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# Sufficient Conditions for Labelled 0–1 Laws

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If  $\mathbf{F}(x) = e^{\mathbf{G}(x)}$ , where  $\mathbf{F}(x) = \sum f(n)x^n$  and  $\mathbf{G}(x) = \sum g(n)x^n$ , with  $0 \leq g(n) = O(n^{\theta n}/n!)$ ,  $\theta \in (0, 1)$ , and  $\gcd(n : g(n) > 0) = 1$ , then  $f(n) = o(f(n-1))$ . This gives an answer to Compton’s request in Question 8.3 [Compton 1987] for an “easily verifiable sufficient condition” to show that an adequate class of structures has a labelled first-order 0–1 law, namely it suffices to show that the labelled component count function is  $O(n^{\theta n})$  for some  $\theta \in (0, 1)$ . It also provides the means to recursively construct an adequate class of structures with a labelled 0–1 law but not an unlabelled 0–1 law, answering Compton’s Question 8.4.

**Keywords:** ratio test, labelled structure, zero-one law

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## 1 Introduction

Exponentiating a power series can have the effect of smoothing out the behavior of the coefficients. In this paper we look at conditions on the growth of the coefficients of  $\mathbf{G}(x) = \sum g(n)x^n$ , where  $g(n) \geq 0$ , which ensure that  $f(n-1)/f(n) \rightarrow \infty$ , where  $\mathbf{F}(x) = e^{\mathbf{G}(x)}$ . One application of this result is to 0-1 laws, where we find, see Theorem 7, that if the labelled component count function for an adequate class of structures is  $O(n^{\theta n})$  for some  $\theta \in (0, 1)$  then the class has a labelled monadic second-order 0-1 law.

Useful notation will be  $f(n) \prec g(n)$  for  $f(n)$  eventually less than  $g(n)$  and  $f(n) \in \text{RT}_\infty$  for  $f(n-1)/f(n) \rightarrow \infty$ ; the notation RT stands for the ratio test.

## 2 The Coefficients of $e^{\text{poly}}$

**Proposition 1** *Given*

$$\begin{aligned} \mathbf{G}(x) &:= g(1)x + \cdots + g(d)x^d, \quad g(i) \geq 0, g(d) > 0, \\ &\text{with } \gcd(j \leq d : g(j) > 0) = 1 \\ \mathbf{F}(x) &:= \sum_{n \geq 0} f(n)x^n = e^{\mathbf{G}(x)}, \end{aligned}$$

*the function  $\mathbf{F}(x)$  is Hayman-admissible. Thus*

$$f(n) \sim \frac{\mathbf{F}(r_n)}{r_n^n \cdot \sqrt{2\pi \mathbf{B}(r_n)}} \quad (1)$$

where  $r_n$  is the unique positive solution to

$$x \cdot \mathbf{G}'(x) = n,$$

and  $\mathbf{B}(x) := x^2 \mathbf{G}''(x) + x \mathbf{G}'(x)$ .

**Proof:** Theorem X of Hayman [5] shows that  $\mathbf{F}(x)$  is Hayman-admissible. Then the rest of the claim is an immediate consequence of Corollary II of [5] where the saddle-point method is applied to find the asymptotics of the coefficients of an admissible function.  $\square$

**Corollary 2** For  $\mathbf{F}(x)$ ,  $\mathbf{G}(x)$  as in the above proposition,

(a)  $f(n) \in \text{RT}_\infty$ ,

(b)  $f(n) = \exp\left(-\frac{n \log n}{d}(1 + o(1))\right)$ .

**Proof:** Item a follows immediately from Corollary IV of Hayman [5].

For item b one uses  $r_n \mathbf{G}'(r_n) = n$  to obtain:

$$\begin{aligned} \left(\frac{n}{cdg(d)}\right)^{1/d} &\leq r_n \leq \left(\frac{n}{dg(d)}\right)^{1/d} \quad \text{for } c > 1 \\ r_n &= (1 + o(1)) \left(\frac{n}{dg(d)}\right)^{1/d} \\ r_n^n &= (1 + o(1))^n \left(\frac{n}{dg(d)}\right)^{n/d} \\ \mathbf{B}(r_n) &= (1 + o(1)) d^2 g(d) \left(\frac{n}{dg(d)}\right) = (1 + o(1)) dn \\ \mathbf{G}(r_n) &= (1 + o(1)) g(d) r_n^d = (1 + o(1)) \frac{n}{d} \\ \mathbf{F}(r_n) &= \exp\left(\frac{n}{d}(1 + o(1))\right). \end{aligned}$$

Apply these results to (1).  $\square$

### 3 Some Technical Lemmas

Now we drop the assumption that  $\mathbf{G}(x)$  is a polynomial, but keep the requirement

$$\gcd(n : g(n) > 0) = 1. \quad (2)$$

This implies that  $f(n) \succ 0$ .

Choose a positive integer  $L \geq 2$  sufficiently large so

$$n > L \Rightarrow [x^n] \exp\left(g(1)x + \cdots + g(L)x^L\right) > 0. \quad (3)$$

Given  $\ell > L$  with  $g(\ell) > 0$  let

$$\begin{aligned}
\mathbf{G}_0(x) &:= \sum_{n \geq 1} g_0(n)x^n := \sum_{1 \leq n \leq \ell} g(n)x^n \\
\mathbf{F}_0(x) &:= \sum_{n \geq 0} f_0(n)x^n := \exp(\mathbf{G}_0(x)) \\
\mathbf{G}_1(x) &:= \sum_{n \geq 1} g_1(n)x^n := \sum_{n \geq \ell+1} g(n)x^n \\
\mathbf{F}_1(x) &:= \sum_{n \geq 0} f_1(n)x^n := \exp(\mathbf{G}_1(x)).
\end{aligned} \tag{4}$$

**Lemma 3** Suppose  $r \geq -1$  is such that

$$ng(n) = O(f_0(n+r)). \tag{5}$$

Then

$$nf_1(n) = O(f(n+r)).$$

**Proof:** In view of (3) and (5) we can choose  $C_r$  such that

$$ng(n) \leq C_r f_0(n+r) \quad \text{for } n+r \geq L+1. \tag{6}$$

Differentiating (4) gives

$$\begin{aligned}
nf_1(n) &= \sum_{j=\ell+1}^n jg(j) \cdot f_1(n-j) \\
&\leq C_r \sum_{j=\ell+1}^n f_0(j+r) \cdot f_1(n-j) \quad \text{by (6)} \\
&\leq C_r \sum_{j=0}^{n+r} f_0(j) \cdot f_1(n+r-j) \\
&= C_r f(n+r),
\end{aligned}$$

the last line following from  $\mathbf{F}(x) = \mathbf{F}_0(x) \cdot \mathbf{F}_1(x)$ .  $\square$

**Lemma 4** Suppose for every integer  $r \geq -1$

$$ng(n) = O(f_0(n+r)).$$

Then  $f(n-1)/f(n) \rightarrow \infty$ .

**Proof:** Since  $f_0(n) \in \text{RT}_\infty$  by Corollary 2 there is a monotone decreasing function  $\varepsilon(n)$  such that for any sufficiently large  $M$  we have  $\varepsilon(n) > f_0(n)/f_0(n-1)$  for  $n \geq M$ , and  $\varepsilon(n) \rightarrow 0$  as  $n \rightarrow \infty$ .

Thus

$$\begin{aligned}
f(n) &= \sum_{0 \leq j \leq n} f_0(j)f_1(n-j) \\
&= \sum_{0 \leq j \leq M-1} f_0(j)f_1(n-j) + \sum_{M \leq j \leq n} f_0(j)f_1(n-j) \\
&\leq o(f(n-1)) + \varepsilon(M) \sum_{M \leq j \leq n} f_0(j-1)f_1(n-j) \\
&\quad \text{by Lemma 3 and the choice of } \varepsilon \\
&\leq o(f(n-1)) + \varepsilon(M)f(n-1).
\end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{f(n)}{f(n-1)} \leq \varepsilon(M),$$

and as  $M$  can be arbitrarily large it follows that

$$\lim_{n \rightarrow \infty} \frac{f(n)}{f(n-1)} = 0.$$

□

## 4 Main Result

We are now in a position to prove the main result, making use of

$$n! = \exp(n \log n \cdot (1 + o(1))),$$

which follows from Stirling's result.

**Theorem 5** Suppose  $\mathbf{F}(x) = \exp(\mathbf{G}(x))$  with  $\mathbf{F}(x) = \sum_{n \geq 0} f(n)x^n$ ,  $\mathbf{G}(x) = \sum_{n \geq 1} g(n)x^n$ , and  $f(n), g(n) \geq 0$ . Suppose also that  $\gcd(n : g(n) > 0) = 1$  and that for some  $\theta \in (0, 1)$

$$g(n) = O(n^{\theta n}/n!).$$

Then

$$f(n) \in \text{RT}_\infty.$$

**Proof:** From Corollary 2, for any integer  $r \geq -1$  and any  $\theta \in (0, 1)$ , by choosing  $\ell > L$  such that  $1/\ell < 1 - \theta$ , we have

$$\begin{aligned}
f_0(n+r) &= \exp\left(-\frac{(n+r)\log(n+r)}{\ell}(1+o(1))\right) \\
&= \exp\left(-\frac{n \log n}{\ell}(1+o(1))\right) \\
&\succ \frac{n^{\theta n}}{(n-1)!}.
\end{aligned}$$

Thus  $ng(n) = O(f_0(n+r))$ . The Theorem then follows from Lemma 4. □

## 5 Best Possible Result

The main result is in a natural sense the best possible.

**Proposition 6** *Suppose  $t(n) \geq 0$  with  $\gcd(n : t(n) > 0) = 1$  is such that for every  $\theta \in (0, 1)$*

$$t(n) \neq O(n^{\theta n}/n!).$$

*Then there is a sequence  $g(n) \geq 0$  with  $\gcd(n : g(n) > 0) = 1$  and  $g(n) \leq t(n)$  but  $f(n) \notin \text{RT}_\infty$ , where one has  $\mathbf{F}(x) = \exp(\mathbf{G}(x))$ .*

**Proof:** For  $\theta \in (0, 1)$  let

$$S(\theta) = \{n \geq 1 : t(n) > n^{\theta n}/n!\}.$$

Then  $S(\theta)$  is an infinite set.

Let  $M$  be such that  $\gcd(n \leq M : t(n) > 0) = 1$ , and let

$$\begin{aligned} g_1(n) &:= \begin{cases} t(n) & \text{if } n \leq M \\ 0 & \text{if } n > M \end{cases} \\ \mathbf{G}_1(x) &:= \sum g_1(n)x^n \\ d_1 &:= \deg(\mathbf{G}_1(x)) \\ \mathbf{F}_1(x) &:= e^{\mathbf{G}_1(x)}. \end{aligned}$$

For  $m \geq 2$  we give a recursive procedure to define polynomials  $\mathbf{G}_m(x)$ ; then letting

$$\begin{aligned} d_m &:= \deg(\mathbf{G}_m(x)) \\ \mathbf{F}_m(x) &:= e^{\mathbf{G}_m(x)}, \end{aligned}$$

by Proposition 1

$$f_m(n) = \exp\left(-\frac{n \log n}{d_m}(1 + o(1))\right).$$

To define  $\mathbf{G}_{m+1}(x)$ , having defined  $\mathbf{G}_m(x)$ , let

$$h_m(n) := \frac{1}{n!} \cdot n^{(1-1/2d_m)n}.$$

Then

$$\frac{h_m(n)}{f_m(n-1)} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Thus we can choose an integer  $d_{m+1} > d_m$  such that

$$\begin{aligned} d_{m+1} &\in S\left(1 - \frac{1}{2d_m}\right) \\ h_m(d_{m+1}) &> f_m(d_{m+1} - 1). \end{aligned}$$

This ensures that  $h_m(d_{m+1}) \leq t(d_{m+1})$ . Let

$$\mathbf{G}_{m+1} := \mathbf{G}_m(x) + h_m(d_{m+1})x^{d_{m+1}}.$$

Then

$$\frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1}-1)} \geq \frac{h_m(d_{m+1})}{f_m(d_{m+1}-1)} > 1.$$

Now let  $\mathbf{G}(x)$  be the nonnegative power series defined by the sequence of polynomials  $\mathbf{G}_m(x)$ ; and let  $\mathbf{F}(x) = e^{\mathbf{G}(x)}$ . Then  $g(n) \leq t(n)$  but  $f(n) \notin \text{RT}_\infty$  as

$$\frac{f(d_{m+1})}{f(d_{m+1}-1)} = \frac{f_{m+1}(d_{m+1})}{f_{m+1}(d_{m+1}-1)} > 1.$$

□

## 6 Application to 0–1 laws

A class  $\mathcal{K}$  of finite relational structures is *adequate* if it is closed under disjoint union and the extraction of components. One can view the structures as being *unlabelled* with the component count function  $p_U(n)$  and the total count function  $a_U(n)$ , both counting up to isomorphism. The corresponding *ordinary* generating series are

$$\mathbf{P}_U(x) := \sum_{n \geq 1} p_U(n)x^n, \quad \mathbf{A}_U(x) := \sum_{n \geq 0} a_U(n)x^n$$

connected by the fundamental equation

$$\mathbf{A}_U(x) = \prod_{j \geq 1} (1 - x^j)^{-p_U(j)}. \quad (7)$$

One can also view the structures as being *labelled* (in all possible ways) with the count functions  $p_L(n)$  for the connected members of  $\mathcal{K}$ , and  $a_L(n)$  for all members of  $\mathcal{K}$ . The corresponding *exponential* generating series are

$$\mathbf{P}_L(x) := \sum_{n \geq 1} p_L(n)x^n/n!, \quad \mathbf{A}_L(x) := \sum_{n \geq 0} a_L(n)x^n/n!$$

connected by the fundamental equation

$$\mathbf{A}_L(x) = e^{\mathbf{P}_L(x)}. \quad (8)$$

All references to Compton in this section are to the two papers [3] and [4].

### 6.1 Unlabelled 0–1 Laws for Adequate Classes

Let  $\mathcal{K}$  be an adequate class with unlabelled count functions and ordinary generating functions as described above. Compton showed that if the radius of convergence  $\rho_U$  of  $\mathbf{A}_U(x)$  is positive then  $\mathcal{K}$  has an unlabelled 0–1 law<sup>(i)</sup> iff  $a_U(n) \in \text{RT}_1$ , that is,

$$\frac{a_U(n-1)}{a_U(n)} \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

$\mathcal{K}$  is finitely generated if  $r = \sum p_U(n) < \infty$ , that is, there are only finitely many connected structures in  $\mathcal{K}$ . In the finitely generated case the asymptotics for the coefficients  $a_U(n)$  have long been known to have the simple polynomial form<sup>(ii)</sup>

$$a_U(n) \sim Cn^{r-1} \tag{9}$$

provided  $\gcd(n : p_U(n) > 0) = 1$ . Item (9) leads to the fact that  $a_U(n) \in \text{RT}_1$ , and hence to an unlabelled 0–1 law. In addition to using this result, Compton notes that the work of Bateman and Erdős [1] shows that if  $p_U(n) \in \{0, 1\}$ , for all  $n$ , then one has  $a_U(n) \in \text{RT}_1$ .

Both of these results were subsumed in the powerful result of Bell [2] which says that if  $p_U(n)$  is polynomially bounded, that is, there is a  $c$  such that  $p_U(n) = O(n^c)$ , then  $a_U(n) \in \text{RT}_1$ .

### 6.2 Labelled 0–1 Laws

Compton shows that if  $\rho_L$ , the radius of convergence of  $\mathbf{A}_L(x)$ , is positive, then  $\mathcal{K}$  has a labelled 0–1 law iff

$$\frac{a_L(n-k)/(n-k)!}{a_L(n)/n!} \rightarrow \infty \quad \text{whenever } p_L(k) > 0. \tag{10}$$

In particular it suffices to show that  $a_L(n)/n! \in \text{RT}_\infty$ .

Compton’s method to show that a given adequate class of finite relational structures  $\mathcal{K}$  has a labelled 0–1 law is to show that its exponential generating function  $\mathbf{A}_L(x) = \sum a_L(n)x^n/n!$  is Hayman-admissible with an infinite radius of convergence. This guarantees that  $a_L(n)/n! \in \text{RT}_\infty$  ([5], Corollary IV). However, as Compton notes, showing that  $\mathbf{A}_L(x)$  is Hayman-admissible can be quite a challenge.

Question 8.3 of [3] first asks if, in the *unlabelled* case, the result of Bateman and Erdős, namely  $p_U(n) \in \{0, 1\}$  implies  $a_U(n) \in \text{RT}_1$ , can be extended to the much more general statement that  $p_U(n) = O(n^k)$  implies  $a_U(n) \in \text{RT}_1$ , yielding an unlabelled 0–1 law. As mentioned earlier, this was proved to be true by Bell. The second part of Question 8.3 asks if there is a simple sufficient condition along similar lines for the labelled case. We can now answer this in the affirmative with a result that is an excellent parallel to Bell’s result for unlabelled structures.

**Theorem 7** *If  $\mathcal{K}$  is an adequate class of structures with*

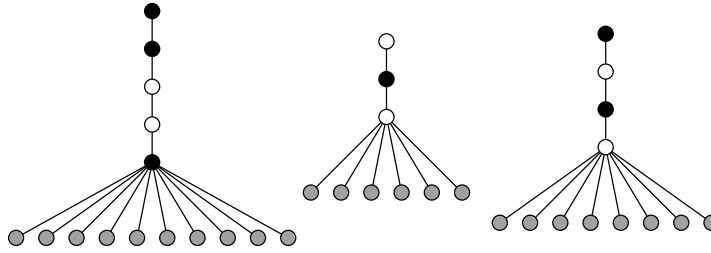
$$p_L(n) = O\left(n^{\theta n}\right) \quad \text{for some } \theta \in (0, 1)$$

*then  $a_L(n)/n! \in \text{RT}_\infty$ , and consequently  $\mathcal{K}$  has a labelled monadic second-order 0–1 law.*

<sup>(i)</sup> Given a logic  $\mathcal{L}$ ,  $\mathcal{K}$  has an unlabelled  $\mathcal{L}$  0–1 law means that for any  $\mathcal{L}$  sentence  $\varphi$ , the probability that  $\varphi$  holds in  $\mathcal{K}$  will be either 0 or 1. In [3] Compton worked with first-order logic, in [4] with monadic second-order logic. In both papers he simply used the phrases “unlabeled 0–1 law” and “labeled 0–1 law”.

<sup>(ii)</sup> This result is usually known as Schur’s Theorem [6, 3.15.2]. One can easily find the asymptotics (9) using a partial fraction decomposition of the right side of (7). The labelled case with finitely many components is more difficult—we needed to invoke Hayman’s treatise [5] just to obtain the asymptotics for  $\log a_L(n)/n!$  (see Corollary 2).





**Fig. 1:** Brooms with two-colored handles

**Proof:** This is an immediate consequence of Theorem 5 and Compton's proof that  $a_L(n)/n! \in \text{RT}_\infty$  guarantees such a 0–1 law.  $\square$

Now we list the examples of classes  $\mathcal{K}$  which Compton shows have a labelled 0–1 law, giving  $p_L(n)$  in each case. It is trivial to check in each case that  $p_L(n) = O(n^{n/2})$ ; thus the 0–1 law in each case follows from our Theorem 7.

- (a) 7.1 Unary Predicates  $p_L(n) = 0$  for  $n > 1$ .
- (b) 7.12 Forests of Rooted Trees of Height 1  $p_L(n) = n$ .
- (c) 7.15 Only Finitely Many Components  $p_L(n)$  is eventually 0.
- (d) 7.16 Equivalence Relations  $p_L(n) = 1$ .
- (e) 7.17 Partitions with a Selection Subset  $p_L(n) = 2^n - 1$ .

We can now augment this list by, in each case, coloring the members of  $\mathcal{K}$  by a fixed set of  $r$  colors in all possible ways. This will increase the original  $p_L(n)$  by a factor of at most  $r^n$ . This will still give  $p_L(n) = O(n^{n/2})$ . Furthermore, in each of these colored cases let  $\mathcal{P}$  be any subset of the connected members, and let  $\mathcal{K}$  be the closure of  $\mathcal{P}$  under disjoint union. Each such  $\mathcal{K}$  has a labelled 0–1 law.

Another application of Theorem 7 is to answer Question 4 of [3] by exhibiting an adequate class  $\mathcal{K}$  such that  $p_L(n) = O(n^{3n/4})$ , hence there is a labelled 0–1 law for  $\mathcal{K}$ ; but also such that  $\rho_U \in (0, 1)$ , so  $\mathcal{K}$  does not have an unlabelled 0–1 law.

Let the components of  $\mathcal{K}$  be the one-element tree  $T_1$  along with rooted trees  $T_{3n}$  of size  $3n$  and height  $n$  consisting of a chain  $C_n$  of  $n$  nodes, with an antichain  $L_{2n}$  of  $2n$  nodes (the leaves of the tree) below the least member of the chain; and the chain  $C_n$  is two-colored while the remaining nodes are uncolored. One can visualize these as brooms with 2-colored handles, see Figure 1.

The number of unlabelled components is given by  $p_U(1) = 1$ ,  $p_U(3n) = 2^n$ . Thus the radius of convergence of the ordinary generating function of  $\mathcal{K}$  is  $\rho_U = \sqrt[3]{2}$ . Since this is positive and not 1 it follows from Theorem 5.9(ii) of [3] that  $\mathcal{K}$  does not have an unlabelled 0–1 law.

For the number  $p_L(3n)$  of labelled components of size  $3n$ :

$$\begin{aligned}
 p_L(3n) &\leq 2^n \binom{3n}{n} n! \\
 &\leq 2^n (3n)^n \exp(n \log n \cdot (1 + o(1))) \\
 &= \exp(2n \log n \cdot (1 + o(1))) \\
 &= (3n)^{(2/3)(3n)(1+o(1))} \\
 &= O\left((3n)^{(3/4)(3n)}\right).
 \end{aligned}$$

Thus  $p_L(n) = O(n^{3n/4})$ , so  $a_L(n)/n! \in \text{RT}_\infty$  by Theorem 7, showing that  $\mathcal{K}$  has a labelled 0–1 law.

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