$ is included in some principal upset $u^{\uparrow} \in \mu(x)$.

To test whether a point x is simple for a fiber S, we have to examine the sets $\mathrm{N}(x)$ and $\mathrm{B}(x)$ in each sheet μ_t where $t \in (\mu(x) \setminus S) \cup (S \setminus \mu(x))$. This implies,*a priori*, to scan the lattice T, check whether each label is in $\mu(x)$ and not in S, or the converse, and, when the answer is positive, to look over the neighborhood of the face x in the selected sheet. If the lattice is the power set of a large amount of proto-labels, this could be long. Fortunately, the following proposition allows us to scan only a subset of the lattice T. The idea is that atoms that cannot be detected in the neighborhood of the point x are not 'useful' for the simplicity test. Hence, we collect all the atoms in the neighborhood of x together with those in S, that is we make an 'OR' on all the bit fields (but the empty set). Let A be the result. Clearly, for each point y in the neighborhood of x and for each minimal element v of $\mu(y)$, we have $\mathcal{A}(v) \subseteq A$. Then, thanks to Proposition 2, we can ignore the atoms that are not in A since, for each label t such that $\mathcal{A}(t) \nsubseteq A$, there exists a label u in the sublattice of T generated by the atoms in A such that the two sheets μ_t and μ_u coincide in the neighborhood of x. In a standard label image, the test falls down to inspect labels which are supremums of very few atoms.

Proposition 2. *Let $A \subset \mathcal{A}(\top)$ be a set of atoms in T and $b \in \mathcal{A}(\top)$ be an atom not in A. Then, for any $t \in T$ such that $b \in \mathcal{A}(t)$, there exists a label $u \in T$ such that*

- $u \in \{\bigvee A' \mid A' \subseteq A\}$ *;*
- *for any fiber S such that, for any minimal element $v \in S$, $\mathcal{A}(v) \subseteq A$, we have $t \in S \Leftrightarrow u \in S$.*

Proof. Let t be a label such that $b \in \mathcal{A}(t)$. We set $\mathcal{A}(t) = A_0 \cup B$ where $A_0 \subseteq A$ and $B \cap A = \emptyset$ (obviously, we have $b \in B$). We set $u = \bigvee A_0$ (thus $u \leqslant t$). It is plain that $u \in \{\bigvee A' \mid A' \subseteq A\}$.

Let S be a fiber such that, for any minimal element $v \in S$, $\mathcal{A}(v) \subseteq A$. If $u \in S$, then $t \in S$ (for $u \leqslant t$ and S is a fiber). Conversely, let us assume that $t \in S$. Let s be a minimal element in S such that $s \leqslant t$. Clearly, we have $\mathcal{A}(s) \subseteq \mathcal{A}(t)$. Since, by hypothesis, $\mathcal{A}(s) \subseteq A$, we derive that $\mathcal{A}(s) \subseteq A_0$. Hence, $s \leqslant u$. Thus, $u \in S$. □

5 Conclusion

The model presented in this paper gives a way to encompass all the topological relations that characterize a label image. It provides a notion of simple point for a label which is based on solid foundations. From a theoretical point of view, it can help us to check what is precisely preserved or modified by a procedure. Furthermore, though the complexity of a label image (with all the intra-labels and inter-labels relations) is much higher than the complexity of a single object, we have seen that the implementation of a space of fibers can be done in an efficient way.

The next step will be to find procedures to get a regular covering image from a non-regular one. This will allow the extraction of a label image defined on $\mathbb{Z}^n$ homotopically equivalent to a given covering image.

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