



**HAL**  
open science

# Hypersingular integral equations over a disc: convergence of a spectral method and connection with Tranter's method

Leandro Farina, Paul A. Martin, Victor Péron

► **To cite this version:**

Leandro Farina, Paul A. Martin, Victor Péron. Hypersingular integral equations over a disc: convergence of a spectral method and connection with Tranter's method. *Journal of Computational and Applied Mathematics*, 2014, 269, pp.118-131. 10.1016/j.cam.2014.03.014 . hal-00971497

**HAL Id: hal-00971497**

**<https://hal.inria.fr/hal-00971497>**

Submitted on 6 Jul 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Hypersingular integral equations over a disc: convergence of a spectral method and connection with Tranter's method

Leandro Farina<sup>a</sup>, P. A. Martin<sup>b</sup>, Victor Péron<sup>c</sup>

<sup>a</sup>*Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brazil (permanent address; email: farina@mat.ufrgs.br) and BCAM–Basque Center for Applied Mathematics, Bilbao, Basque Country, Spain*

<sup>b</sup>*Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, Colorado 80401, USA.*

*Corresponding author. Email: pamartin@mines.edu. Phone: 1-303-273-3895*

<sup>c</sup>*Université de Pau et des Pays de l'Adour, MAGIQUE3D, INRIA-Bordeaux Sud Ouest, LMA, 64013 Pau cedex, France. Email: victor.peron@inria.fr*

---

## Abstract

Two-dimensional hypersingular equations over a disc are considered. A spectral method is developed, using Fourier series in the azimuthal direction and orthogonal polynomials in the radial direction. The method is proved to be convergent. Then, Tranter's method is discussed. This method was devised in the 1950s to solve certain pairs of dual integral equations. It is shown that this method is also convergent because it leads to the same algebraic system as the spectral method.

**Keywords:** hypersingular integral equations; spectral method; Galerkin method; Tranter's method; Jacobi polynomials; screen problems

---

## 1. Introduction

Two-dimensional boundary-value problems involving a Neumann-type boundary condition on a thin plate or crack can often be reduced to one-dimensional hypersingular integral equations. Examples are potential flow past a rigid plate, acoustic scattering by a hard strip, water-wave interaction with thin impermeable barriers [1], and stress fields around cracks [2]; for many additional references, see [3] and [4, §6.7.1]. The basic equation encountered takes the form

$$\int_{-1}^1 \left\{ \frac{1}{(x-t)^2} + K(x,t) \right\} v(t) dt = f(x) \quad \text{for } -1 < x < 1, \quad (1)$$

supplemented by two boundary conditions, which we take to be  $v(-1) = v(1) = 0$ . Here,  $v$  is the unknown function,  $f$  is prescribed and the kernel  $K$  is known. The cross on the integral sign indicates that it is to be interpreted as a two-sided finite-part integral of order two: if  $g'$  is Hölder continuous ( $g \in C^{1,\alpha}$ ),

$$\int_a^b \frac{g(t)}{(x-t)^2} dt = \lim_{\varepsilon \rightarrow 0} \left\{ \int_a^{x-\varepsilon} \frac{g(t)}{(x-t)^2} dt + \int_{x+\varepsilon}^b \frac{g(t)}{(x-t)^2} dt - \frac{2g(x)}{\varepsilon} \right\}. \quad (2)$$

Assuming that  $f$  is sufficiently smooth, the solution  $v$  has square-root zeros at the end-points. This suggests that we write

$$v(x) = w(x) u(x) \quad \text{with} \quad w(x) = \sqrt{1-x^2}.$$

Then, we expand  $u$  using a set of orthogonal polynomials; a good choice is to use Chebyshev polynomials of the second kind,  $U_n$ , defined by [5, 18.5.2]

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad n = 0, 1, 2, \dots$$

This is a good choice because of the formula

$$\frac{1}{\pi} \int_{-1}^1 \frac{w(t)}{(x-t)^2} U_n(t) dt = -(n+1)U_n(x). \quad (3)$$

Thus, we approximate  $u$  by

$$u_N(x) = \sum_{n=0}^N a_n U_n(x),$$

substitute into (1) and evaluate the hypersingular integral analytically, using (3). To find the coefficients  $a_0, a_1, \dots, a_N$ , one can use a collocation method or a Galerkin method. These methods have been used by many authors, and they are known to be very effective. Convergence results are also available; see, for example, [6, 7] and [8, §7.9].

In this paper, we generalize some of these results to *two-dimensional* hypersingular integral equations. Thus, rather than integrating over a finite interval, we now integrate over a circular disc. Such equations arise, for example, in the scattering of acoustic waves by a hard disc; this particular application is described in Appendix A. We develop an appropriate spectral (Galerkin) method, using Fourier expansions in the azimuthal direction and Jacobi polynomials in the radial direction. The Hilbert-space arguments used by Golberg are generalized and a convergence theorem is proved by using tensor-product techniques. Our results are proved in weighted  $L^2$  spaces. It may be possible to obtain results in other spaces, but we have not pursued this. For some results in this direction, see [9]. There is also work by Stephan and his collaborators on Galerkin boundary element methods for hypersingular integral equations over open flat domains; see, for example, [10] and the review [11].

Next, we discuss Tranter's method, a method for solving certain pairs of dual integral equation. This method involves a free parameter (denoted by  $\mu$  in Section 7). In fact, this freedom is illusory: it should be chosen so that the physical fields have the correct behaviour near the edge of the circular disc. Once this choice is made, we find that our spectral method and Tranter's method lead to exactly the same linear system of algebraic equations: thus, Tranter's method is also convergent.

The spectral method and Tranter's method have been used extensively to obtain numerical results; references are given in Section 8. Both methods have been found to converge: we prove that here. We also illustrate the convergence of the spectral method with numerical results for an axisymmetric problem (Section 6). In order to have a concrete example, we have given a detailed study of acoustic scattering by a thin sound-hard screen in Appendix A.

## 2. The hypersingular integral equation

Let  $x$  and  $y$  be Cartesian coordinates. Let  $r$  and  $\theta$  be polar coordinates, so that  $x = r \cos \theta$  and  $y = r \sin \theta$ . The unit disc is

$$D = \{(r, \theta) : 0 \leq r < 1, \quad -\pi < \theta \leq \pi\}.$$

Then, we consider the following hypersingular integral equation

$$\frac{1}{4\pi} \int_D \frac{w(\rho)}{R^3} u(\rho, \varphi) dA + \int_D K(\rho, \varphi; r, \theta) u(\rho, \varphi) w(\rho) dA = f(r, \theta), \quad (4)$$

for  $(r, \theta) \in D$ , where  $dA = \rho d\rho d\varphi$ . Here,  $u$  is to be found,  $f$  is known,  $K(\rho, \varphi; r, \theta)$  is a known weakly-singular kernel, and, as before,

$$w(\rho) = \sqrt{1 - \rho^2}. \quad (5)$$

Also,  $R$  is the distance between two points,  $(r, \theta)$  and  $(\rho, \varphi)$ , on the disc,

$$R = \sqrt{r^2 + \rho^2 - 2r\rho \cos(\theta - \varphi)}.$$

The hypersingular integral in (4) can be defined in several equivalent ways. Thus, if  $g$  is smooth enough ( $g \in C^{1,\alpha}$ ), one natural definition in the context of boundary-value problems is

$$\oint_D g(\rho, \varphi) \frac{dA}{R^3} = \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \int_D g(\rho, \varphi) \left\{ \lim_{\zeta \rightarrow 0} \frac{\partial}{\partial \zeta} \left( \frac{1}{\sqrt{R^2 + (z - \zeta)^2}} \right) \right\} dA;$$

another is

$$\oint_D g(\rho, \varphi) \frac{dA}{R^3} = \nabla_2^2 \int_D g(\rho, \varphi) \frac{dA}{R}, \quad (6)$$

where  $\nabla_2^2$  is the two-dimensional Laplacian; and another (cf. (2)) is

$$\oint_D g(\rho, \varphi) \frac{dA}{R^3} = \lim_{\varepsilon \rightarrow 0} \left\{ \int_{D \setminus D_\varepsilon} g(\rho, \varphi) \frac{dA}{R^3} - \frac{2\pi g(r, \theta)}{\varepsilon} \right\},$$

where  $D_\varepsilon$  is a small disc of radius  $\varepsilon$  centred at the singular point,  $(r, \theta)$ .

We are going to discuss the convergence of a Galerkin method for solving (4). In this method, described in Section 4, we use a Fourier series in  $\theta$  with the coefficients expanded in terms of Jacobi polynomials. Application of such a method to problems in which water waves interact with plane circular discs has been carried out [12]. If  $D$  is replaced by a more general plane domain,  $\Omega$ , the spectral method can still be used, if we first conformally map  $\Omega$  onto a circular disc. For a description of this extended method see [13]. If the disc is nonplanar, it is still possible to use a similar Fourier expansion method [14, 15], in conjunction with a boundary perturbation method [16].

### 3. Fredholm theory

Our objective in this section is to show that a Fredholm theory exists for (4). To do this, we will use a procedure analogous to the one given by Golberg [6].

Define functions  $E_m^\sigma(\theta)$  as follows:  $E_0^e = 1$ ,  $E_0^o = 0$ ,  $E_m^e(\theta) = \sqrt{2} \cos m\theta$ ,  $E_m^o(\theta) = \sqrt{2} \sin m\theta$ ,  $m = 0, 1, 2, \dots$ , where the superscripts e and o indicate even and odd (functions of  $\theta$ ) respectively. These functions are orthogonal:

$$\int_{-\pi}^{\pi} E_m^\sigma(\theta) E_n^\nu(\theta) d\theta = 2\pi \delta_{mn} \delta_{\sigma\nu}. \quad (7)$$

In the radial direction, we are going to expand using

$$\Phi_m^n(\rho) = c_m \rho^n w(\rho) P_m^{(n, 1/2)}(1 - 2\rho^2), \quad \text{with} \quad c_m = \frac{m!}{\Gamma(m + \frac{3}{2})}, \quad (8)$$

where  $P_n^{(\alpha, \beta)}$  is a Jacobi polynomial [5, §18.3] and  $c_m$  has been inserted for later algebraic convenience. The function  $\Phi_m^n(\rho)$  is proportional to  $P_{2m+n+1}^n(w(\rho))$  and to  $\rho^n C_{2m+1}^{n+1/2}(w(\rho))$ , where  $P_n^m$  is an associated Legendre function and  $C_n^\lambda$  is a Gegenbauer polynomial.

As the Jacobi polynomials are orthogonal, so too are the functions  $\Phi_m^n$ :

$$\int_0^1 \Phi_m^n(\rho) \Phi_k^n(\rho) \frac{\rho d\rho}{w(\rho)} = h_m^n \delta_{km}, \quad (9)$$

where

$$h_m^n = \frac{(m+n)! m!}{(4m+2n+3) \Gamma(m+n+\frac{3}{2}) \Gamma(m+\frac{3}{2})}. \quad (10)$$

Next, we define functions of two variables over the unit disc  $D$  by

$$\Psi_m^{n\sigma}(r, \theta) = A_m^n \frac{\Phi_m^n(r)}{w(r)} E_n^\sigma(\theta). \quad (11)$$

We choose the constants  $A_m^n$  so that the set of functions  $\{\Psi_m^{n\sigma}\}$  ( $m, n = 0, 1, 2, \dots, \sigma = e, o$ ) is orthonormal with respect to the weight  $w(\rho) = \sqrt{1 - \rho^2}$ . Thus, using (7) and (9),

$$\int_D \Psi_m^{n\sigma}(\rho, \varphi) \Psi_{m'}^{n'\sigma'}(\rho, \varphi) w(\rho) dA = 2\pi (A_m^n)^2 h_m^n \delta_{mm'} \delta_{nn'} \delta_{\sigma\sigma'}.$$

Hence we take  $2\pi(A_m^n)^2 h_m^n = 1$ .

We define the inner product of two functions,  $f$  and  $g$ , both defined on  $D$ , by

$$\langle f, g \rangle = \int_D f(\rho, \varphi) g^*(\rho, \varphi) w(\rho) dA, \quad (12)$$

where the  $*$  denotes complex conjugation. Thus,

$$\langle \Psi_m^{n\sigma}, \Psi_{m'}^{n'\sigma'} \rangle = \delta_{mm'} \delta_{nn'} \delta_{\sigma\sigma'}. \quad (13)$$

Then we define a weighted  $L^2$  space by

$$L_w^2 = \overline{\text{span}\{\Psi_m^{n\sigma}, m, n = 0, 1, 2, \dots, \sigma = e, o\}}, \quad (14)$$

where the overline denotes closure. Then  $L_w^2$ , with the inner product (12), is a Hilbert space. We also define a norm on  $L_w^2$  by  $\|f\| = \sqrt{\langle f, f \rangle}$ .

For any  $f \in L_w^2$ , we have its generalized Fourier series,

$$f = \sum_{m,n,\sigma} \langle f, \Psi_m^{n\sigma} \rangle \Psi_m^{n\sigma} \quad (15)$$

where the sum is over the ranges given in (14).

The main reason for introducing the functions  $\Psi_m^{n\sigma}$  is that they are eigenfunctions of the basic hypersingular operator. This stems from the following fact. Suppose that  $v$  and  $p$  are related by

$$\frac{1}{4\pi} \int_D \frac{v}{R^3} dA = p(r, \theta), \quad (r, \theta) \in D.$$

Writing

$$v(r, \theta) = \Phi_m^n(r) E_n^\sigma(\theta) \quad \text{and} \quad p(r, \theta) = C_m^n \frac{\Phi_m^n(r)}{\sqrt{1-r^2}} E_n^\sigma(\theta),$$

the coefficient  $C_m^n$  is given by

$$C_m^n = -\frac{\Gamma(m+n+\frac{3}{2})\Gamma(m+\frac{3}{2})}{(m+n)!m!}. \quad (16)$$

Therefore, if we write  $v = w\Psi_m^{n\sigma}$ , we obtain

$$\frac{1}{4\pi} \int_D \frac{w}{R^3} \Psi_m^{n\sigma} dA = C_m^n \Psi_m^{n\sigma}. \quad (17)$$

These results are implicit in Krenk's papers [17, 18] and explicit in [13]. We remark that (17) is the two-dimensional analogue of (3).

We note that  $h_m^n$  and  $C_m^n$ , defined by (10) and (16), respectively, are related by

$$h_m^n C_m^n = -\frac{1}{4m+2n+3}. \quad (18)$$

As  $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$ , we have  $C_0^0 = -\frac{1}{4}\pi$ . Then

$$\frac{C_m^{n+1}}{C_m^n} = \frac{m+n+\frac{3}{2}}{m+n+1} > 1$$

and, similarly,  $C_{m+1}^n/C_m^n > 1$ . Thus, as  $|C_m^n|$  is an increasing function of both  $m$  and  $n$ , we infer that

$$|C_m^n| \geq \pi/4, \quad m, n = 0, 1, 2, \dots \quad (19)$$

This bound will be used later.

### 3.1. The dominant equation

Suppose that  $K \equiv 0$ . Thus (4) reduces to the *dominant equation*

$$Hu = f, \quad (20)$$

where the hypersingular operator  $H : V \rightarrow L_w^2$  is defined on the space

$$V = \{u \in L_w^2 \mid \sum_{m,n,\sigma} \langle u, \Psi_m^{n\sigma} \rangle^2 (\mathcal{C}_m^n)^2 < \infty\}$$

by

$$(Hu)(r, \theta) = \frac{1}{4\pi} \int_D \frac{w(\rho)}{R^3} u(\rho, \varphi) dA.$$

Substituting

$$u = \sum_{m,n,\sigma} \langle u, \Psi_m^{n\sigma} \rangle \Psi_m^{n\sigma} \quad (21)$$

and (15) into (20), we have

$$\begin{aligned} & \frac{1}{4\pi} \int_D \frac{w(\rho)}{R^3} \sum_{m,n,\sigma} \langle u, \Psi_m^{n\sigma} \rangle \Psi_m^{n\sigma}(\rho, \varphi) dA \\ &= \frac{1}{4\pi} \sum_{m,n,\sigma} \langle u, \Psi_m^{n\sigma} \rangle \int_D \frac{w(\rho)}{R^3} \Psi_m^{n\sigma}(\rho, \varphi) dA \\ &= \sum_{m,n,\sigma} \langle u, \Psi_m^{n\sigma} \rangle \mathcal{C}_m^n \Psi_m^{n\sigma} = \sum_{m,n,\sigma} \langle f, \Psi_m^{n\sigma} \rangle \Psi_m^{n\sigma}, \end{aligned}$$

where we have used (17). Thus

$$\langle u, \Psi_m^{n\sigma} \rangle = (\mathcal{C}_m^n)^{-1} \langle f, \Psi_m^{n\sigma} \rangle.$$

Since  $f \in L_w^2$ , we have

$$\|f\|^2 = \sum_{m,n,\sigma} \langle f, \Psi_m^{n\sigma} \rangle^2 < \infty.$$

Then,

$$\|u\|^2 = \int_D |u|^2 w dA = \sum_{m,n,\sigma} \frac{\langle f, \Psi_m^{n\sigma} \rangle^2}{(\mathcal{C}_m^n)^2} \leq \frac{16}{\pi^2} \sum_{m,n,\sigma} \langle f, \Psi_m^{n\sigma} \rangle^2 = \frac{16}{\pi^2} \|f\|^2, \quad (22)$$

where we have used the bound (19). Thus, we see that  $H^{-1} : L_w^2 \rightarrow V$ , given by

$$H^{-1}u = \sum_{m,n,\sigma} \frac{\langle u, \Psi_m^{n\sigma} \rangle}{\mathcal{C}_m^n} \Psi_m^{n\sigma}, \quad (23)$$

is a bounded right inverse for  $H$ . From (23) it follows that the nullspace of  $H^{-1}$  is equal to 0, and therefore for every  $f \in L_w^2$ , (20) has a unique solution  $u \in V$ .

### 3.2. The general equation

Now suppose that  $K(\rho, \varphi; r, \theta) \not\equiv 0$ . Thus, we consider (4), written as

$$Hu + Ku = f, \quad (24)$$

where  $f \in L_w^2$  and the integral operator  $K : V \rightarrow L_w^2$  is defined by

$$(Ku)(r, \theta) = \int_D K(\rho, \varphi; r, \theta) u(\rho, \varphi) w(\rho) dA. \quad (25)$$

If (24) has a solution  $u \in V$ , then  $Ku \in L_w^2$ , so that the right-hand side of  $Hu = f - Ku$  is also in  $L_w^2$ . Thus

$$u + H^{-1}Ku = H^{-1}f. \quad (26)$$

Solving (24) is equivalent to solving (26), an equation of the second kind.

Since  $K(\rho, \varphi; r, \theta)$  is weakly singular,  $K$  is a compact operator on  $V$ , and the boundedness of  $H^{-1}$  implies that  $H^{-1}K$  is compact also. Thus the solvability of (24) can be determined from the classical Fredholm theory. In particular, (26) has a unique solution if and only if the nullspace of  $I + H^{-1}K$  is equal to 0. We assume that this condition holds, and this shows that (24) has a unique solution  $u \in V$  for every  $f \in L_w^2$ .

#### 4. The spectral method

Now we consider an approximation to  $u$ . First we define the space  $V_N \subset V$  as

$$V_N = \text{span}\{\Psi_m^{n\sigma}, \quad m = 0, 1, \dots, N_1, \quad n = 0, 1, \dots, N_2, \quad \sigma = e, o\},$$

where  $N$  is the dimension of  $V_N$ . We look for approximate solutions of (24),  $u_N \in V_N$ . Then  $u_N$  can be expressed as a linear combination of functions  $\Psi_m^{n\sigma}$ . For brevity, we write

$$u_N = \sum_{m,n,\sigma}^N a_m^{n\sigma} \Psi_m^{n\sigma} \equiv \sum_{m=0}^{N_1} \sum_{n=0}^{N_2} \sum_{\sigma=e,o} a_m^{n\sigma} \Psi_m^{n\sigma}. \quad (27)$$

Define the residual  $\mathcal{R}_N$  by  $\mathcal{R}_N = (H + K)u_N - f$ . Thus, from (27),

$$\mathcal{R}_N = \sum_{m,n,\sigma}^N a_m^{n\sigma} (H + K)\Psi_m^{n\sigma} - f = \sum_{m,n,\sigma}^N a_m^{n\sigma} (\mathcal{C}_m^n \Psi_m^{n\sigma} + K\Psi_m^{n\sigma}) - f,$$

using (17). To determine the coefficients  $a_m^{n\sigma}$ , we impose the condition

$$\langle \mathcal{R}_N, \Psi_m^{n\sigma} \rangle = 0, \quad m = 0, 1, \dots, N_1, \quad n = 0, 1, \dots, N_2, \quad \sigma = e, o, \quad (28)$$

which generates

$$\sum_{m,n,\sigma}^N a_m^{n\sigma} (\mathcal{C}_m^n \langle \Psi_m^{n\sigma}, \Psi_l^{k\nu} \rangle + \langle K\Psi_m^{n\sigma}, \Psi_l^{k\nu} \rangle) = \langle f, \Psi_l^{k\nu} \rangle, \quad \begin{array}{l} l = 0, 1, \dots, N_1, \\ k = 0, 1, \dots, N_2, \\ \nu = e, o. \end{array}$$

Using (13), we obtain

$$\mathcal{C}_l^k a_l^{k\nu} + \sum_{m,n,\sigma}^N \langle K\Psi_m^{n\sigma}, \Psi_l^{k\nu} \rangle a_m^{n\sigma} = \langle f, \Psi_l^{k\nu} \rangle, \quad \begin{array}{l} l = 0, 1, \dots, N_1, \\ k = 0, 1, \dots, N_2, \\ \nu = e, o. \end{array} \quad (29)$$

The spectral method for solving (4) consists of solving equation (29), which in turn can be seen as a linear system for the coefficients  $a_m^{n\sigma}$ , followed by use of (27). We will show that this method converges in mean.

To do this, we introduce orthogonal projection operators  $\mathcal{P}_N : L_w^2 \rightarrow V_N$ , defined by

$$\mathcal{P}_N f = \sum_{m,n,\sigma}^N \langle f, \Psi_m^{n\sigma} \rangle \Psi_m^{n\sigma}.$$

We will write  $N \rightarrow \infty$  as a shorthand for  $N_1 \rightarrow \infty$  and  $N_2 \rightarrow \infty$ . Evidently,  $\|\mathcal{P}_N f - f\| \rightarrow 0$  as  $N \rightarrow \infty$  (Parseval).

From (28), we have  $\mathcal{P}_N \mathcal{R}_N = 0$ , giving

$$\mathcal{P}_N H u_N + \mathcal{P}_N K u_N = \mathcal{P}_N f. \quad (30)$$

By construction,  $H u_N$  is in  $V_N$  whence  $\mathcal{P}_N H u_N = H u_N$ . Thus (30) simplifies to

$$H u_N + \mathcal{P}_N K u_N = \mathcal{P}_N f, \quad (31)$$

which is in the form of equations treated in the book by Golberg and Chen [8, §4.14.2]. Then we can appeal to a general result [8, Theorem 4.42] and deduce that  $u_N \rightarrow u$  as  $N \rightarrow \infty$ , where  $u$  solves (24).

## 5. Special cases of the spectral method

In many applications, the kernel  $K(\rho, \varphi; r, \theta)$  is an even function of  $\varphi - \theta$  and so it can be expanded as

$$K(\rho, \varphi; r, \theta) = \sum_{n=0}^{\infty} \epsilon_n K_n(\rho, r) \cos n(\varphi - \theta) = \sum_{n,\sigma} K_n(\rho, r) E_n^\sigma(\varphi) E_n^\sigma(\theta), \quad (32)$$

where  $\sum_{n,\sigma} \equiv \sum_{n=0}^{\infty} \sum_{\sigma=e,o}$ ,  $\epsilon_0 = 1$  and  $\epsilon_n = 1$  for  $n \geq 1$ . Then

$$(K \Psi_m^{n\sigma})(r, \theta) = 2\pi A_m^n E_n^\sigma(\theta) \int_0^1 K_n(\rho, r) \Phi_m^n(\rho) \rho d\rho,$$

using (7), (11) and (25). Integrating again gives

$$\langle K \Psi_m^{n\sigma}, \Psi_l^{k\nu} \rangle = (2\pi)^2 A_m^n A_l^k \delta_{nk} \delta_{\sigma\nu} I_{ml}^n$$

where

$$I_{ml}^n = \int_0^1 \int_0^1 K_n(\rho, r) \Phi_m^n(\rho) \Phi_l^n(r) \rho r dr d\rho. \quad (33)$$

Thus, the linear system (29) becomes

$$C_l^k a_l^{k\nu} + (2\pi)^2 A_l^k \sum_{m=0}^{N_1} I_{ml}^k A_m^k a_m^{k\nu} = \langle f, \Psi_l^{k\nu} \rangle, \quad l = 0, 1, \dots, N_1, \quad (34)$$

for each  $k = 0, 1, \dots, N_2$  and  $\nu = e, o$ : the system (29) has decoupled into many smaller systems.

Suppose, now, that

$$K_n(\rho, r) = K_n(r, \rho) = \frac{1}{4\pi} \int_0^\infty p(\kappa) J_n(\kappa\rho) J_n(\kappa r) d\kappa \quad (35)$$

for some function  $p$ , where  $J_n$  is a Bessel function; see (A.10) for such a representation in the context of acoustic scattering. Then

$$I_{ml}^n = \frac{1}{4\pi} \int_0^\infty p(\kappa) \mathcal{J}_l^n \mathcal{J}_m^n d\kappa,$$

where

$$\mathcal{J}_m^n(\kappa) = \int_0^1 J_n(\kappa r) \Phi_m^n(r) r dr = \frac{2}{\kappa\sqrt{\pi}} j_{n+2m+1}(\kappa) \quad (36)$$

and  $j_n(x) = \sqrt{\pi/(2x)} J_{n+1/2}(x)$  is a spherical Bessel function. (The integral has been evaluated using Tranter's integral; see (58) below.) Then, multiply (34) by  $h_l^k A_l^k$ , use  $2\pi h_l^k (A_l^k)^2 = 1$  and (18), and define  $\mathcal{A}_l^{k\nu} = a_l^{k\nu} A_l^k$ . The result is

$$\frac{\mathcal{A}_l^{k\nu}}{2k+4l+3} - \frac{2}{\pi} \sum_{m=0}^{N_1} \mathcal{A}_m^{k\nu} \int_0^\infty \frac{p(\kappa)}{\kappa^2} j_{k+2l+1}(\kappa) j_{k+2m+1}(\kappa) d\kappa = -h_l^k A_l^k \langle f, \Psi_l^{k\nu} \rangle, \quad (37)$$



where the constants on the right-hand side are given explicitly by

$$h_l^k A_l^k \langle f, \Psi_l^{k\nu} \rangle = \frac{1}{2\pi} \int_D f(\rho, \varphi) \Phi_l^k(\rho) E_l^\nu(\varphi) dA. \quad (38)$$

It turns out that exactly the same system, (37), arises when Tranter's method is used to solve a related pair of dual integral equations. This approach is described in Section 7.

## 6. Numerical examples

The spectral method can be readily implemented. For a specific example, we apply it to a simple but non-trivial axisymmetric problem, described in Section 6.1. The numerical results are then given in Section 6.2.

### 6.1. An example

Let us consider a simple axisymmetric problem, with a symmetric separable kernel. Thus

$$K(\rho, \varphi; r, \theta) = Q(\rho)Q(r) \quad \text{and} \quad f(r, \theta) = f_0(r)$$

for some functions  $Q$  and  $f_0$ . Then, from (34), there is just one non-trivial system to solve, that with  $k = 0$  and  $\nu = e$ ,

$$\mathcal{C}_l^0 a_l^N + (2\pi)^2 A_l^0 \sum_{m=0}^N I_{ml}^0 A_m^0 a_m^N = \langle f, \Psi_l^{0e} \rangle, \quad l = 0, 1, \dots, N, \quad (39)$$

where  $N = N_1$  and we have written  $a_m^N \equiv a_m^{0e}$  to emphasise the dependence on  $N$ . The separability of  $K$  implies that we can solve (39) explicitly, because, from (33),

$$I_{ml}^0 = Q_m Q_l \quad \text{with} \quad Q_m = \int_0^1 Q(r) \Phi_m^0(r) r dr.$$

Thus, from (39),

$$\mathcal{C}_l^0 a_l^N = \langle f, \Psi_l^{0e} \rangle - (2\pi)^2 A_l^0 Q_l S_N, \quad l = 0, 1, \dots, N, \quad (40)$$

where

$$S_N = \sum_{m=0}^N A_m^0 Q_m a_m^N$$

is easily determined: multiply (40) by  $A_l^0 Q_l / \mathcal{C}_l^0$  and sum over  $l$  to give

$$S_N = F_N / (1 - T_N) \quad (41)$$

where

$$T_N = 2\pi \sum_{m=0}^N (4m+3) Q_m^2, \quad F_N = - \sum_{m=0}^N (4m+3) Q_m \int_0^1 f_0(r) \Phi_m^0(r) r dr$$

and we have used (38) together with relations between  $\mathcal{C}_m^0$ ,  $A_m^0$  and  $h_m^0$ .

Having determined  $a_l^N$  by solving (39), we have

$$u_N = \sum_{l=0}^N a_l^N \Psi_l^{0e} \quad \text{and} \quad u = \sum_{l=0}^{\infty} a_l \Psi_l^{0e},$$

where  $a_l$  solves (39) with  $N = \infty$ . Thus

$$\mathcal{C}_l^0 a_l = \langle f, \Psi_l^{0e} \rangle - (2\pi)^2 A_l^0 Q_l S_\infty, \quad l = 0, 1, \dots, \quad (42)$$

where  $S_\infty$  is defined by (41) with  $N = \infty$ . Hence

$$\|u - u_N\|^2 = \sum_{l=0}^N (a_l - a_l^N)^2 + \sum_{l=N+1}^{\infty} (a_l)^2. \quad (43)$$

Also, subtracting (40) from (42) gives

$$a_l - a_l^N = (2\pi)^2 (A_l^0 / \mathcal{C}_l^0) Q_l (S_N - S_\infty), \quad l = 0, 1, \dots, N. \quad (44)$$

For a specific example, let us choose  $Q(r) = \lambda J_0(\zeta r)$  and  $f_0(r) = J_0(\zeta r)$ , where  $\lambda$  and  $\zeta$  are constants. Then  $Q_m = \lambda \mathcal{J}_m^0(\zeta)$ ,  $\langle f, \Psi_m^{0e} \rangle = 2\pi A_m^0 \mathcal{J}_m^0(\zeta)$ ,  $T_N = -2\pi \lambda F_N$  and

$$F_N = -\lambda \sum_{m=0}^N (4m+3) \{\mathcal{J}_m^0(\zeta)\}^2 = -\frac{4\lambda}{\pi \zeta^2} \sum_{m=0}^N (4m+3) j_{2m+1}^2(\zeta),$$

with  $\mathcal{J}_l^0$  defined by (36). In the limit as  $N \rightarrow \infty$ , we have

$$F_\infty = -\frac{2\lambda}{\pi \zeta^2} \left(1 - \frac{\sin 2\zeta}{2\zeta}\right),$$

using [5, 10.60.12 and 10.60.13]. From (40) and (42), we obtain

$$\mathcal{C}_m^0 a_m^N = 2\pi A_m^0 (1 - 2\pi \lambda S_N) \mathcal{J}_m^0(\zeta) \quad \text{and} \quad \mathcal{C}_m^0 a_m = 2\pi A_m^0 (1 - 2\pi \lambda S_\infty) \mathcal{J}_m^0(\zeta). \quad (45)$$

For (44), we want

$$S_N - S_\infty = \frac{F_N - F_\infty}{(1 + 2\pi \lambda F_N)(1 + 2\pi \lambda F_\infty)},$$

which decays rapidly to zero as  $N$  increases. Equation (43) reduces to

$$\begin{aligned} \|u - u_N\|^2 &= -2\pi (2\pi \lambda)^2 (S_N - S_\infty)^2 \sum_{l=0}^N \frac{4l+3}{\mathcal{C}_l^0} \{\mathcal{J}_l^0(\zeta)\}^2 \\ &\quad - 2\pi (1 - 2\pi \lambda S_\infty)^2 \sum_{l=N+1}^{\infty} \frac{4l+3}{\mathcal{C}_l^0} \{\mathcal{J}_l^0(\zeta)\}^2, \end{aligned} \quad (46)$$

using  $2\pi (A_l^0)^2 / \mathcal{C}_l^0 = -(4l+3)$ . The coefficients  $\mathcal{C}_l^0$  are given by (16), and they are negative.

We can estimate the second term on the right-hand side of (46) since, using (19), we have

$$\left| \sum_{l=N+1}^{\infty} \frac{4l+3}{\mathcal{C}_l^0} \{\mathcal{J}_l^0(\zeta)\}^2 \right| \leq \frac{4}{\pi} \sum_{l=N+1}^{\infty} (4l+3) \{\mathcal{J}_l^0(\zeta)\}^2 = \frac{4}{\lambda \pi} (F_N - F_\infty).$$

## 6.2. Numerical results

To illustrate the numerical implementation, a Matlab script for solving (39) was written. We present numerical results for different choices of  $f$  (the right-hand side) and  $Q$  (the function determining the kernel  $K$ ), for the simple axisymmetric problems considered in Section 6.1.

First, let us consider the case of  $Q(r) = J_0(r)$  and  $f_0(r) = J_0(r)$  (i.e.,  $\lambda = \zeta = 1$ ), for which an analytical solution was given in Section 6.1. The numerical results show that the quantity  $S_\infty - S_N$  is as small as  $5.2009 \times 10^{-4}$  for  $N = 0$  and converges numerically to  $1.1102 \times 10^{-16}$  for  $N = 3$ . Analogously, the first term on the right hand side of (46) is  $2.9597 \times 10^{-5}$  for  $N = 0$  and  $1.3500 \times 10^{-30}$  for  $N = 3$ .

In Table 1, the coefficients  $a_m^N$  are shown for  $N = 5, 10, 30$ . It is seen that each sequence  $(a_m^N)_m$  decays to zero very rapidly and the stability and fast convergence of the spectral method are apparent. The same characteristics are observed for other choices of  $f$  and  $Q$ . For instance, for  $f_0(r) = Q(r) = J_0(4r)$ , the coefficients are shown in Table 2.

In Figure 1, we can see, graphically, the decay of  $a_m^{15}$  for other cases involving Bessel functions. An oscillatory behaviour of  $a_m^N$  occurred when  $Q$  is singular at the edge of the disc,  $Q(\rho) = 1/(5[\rho - 1]) + J_0(\rho)$ , with  $f_0(\rho) = J_0(\rho)$ , see Figure 2.

$m$	$a_m^5$	$a_m^{10}$	$a_m^{30}$
0	1.409196726786439	1.409196726786439	1.409196726786440
1	0.042915968733271	0.042915968733271	0.042915968733271
2	0.000442308257273	0.000442308257273	0.000442308257273
3	0.000002291112648	0.000002291112648	0.000002291112648
4	0.00000007136162	0.00000007136162	0.00000007136162
5	0.00000000014834	0.00000000014834	0.00000000014834
6		0.000000000000022	0.000000000000022
7		0.000000000000000	0.000000000000000

Table 1: The coefficients  $a_m^N$  for  $N = 5, 10, 30$  when  $f = J_0(\rho)$  and  $Q = J_0(\rho)$ .

$m$	$a_m^5$	$a_m^{10}$	$a_m^{30}$
0	-0.205479259329904	-0.205479259330105	-0.205479259330105
1	-0.413133494237696	-0.413133494238101	-0.413133494238101
2	-0.093555808001548	-0.093555808001639	-0.093555808001639
3	-0.009020467889063	-0.009020467889072	-0.009020467889072
4	-0.000491797018978	-0.000491797018978	-0.000491797018978
5	-0.000017365405743	-0.000017365405743	-0.000017365405743
6		-0.000000430856651	-0.000000430856651
7		-0.000000007935952	-0.000000007935952
8		-0.000000000112917	-0.000000000112917
9		-0.000000000001279	-0.000000000001279
10		-0.000000000000012	-0.000000000000012
11		-0.000000000000000	-0.000000000000000

Table 2: The coefficients  $a_m^N$  for  $N = 5, 10, 30$  when  $f = J_0(4\rho)$  and  $Q = J_0(4\rho)$ .

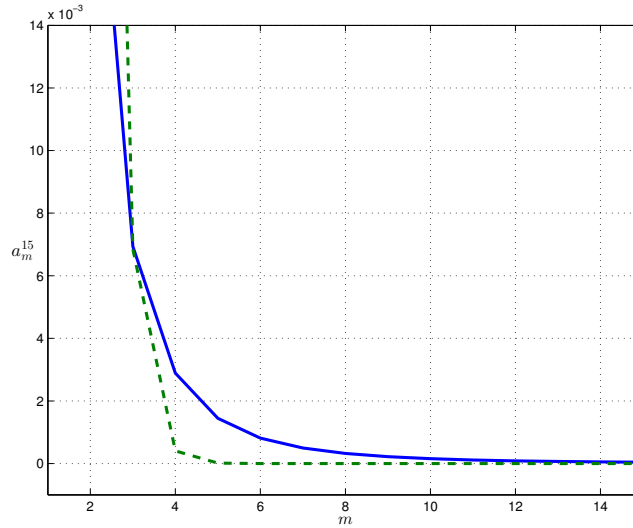


Figure 1: The coefficients  $a_m^{15}$  for the case  $f = \rho$ ,  $Q = J_4(2\rho)$  (solid line) and for  $f = J_0(\rho)$ ,  $Q = J_2(4\rho)$  (dashed line).

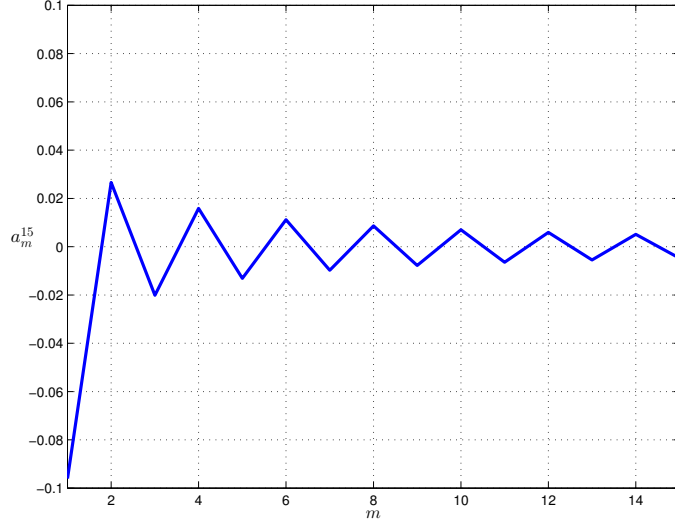


Figure 2: The coefficients  $a_m^{15}$  for the case  $f = J_0(\rho)$ ,  $Q = 1/(5[\rho - 1]) + J_0(\rho)$ .

## 7. Dual integral equations and Tranter's method

Boundary value problems that lead to the hypersingular integral equation (4) can often be treated by reduction to a pair of dual integral equations for an auxiliary function  $B$ . These equations have the form

$$\frac{1}{4\pi^2} \iint \kappa(1 + h(\kappa))B e^{-i(\xi x + \eta y)} d\xi d\eta = g(x, y), \quad (x, y) \in D, \quad (47)$$

$$\iint B e^{-i(\xi x + \eta y)} d\xi d\eta = 0, \quad (x, y) \in D', \quad (48)$$

where  $D$  is the unit disc in the  $xy$ -plane and  $D'$  is the rest of that plane. The double integrals are over the whole  $\xi\eta$ -plane,  $\kappa = \sqrt{\xi^2 + \eta^2}$  and  $h(\kappa)$  is a given function satisfying  $h(\kappa) \rightarrow 0$  as  $\kappa \rightarrow \infty$ . For acoustic scattering by a hard disc,  $h(\kappa) = (\gamma/\kappa) - 1$  with  $\gamma(\kappa)$  defined by (A.3).

Expand  $B$  and  $g$  in Fourier series,

$$B(\xi, \eta) = 2\pi \sum_{n,\sigma} i^n B_n^\sigma(\kappa) E_n^\sigma(\beta), \quad g(x, y) = \sum_{n,\sigma} g_n^\sigma(r) E_n^\sigma(\theta),$$

where we have introduced polar coordinates, defined by

$$x = r \cos \theta, \quad y = r \sin \theta, \quad \xi = \kappa \cos \beta, \quad \eta = \kappa \sin \beta. \quad (49)$$

Then, using the expansion [5, 10.12.3]

$$e^{\pm i(\xi x + \eta y)} = \sum_{n=0}^{\infty} \epsilon_n (\pm i)^n J_n(\kappa r) \cos n(\theta - \beta) = \sum_{n,\sigma} (\pm i)^n J_n(\kappa r) E_n^\sigma(\theta) E_n^\sigma(\beta), \quad (50)$$

we can integrate with respect to  $\beta$ . The result is

$$\int_0^\infty B_n^\sigma(\kappa) J_n(\kappa r) (1 + h(\kappa)) \kappa^2 d\kappa = g_n^\sigma(r), \quad 0 \leq r < 1, \quad (51)$$

$$\int_0^\infty B_n^\sigma(\kappa) J_n(\kappa r) \kappa d\kappa = 0, \quad r > 1, \quad (52)$$

with separate pairs of dual integral equations for each pair,  $n$  and  $\sigma$ .

These equations are amenable to Tranter's method [19]. Thus, write

$$B_n^\sigma(\kappa) = \kappa^{-\mu} \sum_{m=0}^{\infty} \alpha_m^{n\sigma} J_{n+2m+\mu}(\kappa), \quad (53)$$

where  $\mu$  is a positive parameter and the coefficients  $\alpha_m^{n\sigma}$  are to be found. The expansion (53) ensures that (52) is satisfied automatically (use [5, 10.22.56]) whereas (51) becomes

$$\sum_{m=0}^{\infty} \alpha_m^{n\sigma} \int_0^{\infty} (1+h(\kappa)) \kappa^{2-\mu} J_{n+2m+\mu}(\kappa) J_n(\kappa r) d\kappa = g_n^\sigma(r), \quad 0 \leq r < 1. \quad (54)$$

Multiply this equation by  $r^{n+1}(1-r^2)^{\mu-1} \mathcal{F}_l(n+\mu, n+1, r^2)$  and integrate over  $0 < r < 1$ . Here,  $\mathcal{F}_m(a, c, x) = {}_2F_1(-m, m+a; c; x)$  is the notation for Jacobi polynomials used by some authors [20, p. 83], including Tranter [19, 21]; specifically, from (8) and [5, 18.5.7],

$$\Phi_m^n(r) = r^n \sqrt{1-r^2} \frac{(n+m)!}{n! \Gamma(m + \frac{3}{2})} \mathcal{F}_m(n+3/2, n+1, r^2).$$

To effect the integration, we use *Tranter's integral* [19, 21],

$$\frac{2^{1-\mu} \Gamma(n+m+1)}{\Gamma(n+1) \Gamma(m+\mu)} \int_0^1 x^{n+1} \mathcal{F}_m(n+\mu, n+1, x^2) \frac{J_n(\xi x)}{(1-x^2)^{1-\mu}} dx = \xi^{-\mu} J_{n+2m+\mu}(\xi), \quad (55)$$

with  $n > -1$  and  $\mu > 0$ . Thus, (54) gives

$$\sum_{m=0}^{\infty} \alpha_m^{n\sigma} \int_0^{\infty} (1+h(\kappa)) \kappa^{2-2\mu} J_{n+2m+\mu}(\kappa) J_{n+2l+\mu}(\kappa) d\kappa = E(n, l, \mu), \quad (56)$$

where the constants on the right-hand side are given by

$$E(n, l, \mu) = \frac{2^{1-\mu} (n+l)!}{n! \Gamma(l+\mu)} \int_0^1 r^{n+1} (1-r^2)^{\mu-1} \mathcal{F}_l(n+\mu, n+1, r^2) g_n^\sigma(r) dr.$$

Now, how should we select  $\mu$ ? Tranter [19, p. 319] observes that the system (56) can be solved explicitly if the term  $(1+h)\kappa^{2-2\mu}$  in the integrand is replaced by  $\kappa^{-1}$ , and then suggests that the difference between these two terms should be made 'fairly small', if possible, by the choice of  $\mu$ . (There is a similar suggestion in Tranter's book [21, p. 116] and in Duffy's book [22, p. 248].) If we interpret Tranter's prescription as meaning in the limit  $\kappa \rightarrow \infty$ , we find that  $\mu = \frac{3}{2}$ . With this choice, (56) reduces to

$$\begin{aligned} \sum_{m=0}^{\infty} \alpha_m^{n\sigma} \int_0^{\infty} (1+h(\kappa)) j_{n+2m+1}(\kappa) j_{n+2l+1}(\kappa) d\kappa &= \frac{\pi}{2} E(n, l, 3/2) \\ &= \frac{\pi}{2} \frac{2^{-1/2} (n+l)!}{n! \Gamma(l + \frac{3}{2})} \int_0^1 r^{n+1} \sqrt{1-r^2} \mathcal{F}_l(n+3/2, n+1, r^2) g_n^\sigma(r) dr \\ &= \frac{\pi}{2\sqrt{2}} \int_0^1 \Phi_l^n(r) g_n^\sigma(r) r dr, \end{aligned} \quad (57)$$

where  $j_n$  is a spherical Bessel function, and (55) gives

$$\int_0^1 \Phi_m^n(x) J_n(\xi x) x dx = \frac{2}{\xi \sqrt{\pi}} j_{n+2m+1}(\xi). \quad (58)$$

Then, as the spherical Bessel functions are orthogonal in the following sense [5, 10.22.55],

$$\int_0^\infty j_{n+2m+1}(\kappa)j_{n+2l+1}(\kappa) d\kappa = \frac{\pi\delta_{lm}}{2(2n+4l+3)}, \quad (59)$$

the system (57) becomes

$$\frac{\alpha_l^{n\sigma}}{2n+4l+3} + \frac{2}{\pi} \sum_{m=0}^{\infty} \alpha_m^{n\sigma} \int_0^\infty h(\kappa)j_{n+2m+1}(\kappa)j_{n+2l+1}(\kappa) d\kappa = \frac{1}{\sqrt{2}} \int_0^1 \Phi_l^n(r)g_n^\sigma(r) dr.$$

This system is the same as (37). As the spectral method leading to (37) has been shown to be convergent, we infer that truncated forms of Tranter's method are convergent.

Notice the choice  $\mu = \frac{3}{2}$  made with Tranter's method. This choice is not arbitrary. Indeed, with any particular application, the quantity  $B$  can be related to a physical quantity,  $v$ , a quantity that has a known behaviour near the edge of the disc,  $D$ . This behaviour is enforced by the correct choice for  $\mu$ . Similar remarks can be made when Tranter's method is used for other boundary value problems, such as acoustic scattering by a sound-soft disc (Dirichlet condition).

## 8. Conclusion

We have shown that two apparently different numerical methods are convergent. The first is a spectral method for solving two-dimensional hypersingular integral equations over a disc. The unknown function is expanded in a tensor-product manner, with trigonometric functions of the angular variable and orthogonal polynomials in the radial variable. The second method, Tranter's method, is older and arises when the underlying boundary value problem is reduced to dual integral equations instead of a hypersingular integral equation. In this method, a different unknown function is expanded in a series of Bessel functions of various orders (a Neumann series). Although the two methods appear to be unrelated, they are not: they both lead to the same linear algebraic system.

Both methods have been used in the literature to generate numerical results for a variety of physical problems involving discs. For the spectral method, see [23, 12, 14, 15] and [24, §5.2]. For Tranter's method, see [25, 26, 27] and [22, p. 251]. The observed convergence accords with our theoretical analysis. It would be useful to estimate the rate of convergence, but we have not done that yet. Extensions to systems of integral equations may also be feasible; certainly, there are relevant numerical results in the literature obtained using variants of the spectral method [23, 24] and of Tranter's method [29, 26, 27, 28]. Again, numerical convergence has been observed but the algorithms have not been analysed.

## Acknowledgments

LF and VP acknowledge support from EU project FP7-295217-HPC-GA.

## Appendix A. An example: acoustic scattering by a hard screen

Consider a flat, sound-hard screen,  $\Omega$ , in the plane  $z = 0$ . There is an incident field,  $u_{\text{in}}$ , and the problem is to compute the scattered field,  $u$ . Thus, we seek a bounded solution of  $(\nabla^2 + k^2)u = 0$ , satisfying the Sommerfeld radiation condition at infinity and the boundary condition

$$\frac{\partial u}{\partial z} = g_{\text{in}} \quad \text{on both sides of } \Omega, \quad (\text{A.1})$$

where  $g_{\text{in}}(x, y) = -\partial u_{\text{in}}/\partial z$  evaluated at  $z = 0$ . It can be shown that the solution must be an odd function of  $z$ , so the problem can be reduced to one in the half-space  $z > 0$ .

*Derivation of a hypersingular integral equation using Fourier transforms*

Take the Fourier transform of  $(\nabla^2 + k^2)u = 0$  with respect to  $x$  and  $y$ , with, for example,

$$U(\xi, \eta, z) = \mathcal{F}\{u\} = \iint u(x, y, z) e^{i(\xi x + \eta y)} dx dy;$$

the integration is over the whole  $xy$ -plane. Then, with  $\kappa^2 = \xi^2 + \eta^2$ , and writing, for example  $U' = \partial U / \partial z$ , we obtain  $U'' + (k^2 - \kappa^2)U = 0$ . Hence,

$$U(\xi, \eta, z) = B(\xi, \eta) e^{-\gamma z}, \quad z > 0, \quad (\text{A.2})$$

for some function  $B$ , where

$$\gamma = (\kappa^2 - k^2)^{1/2} = \begin{cases} \sqrt{\kappa^2 - k^2}, & |\kappa| > k, \\ -i\sqrt{k^2 - \kappa^2}, & |\kappa| < k; \end{cases} \quad (\text{A.3})$$

thus,  $\text{Re } \gamma \geq 0$  and the branch has been chosen so that the radiation condition is satisfied with a time dependence of  $e^{-i\omega t}$ .

We have a screen,  $\Omega$ , in the  $xy$ -plane. The rest of the  $xy$ -plane is denoted by  $\Omega'$ . As we have split the problem into two half-space problems, we must also impose continuity of  $u$  across  $\Omega'$ . Thus, let

$$v(x, y) = u(x, y, 0+) - u(x, y, 0-);$$

this gives the discontinuity in  $u$  across the plane  $z = 0$ . Hence  $v(x, y) = 0$  for  $(x, y) \in \Omega'$ . We regard  $v$  on  $\Omega$  as our basic unknown. Its Fourier transform is

$$V(\xi, \eta) = \int_{\Omega} v(x, y) e^{i(\xi x + \eta y)} dx dy = 2B = 2U(\xi, \eta, 0). \quad (\text{A.4})$$

We obtain an integral equation by inverting,  $u = \mathcal{F}^{-1}U$ , and imposing (A.1),

$$\frac{1}{4\pi^2} \iint U'(\xi, \eta, 0) e^{-i(\xi x + \eta y)} d\xi d\eta = g_{\text{in}}(x, y), \quad (x, y) \in \Omega, \quad (\text{A.5})$$

where  $U$  is given by (A.2). Symbolically, we have

$$\mathcal{F}^{-1} \{ \gamma \mathcal{F}\{v\} \} = -2g_{\text{in}}. \quad (\text{A.6})$$

This integral equation holds for flat screens  $\Omega$  of any shape.

If  $\Omega$  had been sound-soft, with a Dirichlet boundary condition on  $\Omega$  instead of (A.1), we would have obtained

$$\mathcal{F}^{-1} \{ \gamma^{-1} \mathcal{F}\{u_z\} \} = \tilde{g}, \quad (\text{A.7})$$

an equation for the normal derivative of  $u$  on  $\Omega$ ,  $u_z(x, y)$ , where  $\tilde{g}$  is known from the boundary condition. We remark that Penzel [9] has given a detailed analysis of a Galerkin method for (A.7), with expansion functions similar to our  $\Psi_m^{n\sigma}$ , and he mentions that similar methods apply to (A.6).

Returning to (A.6), this equation can be written as a hypersingular integral equation, as follows. Let  $\mathcal{L} = \nabla_2^2 + k^2$ , where  $\nabla_2^2$  is the two-dimensional Laplacian with respect to  $x$  and  $y$ . We have  $\mathcal{L}e^{i(\xi x + \eta y)} = -\gamma^2 e^{i(\xi x + \eta y)}$ . Thus

$$\mathcal{F}^{-1} \{ \gamma \mathcal{F}\{v\} \} = -\mathcal{L}\mathcal{F}^{-1} \{ \gamma^{-1} \mathcal{F}\{v\} \}.$$

Then, changing the order of integration (which is now permissible), we obtain

$$(\mathcal{F}^{-1} \{ \gamma^{-1} \mathcal{F}\{v\} \})(x, y) = \int_{\Omega} M(x - x', y - y') v(x', y') dA' \quad (\text{A.8})$$

where  $dA' = dx' dy'$  and

$$M(x, y) = \iint \frac{1}{4\pi^2\gamma} e^{-i(\xi x + \eta y)} d\xi d\eta = \frac{1}{2\pi} \int_0^\infty \frac{\kappa}{\gamma} J_0(\kappa r) d\kappa = \frac{e^{ikr}}{2\pi r}. \quad (\text{A.9})$$

To evaluate  $M$ , we used polar coordinates, (49),  $e^{i\kappa x} = \sum_{n=0}^\infty \epsilon_n i^n J_n(\kappa r) \cos n\theta$  and [30, 6.554 (2) and (3)]. Thus, (A.6) becomes

$$\mathcal{L} \int_\Omega \frac{e^{ikR}}{4\pi R} v(x', y') dA' = g_{\text{in}}(x, y), \quad (x, y) \in \Omega,$$

with  $R = \{(x - x')^2 + (y - y')^2\}^{1/2}$ . We have

$$\begin{aligned} \mathcal{L} \left( \frac{e^{ikR}}{R} \right) &= \nabla_2^2 \left( \frac{1}{R} \right) + \nabla_2^2 \left( \frac{e^{ikR} - 1}{R} \right) + k^2 \frac{e^{ikR}}{R} \\ &= \nabla_2^2 \left( \frac{1}{R} \right) + \frac{(1 - ikR)e^{ikR} - 1}{R^3}, \end{aligned}$$

where the last term is  $O(R^{-1})$  as  $R \rightarrow 0$ . Hence, using (6), we see that  $v$  solves a hypersingular integral equation of the form (4).

The hypersingular part does not depend on  $k$ . Thus, if we write  $\gamma = \kappa + (\gamma - \kappa)$ , (A.6) becomes

$$-\mathcal{F}^{-1} \{ \kappa \mathcal{F}\{v\} \} - \mathcal{F}^{-1} \{ (\gamma - \kappa) \mathcal{F}\{v\} \} = 2g_{\text{in}}$$

and a calculation similar to that leading to (A.8) with (A.9) gives

$$-\mathcal{F}^{-1} \{ \kappa \mathcal{F}\{v\} \} = \nabla_2^2 \mathcal{F}^{-1} \{ \kappa^{-1} \mathcal{F}\{v\} \} = \nabla_2^2 \int_\Omega \frac{v dA'}{2\pi R}.$$

Thus, we have

$$\frac{1}{4\pi} \int_\Omega \frac{v}{R^3} dA' + (\mathcal{K}_\Omega v)(x, y) = g_{\text{in}}(x, y), \quad (x, y) \in \Omega$$

where

$$(\mathcal{K}_\Omega v)(x, y) = -\frac{1}{2} (\mathcal{F}^{-1} \{ (\gamma - \kappa) \mathcal{F}\{v\} \}) (x, y) = \int_\Omega \mathcal{K}(x - x', y - y') v(x', y') dA'$$

and

$$\mathcal{K}(X, Y) = \frac{1}{8\pi^2} \iint (\kappa - \gamma) e^{-i(\xi X + \eta Y)} d\xi d\eta = \frac{1}{4\pi} \int_0^\infty \kappa(\kappa - \gamma) J_0(\kappa R) d\kappa. \quad (\text{A.10})$$

Using the addition theorem [4, Theorem 2.10], [5, 10.23.7],

$$J_0(\kappa R) = \sum_{n,\sigma} J_n(\kappa r) J_n(\kappa \rho) E_n^\sigma(\theta) E_n^\sigma(\varphi),$$

we can write  $\mathcal{K} \equiv K$  in the form (32) with the representation (35) and  $p(\kappa) = \kappa(\kappa - \gamma)$ .

Adopting a similar procedure for the sound-soft problem, we find that (A.7) can be written as a Fredholm integral equation of the first kind over  $\Omega$  with a weakly-singular kernel.

#### *Use of the free-space Green's function*

There are well-known integral representations for scattering by bounded obstacles of finite volume, and these can be specialised for a flat screen [4, §6.7]. For a sound-hard screen, we obtain

$$u(x, y, z) = \int_\Omega v(x', y') \left\{ \lim_{z' \rightarrow 0} \frac{\partial}{\partial z'} \left( \frac{e^{ik\mathcal{R}}}{4\pi\mathcal{R}} \right) \right\} dA',$$

where  $\mathcal{R}^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ . This representation makes use of the free-space Green's function,  $e^{ik\mathcal{R}}/\mathcal{R}$ , and so the radiation condition is satisfied. On the screen (where  $z = 0$ ), we can use (A.1), giving precisely the same integral equation, (4), as obtained above.

The main virtue of the Fourier-transform derivation is that it does not require access to the free-space Green's function. On the other hand, Green's function techniques can be used for non-flat screens.



### Dual integral equations

Instead of working with a physical unknown such as  $v$ , we can use  $B(\xi, \eta) = U(\xi, \eta, 0)$ , and then impose (A.1) on  $\Omega$  and  $v = 0$  on  $\Omega'$ . This gives a pair of dual integral equations for  $B$ :

$$\frac{1}{4\pi^2} \iint \gamma B e^{-i(\xi x + \eta y)} d\xi d\eta = -g_{\text{in}}(x, y), \quad (x, y) \in \Omega, \quad (\text{A.11})$$

$$\frac{1}{4\pi^2} \iint B e^{-i(\xi x + \eta y)} d\xi d\eta = 0, \quad (x, y) \in \Omega'. \quad (\text{A.12})$$

These are in the form of (47) and (48) with  $h(\kappa) = (\gamma/\kappa) - 1 = O(\kappa^{-2})$  as  $\kappa \rightarrow \infty$ .

### References

- [1] N.F. Parsons, P.A. Martin, Trapping of water waves by submerged plates using hypersingular integral equations, *J. Fluid Mech.* 284 (1995) 359–375.
- [2] P.A. Martin, Perturbed cracks in two dimensions: an integral equation approach, *Int. J. Fract.* 104 (2000) 315–325.
- [3] G. Monegato, Definitions, properties and applications of finite-part integrals, *J. Comput. Appl. Math.* 229 (2009) 425–439.
- [4] P.A. Martin, *Multiple Scattering*, Cambridge University Press, Cambridge, 2006.
- [5] NIST Digital Library of Mathematical Functions, <http://dlmf.nist.gov/>.
- [6] M.A. Golberg, The convergence of several algorithms for solving integral equations with finite-part integrals, *J. Integral Eqns* 5 (1983) 329–340.
- [7] V.J. Ervin, E.P. Stephan, Collocation with Chebyshev polynomials for a hypersingular integral equation on an interval, *J. Comput. Appl. Math.* 43 (1992) 221–229.
- [8] M.A. Golberg, C.S. Chen, *Discrete Projection Methods for Integral Equations*, Computational Mechanics Publications, Southampton, 1997.
- [9] F. Penzel, On the solution of integral equations on the circular disk by use of orthogonal polynomials, in: S. Samko, A. Lebre, A.F. dos Santos (Eds.), *Factorization, Singular Operators and Related Problems*, Kluwer, Dordrecht, 2003, pp. 205–217.
- [10] N. Heuer, M.E. Mellado, E. P. Stephan, A  $p$ -adaptive algorithm for the BEM with the hypersingular operator on the plane screen, *Int. J. Numer. Meth. Eng.* 53 (2002) 85–104.
- [11] E.P. Stephan, The  $hp$ -version of BEM—fast convergence, adaptivity and efficient preconditioning, *J. Computational Math.* 27 (2009) 348–359.
- [12] L. Farina, P.A. Martin, Scattering of water waves by a submerged disc using a hypersingular integral equation, *Appl. Ocean Res.* 20 (1998) 121–134. Erratum: 21 (1999) 157–158.
- [13] P.A. Martin, Mapping flat cracks onto penny-shaped cracks, with applications to somewhat circular tensile cracks, *Q. Appl. Math.* 54 (1996) 663–653.
- [14] J.S. Ziebell, L. Farina, Water wave radiation by a submerged rough disc, *Wave Motion* 49 (2012) 34–49.
- [15] L. Farina, J.S. Ziebell, Solutions of hypersingular integral equations over circular domains by a spectral method, in J. Brandts, S. Korotov, M. Křížek, J. Šístek, T. Vejchodský (Eds.), *Applications of Mathematics 2013, In Honor of the 70th Birthday of Karel Segeth*, Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague, 2013, pp. 51–66.
- [16] P.A. Martin, On potential flow past wrinkled discs, *Proc. Roy. Soc. A* 454 (1998) 2209–2221.
- [17] S. Krenk, A circular crack under asymmetric loads and some related integral equations, *J. Appl. Mech.* 46 (1979) 821–826.
- [18] S. Krenk, Some integral relations of Hankel transform type and applications to elasticity theory, *Integral Eqns. & Operator Theory* 5 (1982) 548–561.
- [19] C.J. Tranter, A further note on dual integral equations and an application to the diffraction of electromagnetic waves, *Quart. J. Mech. Appl. Math.* 7 (1954) 317–325.
- [20] W. Magnus, F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics*, Chelsea, New York, 1949.
- [21] C.J. Tranter, *Integral Transforms in Mathematical Physics*, third ed., Methuen, London, 1966.
- [22] D.G. Duffy, *Mixed Boundary Value Problems*, Chapman & Hall/CRC, Boca Raton, 2008.
- [23] S. Krenk, H. Schmidt, Elastic wave scattering by a circular crack, *Phil. Trans. Roy. Soc. A* 308 (1982) 167–198.
- [24] A. Boström, Review of hypersingular integral equation method for crack scattering and application to modeling of ultrasonic nondestructive evaluation, *Appl. Mech. Rev.* 56 (2003) 383–405.
- [25] W. Zhang, H.A. Stone, Oscillatory motions of circular disks and nearly spherical particles in viscous flows, *J. Fluid Mech.* 367 (1998) 329–358.
- [26] A.M.J. Davis, R.J. Nagem, Effect of viscosity on acoustic diffraction by a circular disk, *J. Acoust. Soc. Amer.* 115 (2004) 2738–2748.
- [27] J.D. Sherwood, Resistance coefficients for Stokes flow around a disk with a Navier slip condition, *Phys. Fluids* 24 (2012) 093103.
- [28] J.P. Tanzosh, H.A. Stone, Transverse motion of a disk through a rotating viscous fluid, *J. Fluid Mech.* 301 (1995) 295–324.

- [29] A.M.J. Davis, S.G. Llewellyn Smith, Tangential oscillations of a circular disk in a viscous stratified fluid, *J. Fluid Mech.* 656 (2010) 342–359.
- [30] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, fifth ed., Academic Press, San Diego, 1994.