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Sufficient Conditions for Labelled 0–1 Laws

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If $\mathbf{F}(x) = e^{\mathbf{G}(x)}$, where $\mathbf{F}(x) = \sum f(n)x^n$ and $\mathbf{G}(x) = \sum g(n)x^n$, with $0 \leq g(n) = O(n^{\theta n}/n!)$, $\theta \in (0, 1)$, and $\gcd(n : g(n) > 0) = 1$, then $f(n) = o(f(n-1))$. This gives an answer to Compton’s request in Question 8.3 [Compton 1987] for an “easily verifiable sufficient condition” to show that an adequate class of structures has a labelled first-order 0–1 law, namely it suffices to show that the labelled component count function is $O(n^{\theta n})$ for some $\theta \in (0, 1)$. It also provides the means to recursively construct an adequate class of structures with a labelled 0–1 law but not an unlabelled 0–1 law, answering Compton’s Question 8.4.

Keywords: ratio test, labelled structure, zero-one law

1 Introduction

Exponentiating a power series can have the effect of smoothing out the behavior of the coefficients. In this paper we look at conditions on the growth of the coefficients of $\mathbf{G}(x) = \sum g(n)x^n$, where $g(n) \geq 0$, which ensure that $f(n-1)/f(n) \rightarrow \infty$, where $\mathbf{F}(x) = e^{\mathbf{G}(x)}$. One application of this result is to 0-1 laws, where we find, see Theorem 7, that if the labelled component count function for an adequate class of structures is $O(n^{\theta n})$ for some $\theta \in (0, 1)$ then the class has a labelled monadic second-order 0-1 law.

Useful notation will be $f(n) \prec g(n)$ for $f(n)$ eventually less than $g(n)$ and $f(n) \in \text{RT}_\infty$ for $f(n-1)/f(n) \rightarrow \infty$; the notation RT stands for the ratio test.

2 The Coefficients of e^{poly}

Proposition 1 *Given*

$$\begin{aligned} \mathbf{G}(x) &:= g(1)x + \cdots + g(d)x^d, \quad g(i) \geq 0, g(d) > 0, \\ &\text{with } \gcd(j \leq d : g(j) > 0) = 1 \\ \mathbf{F}(x) &:= \sum_{n \geq 0} f(n)x^n = e^{\mathbf{G}(x)}, \end{aligned}$$

the function $\mathbf{F}(x)$ is Hayman-admissible. Thus

$$f(n) \sim \frac{\mathbf{F}(r_n)}{r_n^n \cdot \sqrt{2\pi \mathbf{B}(r_n)}} \quad (1)$$

where r_n is the unique positive solution to

$$x \cdot \mathbf{G}'(x) = n,$$

and $\mathbf{B}(x) := x^2 \mathbf{G}''(x) + x \mathbf{G}'(x)$.

Proof: Theorem X of Hayman [5] shows that $\mathbf{F}(x)$ is Hayman-admissible. Then the rest of the claim is an immediate consequence of Corollary II of [5] where the saddle-point method is applied to find the asymptotics of the coefficients of an admissible function. \square

Corollary 2 For $\mathbf{F}(x)$, $\mathbf{G}(x)$ as in the above proposition,

(a) $f(n) \in \text{RT}_\infty$,

(b) $f(n) = \exp\left(-\frac{n \log n}{d}(1 + o(1))\right)$.

Proof: Item a follows immediately from Corollary IV of Hayman [5].

For item b one uses $r_n \mathbf{G}'(r_n) = n$ to obtain:

$$\begin{aligned} \left(\frac{n}{cdg(d)}\right)^{1/d} &\leq r_n \leq \left(\frac{n}{dg(d)}\right)^{1/d} \quad \text{for } c > 1 \\ r_n &= (1 + o(1)) \left(\frac{n}{dg(d)}\right)^{1/d} \\ r_n^n &= (1 + o(1))^n \left(\frac{n}{dg(d)}\right)^{n/d} \\ \mathbf{B}(r_n) &= (1 + o(1)) d^2 g(d) \left(\frac{n}{dg(d)}\right) = (1 + o(1)) dn \\ \mathbf{G}(r_n) &= (1 + o(1)) g(d) r_n^d = (1 + o(1)) \frac{n}{d} \\ \mathbf{F}(r_n) &= \exp\left(\frac{n}{d}(1 + o(1))\right). \end{aligned}$$

Apply these results to (1). \square

3 Some Technical Lemmas

Now we drop the assumption that $\mathbf{G}(x)$ is a polynomial, but keep the requirement

$$\gcd(n : g(n) > 0) = 1. \quad (2)$$

This implies that $f(n) \succ 0$.

Choose a positive integer $L \geq 2$ sufficiently large so

$$n > L \Rightarrow [x^n] \exp\left(g(1)x + \cdots + g(L)x^L\right) > 0. \quad (3)$$

Given $\ell > L$ with $g(\ell) > 0$ let

$$\begin{aligned}
\mathbf{G}_0(x) &:= \sum_{n \geq 1} g_0(n)x^n := \sum_{1 \leq n \leq \ell} g(n)x^n \\
\mathbf{F}_0(x) &:= \sum_{n \geq 0} f_0(n)x^n := \exp(\mathbf{G}_0(x)) \\
\mathbf{G}_1(x) &:= \sum_{n \geq 1} g_1(n)x^n := \sum_{n \geq \ell+1} g(n)x^n \\
\mathbf{F}_1(x) &:= \sum_{n \geq 0} f_1(n)x^n := \exp(\mathbf{G}_1(x)).
\end{aligned} \tag{4}$$

Lemma 3 Suppose $r \geq -1$ is such that

$$ng(n) = O(f_0(n+r)). \tag{5}$$

Then

$$nf_1(n) = O(f(n+r)).$$

Proof: In view of (3) and (5) we can choose C_r such that

$$ng(n) \leq C_r f_0(n+r) \quad \text{for } n+r \geq L+1. \tag{6}$$

Differentiating (4) gives

$$\begin{aligned}
nf_1(n) &= \sum_{j=\ell+1}^n jg(j) \cdot f_1(n-j) \\
&\leq C_r \sum_{j=\ell+1}^n f_0(j+r) \cdot f_1(n-j) \quad \text{by (6)} \\
&\leq C_r \sum_{j=0}^{n+r} f_0(j) \cdot f_1(n+r-j) \\
&= C_r f(n+r),
\end{aligned}$$

the last line following from $\mathbf{F}(x) = \mathbf{F}_0(x) \cdot \mathbf{F}_1(x)$. \square

Lemma 4 Suppose for every integer $r \geq -1$

$$ng(n) = O(f_0(n+r)).$$

Then $f(n-1)/f(n) \rightarrow \infty$.

Proof: Since $f_0(n) \in \text{RT}_\infty$ by Corollary 2 there is a monotone decreasing function $\varepsilon(n)$ such that for any sufficiently large M we have $\varepsilon(n) > f_0(n)/f_0(n-1)$ for $n \geq M$, and $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$.

Thus

$$\begin{aligned}
f(n) &= \sum_{0 \leq j \leq n} f_0(j)f_1(n-j) \\
&= \sum_{0 \leq j \leq M-1} f_0(j)f_1(n-j) + \sum_{M \leq j \leq n} f_0(j)f_1(n-j) \\
&\leq o(f(n-1)) + \varepsilon(M) \sum_{M \leq j \leq n} f_0(j-1)f_1(n-j) \\
&\quad \text{by Lemma 3 and the choice of } \varepsilon \\
&\leq o(f(n-1)) + \varepsilon(M)f(n-1).
\end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{f(n)}{f(n-1)} \leq \varepsilon(M),$$

and as M can be arbitrarily large it follows that

$$\lim_{n \rightarrow \infty} \frac{f(n)}{f(n-1)} = 0.$$

□

4 Main Result

We are now in a position to prove the main result, making use of

$$n! = \exp(n \log n \cdot (1 + o(1))),$$

which follows from Stirling's result.

Theorem 5 Suppose $\mathbf{F}(x) = \exp(\mathbf{G}(x))$ with $\mathbf{F}(x) = \sum_{n \geq 0} f(n)x^n$, $\mathbf{G}(x) = \sum_{n \geq 1} g(n)x^n$, and $f(n), g(n) \geq 0$. Suppose also that $\gcd(n : g(n) > 0) = 1$ and that for some $\theta \in (0, 1)$

$$g(n) = O(n^{\theta n}/n!).$$

Then

$$f(n) \in \text{RT}_\infty.$$

Proof: From Corollary 2, for any integer $r \geq -1$ and any $\theta \in (0, 1)$, by choosing $\ell > L$ such that $1/\ell < 1 - \theta$, we have

$$\begin{aligned}
f_0(n+r) &= \exp\left(-\frac{(n+r)\log(n+r)}{\ell}(1+o(1))\right) \\
&= \exp\left(-\frac{n \log n}{\ell}(1+o(1))\right) \\
&\succ \frac{n^{\theta n}}{(n-1)!}.
\end{aligned}$$

Thus $ng(n) = O(f_0(n+r))$. The Theorem then follows from Lemma 4. □

5 Best Possible Result

The main result is in a natural sense the best possible.

Proposition 6 *Suppose $t(n) \geq 0$ with $\gcd(n : t(n) > 0) = 1$ is such that for every $\theta \in (0, 1)$*

$$t(n) \neq O(n^{\theta n}/n!).$$

Then there is a sequence $g(n) \geq 0$ with $\gcd(n : g(n) > 0) = 1$ and $g(n) \leq t(n)$ but $f(n) \notin \text{RT}_\infty$, where one has $\mathbf{F}(x) = \exp(\mathbf{G}(x))$.

Proof: For $\theta \in (0, 1)$ let

$$S(\theta) = \{n \geq 1 : t(n) > n^{\theta n}/n!\}.$$

Then $S(\theta)$ is an infinite set.

Let M be such that $\gcd(n \leq M : t(n) > 0) = 1$, and let

$$\begin{aligned} g_1(n) &:= \begin{cases} t(n) & \text{if } n \leq M \\ 0 & \text{if } n > M \end{cases} \\ \mathbf{G}_1(x) &:= \sum g_1(n)x^n \\ d_1 &:= \deg(\mathbf{G}_1(x)) \\ \mathbf{F}_1(x) &:= e^{\mathbf{G}_1(x)}. \end{aligned}$$

For $m \geq 2$ we give a recursive procedure to define polynomials $\mathbf{G}_m(x)$; then letting

$$\begin{aligned} d_m &:= \deg(\mathbf{G}_m(x)) \\ \mathbf{F}_m(x) &:= e^{\mathbf{G}_m(x)}, \end{aligned}$$

by Proposition 1

$$f_m(n) = \exp\left(-\frac{n \log n}{d_m}(1 + o(1))\right).$$

To define $\mathbf{G}_{m+1}(x)$, having defined $\mathbf{G}_m(x)$, let

$$h_m(n) := \frac{1}{n!} \cdot n^{(1-1/2d_m)n}.$$

Then

$$\frac{h_m(n)}{f_m(n-1)} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Thus we can choose an integer $d_{m+1} > d_m$ such that

$$\begin{aligned} d_{m+1} &\in S\left(1 - \frac{1}{2d_m}\right) \\ h_m(d_{m+1}) &> f_m(d_{m+1} - 1). \end{aligned}$$

This ensures that $h_m(d_{m+1}) \leq t(d_{m+1})$. Let

$$\mathbf{G}_{m+1} := \mathbf{G}_m(x) + h_m(d_{m+1})x^{d_{m+1}}.$$

Then

$$\frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1}-1)} \geq \frac{h_m(d_{m+1})}{f_m(d_{m+1}-1)} > 1.$$

Now let $\mathbf{G}(x)$ be the nonnegative power series defined by the sequence of polynomials $\mathbf{G}_m(x)$; and let $\mathbf{F}(x) = e^{\mathbf{G}(x)}$. Then $g(n) \leq t(n)$ but $f(n) \notin \text{RT}_\infty$ as

$$\frac{f(d_{m+1})}{f(d_{m+1}-1)} = \frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1}-1)} > 1.$$

□

6 Application to 0–1 laws

A class \mathcal{K} of finite relational structures is *adequate* if it is closed under disjoint union and the extraction of components. One can view the structures as being *unlabelled* with the component count function $p_U(n)$ and the total count function $a_U(n)$, both counting up to isomorphism. The corresponding *ordinary* generating series are

$$\mathbf{P}_U(x) := \sum_{n \geq 1} p_U(n)x^n, \quad \mathbf{A}_U(x) := \sum_{n \geq 0} a_U(n)x^n$$

connected by the fundamental equation

$$\mathbf{A}_U(x) = \prod_{j \geq 1} (1 - x^j)^{-p_U(j)}. \quad (7)$$

One can also view the structures as being *labelled* (in all possible ways) with the count functions $p_L(n)$ for the connected members of \mathcal{K} , and $a_L(n)$ for all members of \mathcal{K} . The corresponding *exponential* generating series are

$$\mathbf{P}_L(x) := \sum_{n \geq 1} p_L(n)x^n/n!, \quad \mathbf{A}_L(x) := \sum_{n \geq 0} a_L(n)x^n/n!$$

connected by the fundamental equation

$$\mathbf{A}_L(x) = e^{\mathbf{P}_L(x)}. \quad (8)$$

All references to Compton in this section are to the two papers [3] and [4].

6.1 Unlabelled 0–1 Laws for Adequate Classes

Let \mathcal{K} be an adequate class with unlabelled count functions and ordinary generating functions as described above. Compton showed that if the radius of convergence ρ_U of $\mathbf{A}_U(x)$ is positive then \mathcal{K} has an unlabelled 0–1 law⁽ⁱ⁾ iff $a_U(n) \in \text{RT}_1$, that is,

$$\frac{a_U(n-1)}{a_U(n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

\mathcal{K} is finitely generated if $r = \sum p_U(n) < \infty$, that is, there are only finitely many connected structures in \mathcal{K} . In the finitely generated case the asymptotics for the coefficients $a_U(n)$ have long been known to have the simple polynomial form⁽ⁱⁱ⁾

$$a_U(n) \sim Cn^{r-1} \tag{9}$$

provided $\gcd(n : p_U(n) > 0) = 1$. Item (9) leads to the fact that $a_U(n) \in \text{RT}_1$, and hence to an unlabelled 0–1 law. In addition to using this result, Compton notes that the work of Bateman and Erdős [1] shows that if $p_U(n) \in \{0, 1\}$, for all n , then one has $a_U(n) \in \text{RT}_1$.

Both of these results were subsumed in the powerful result of Bell [2] which says that if $p_U(n)$ is polynomially bounded, that is, there is a c such that $p_U(n) = O(n^c)$, then $a_U(n) \in \text{RT}_1$.

6.2 Labelled 0–1 Laws

Compton shows that if ρ_L , the radius of convergence of $\mathbf{A}_L(x)$, is positive, then \mathcal{K} has a labelled 0–1 law iff

$$\frac{a_L(n-k)/(n-k)!}{a_L(n)/n!} \rightarrow \infty \quad \text{whenever } p_L(k) > 0. \tag{10}$$

In particular it suffices to show that $a_L(n)/n! \in \text{RT}_\infty$.

Compton’s method to show that a given adequate class of finite relational structures \mathcal{K} has a labelled 0–1 law is to show that its exponential generating function $\mathbf{A}_L(x) = \sum a_L(n)x^n/n!$ is Hayman-admissible with an infinite radius of convergence. This guarantees that $a_L(n)/n! \in \text{RT}_\infty$ ([5], Corollary IV). However, as Compton notes, showing that $\mathbf{A}_L(x)$ is Hayman-admissible can be quite a challenge.

Question 8.3 of [3] first asks if, in the *unlabelled* case, the result of Bateman and Erdős, namely $p_U(n) \in \{0, 1\}$ implies $a_U(n) \in \text{RT}_1$, can be extended to the much more general statement that $p_U(n) = O(n^k)$ implies $a_U(n) \in \text{RT}_1$, yielding an unlabelled 0–1 law. As mentioned earlier, this was proved to be true by Bell. The second part of Question 8.3 asks if there is a simple sufficient condition along similar lines for the labelled case. We can now answer this in the affirmative with a result that is an excellent parallel to Bell’s result for unlabelled structures.

Theorem 7 *If \mathcal{K} is an adequate class of structures with*

$$p_L(n) = O\left(n^{\theta n}\right) \quad \text{for some } \theta \in (0, 1)$$

then $a_L(n)/n! \in \text{RT}_\infty$, and consequently \mathcal{K} has a labelled monadic second-order 0–1 law.

⁽ⁱ⁾ Given a logic \mathcal{L} , \mathcal{K} has an unlabelled \mathcal{L} 0–1 law means that for any \mathcal{L} sentence φ , the probability that φ holds in \mathcal{K} will be either 0 or 1. In [3] Compton worked with first-order logic, in [4] with monadic second-order logic. In both papers he simply used the phrases “unlabeled 0–1 law” and “labeled 0–1 law”.

⁽ⁱⁱ⁾ This result is usually known as Schur’s Theorem [6, 3.15.2]. One can easily find the asymptotics (9) using a partial fraction decomposition of the right side of (7). The labelled case with finitely many components is more difficult—we needed to invoke Hayman’s treatise [5] just to obtain the asymptotics for $\log a_L(n)/n!$ (see Corollary 2).

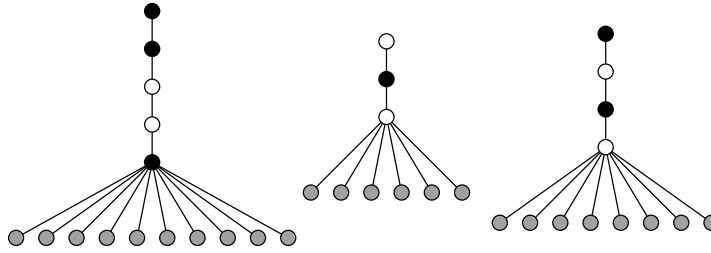


Fig. 1: Brooms with two-colored handles

Proof: This is an immediate consequence of Theorem 5 and Compton's proof that $a_L(n)/n! \in \text{RT}_\infty$ guarantees such a 0–1 law. \square

Now we list the examples of classes \mathcal{K} which Compton shows have a labelled 0–1 law, giving $p_L(n)$ in each case. It is trivial to check in each case that $p_L(n) = O(n^{n/2})$; thus the 0–1 law in each case follows from our Theorem 7.

- (a) 7.1 Unary Predicates $p_L(n) = 0$ for $n > 1$.
- (b) 7.12 Forests of Rooted Trees of Height 1 $p_L(n) = n$.
- (c) 7.15 Only Finitely Many Components $p_L(n)$ is eventually 0.
- (d) 7.16 Equivalence Relations $p_L(n) = 1$.
- (e) 7.17 Partitions with a Selection Subset $p_L(n) = 2^n - 1$.

We can now augment this list by, in each case, coloring the members of \mathcal{K} by a fixed set of r colors in all possible ways. This will increase the original $p_L(n)$ by a factor of at most r^n . This will still give $p_L(n) = O(n^{n/2})$. Furthermore, in each of these colored cases let \mathcal{P} be any subset of the connected members, and let \mathcal{K} be the closure of \mathcal{P} under disjoint union. Each such \mathcal{K} has a labelled 0–1 law.

Another application of Theorem 7 is to answer Question 4 of [3] by exhibiting an adequate class \mathcal{K} such that $p_L(n) = O(n^{3n/4})$, hence there is a labelled 0–1 law for \mathcal{K} ; but also such that $\rho_U \in (0, 1)$, so \mathcal{K} does not have an unlabelled 0–1 law.

Let the components of \mathcal{K} be the one-element tree T_1 along with rooted trees T_{3n} of size $3n$ and height n consisting of a chain C_n of n nodes, with an antichain L_{2n} of $2n$ nodes (the leaves of the tree) below the least member of the chain; and the chain C_n is two-colored while the remaining nodes are uncolored. One can visualize these as brooms with 2-colored handles, see Figure 1.

The number of unlabelled components is given by $p_U(1) = 1$, $p_U(3n) = 2^n$. Thus the radius of convergence of the ordinary generating function of \mathcal{K} is $\rho_U = \sqrt[3]{2}$. Since this is positive and not 1 it follows from Theorem 5.9(ii) of [3] that \mathcal{K} does not have an unlabelled 0–1 law.

For the number $p_L(3n)$ of labelled components of size $3n$:

$$\begin{aligned}
 p_L(3n) &\leq 2^n \binom{3n}{n} n! \\
 &\leq 2^n (3n)^n \exp(n \log n \cdot (1 + o(1))) \\
 &= \exp(2n \log n \cdot (1 + o(1))) \\
 &= (3n)^{(2/3)(3n)(1+o(1))} \\
 &= O\left((3n)^{(3/4)(3n)}\right).
 \end{aligned}$$

Thus $p_L(n) = O(n^{3n/4})$, so $a_L(n)/n! \in \text{RT}_\infty$ by Theorem 7, showing that \mathcal{K} has a labelled 0–1 law.

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