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► **To cite this version:**

Odile Favaron, H. Karami, S. M. Sheikholeslami. Total domination in K - and K -covered graphs. Discrete Mathematics and Theoretical Computer Science, DMTCS, 2008, 10 (1), pp.35–42. <hal-00972309>

HAL Id: hal-00972309

<https://hal.inria.fr/hal-00972309>

Submitted on 3 Apr 2014

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Total domination in K_5 - and K_6 -covered graphs

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received March 31, 2006, revised Dec 20, 2007, accepted Jan 18, 2008.

A graph G is K_r -covered if each vertex of G is contained in a K_r -clique. Let $\gamma_t(G)$ denote the total domination number of G . It has been conjectured that every K_r -covered graph of order n with no K_r -component satisfies $\gamma_t(G) \leq \frac{2n}{r+1}$. We prove that this conjecture is true for $r = 5$ and 6 .

Keywords: total domination, clique cover

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [6] for terminology and notation which are not defined here. The *open neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set of all vertices adjacent to v . Its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. If S is a set of vertices of G , then $N(S) = \cup_{u \in S} N(u)$ and $N[S] = N(S) \cup S$. For the sake of simplicity we write $N(u, v)$ instead of $N(\{u, v\})$. For each nonempty set S of vertices, the subgraph induced by S is denoted by $G[S]$. For an integer $p \geq 3$, a *multitriangle* of order p is the graph consisting of $p - 2$ triangles sharing one edge. A multitriangle of order 3 is thus a triangle.

A set $D \subseteq V$ is a *total dominating set* if every vertex in V is adjacent to a vertex of D . Obviously every graph without isolated vertices has a total dominating set. The *total domination number*, $\gamma_t(G)$, is the minimum cardinality of a total dominating set. If G has q components G_i , then $\gamma_t(G) = \sum_{i=1}^q \gamma_t(G_i)$. As for the domination number, the determination of the total domination number of a graph is NP-hard and it is interesting to determine good upper bounds on $\gamma_t(G)$. Conditions on the density of the graph allow us to lower such bounds. Here we consider the condition that every vertex is contained in a sufficiently large clique.

A K_r -*component* of G is a component isomorphic to a clique K_r . A graph G is K_r -*covered*, $r \geq 2$, if every vertex of G is contained in a clique K_r , and *minimally K_r -covered* if it is K_r -covered but $G - e$ is not K_r -covered for any edge of G . These properties were already considered by Henning and Swart in [5] under the terms “with no K_r -isolated vertex” or “Property $C(1, r)$ ”, and “Property $C(2, r)$ ”, respectively.

In a K_r -covered graph, a *good vertex* is a vertex of degree $r - 1$ and a *good clique* is a clique K_r containing a good vertex. If z is a good vertex, we denote by \mathcal{C}_z the good clique containing it. The following results, independently proved in [3] and in [4], will be constantly used throughout the paper.

Theorem A [3, 4] *Every edge of a minimally K_r -covered graph is contained in a good clique.*

In 2004, Cockayne, Favaron and Mynhardt [1] conjectured that every K_r -covered graph G of order n with no K_r -component satisfies $\gamma_t(G) \leq \frac{2n}{r+1}$. They proved this conjecture for $r = 3, 4$. In this paper we prove it for $r = 5$ and 6 .

2 Proof of the conjecture for $r = 5$ and $r = 6$

The proof uses a particular family \mathcal{F}_r of minimally K_r -covered graphs which was already considered in [1, 4]. Recall that the corona of a graph H is obtained from H by adding a pendant edge at each vertex of H and that the middle graph of H is the line graph of the corona of H (see for instance [2]).

Definition 1 \mathcal{F}_r is the family of middle graphs of $(r - 1)$ -regular graphs.

From this definition \mathcal{F}_r is the collection of graphs consisting of edge-disjoint cliques of order r , where each such clique contains exactly one vertex of degree $r - 1$ and the remaining $r - 1$ vertices have degree $2(r - 1)$. Let \mathcal{S} be the set of these edge-disjoint cliques. Then each vertex of G of degree $r - 1$ belongs to exactly one K_r in \mathcal{S} and each vertex of degree $2(r - 1)$ belongs to exactly two K_r 's in \mathcal{S} .

The following result is proved in [1].

Theorem B (See [1]) *For $r \geq 3$, every graph of order n of \mathcal{F}_r satisfies $\gamma_t(G) < \frac{2n}{r+1}$.*

We can now prove the conjecture for $r = 5$ and $r = 6$.

Theorem 1 *For $r=5$ or 6 , every K_r -covered graph G of order n with no K_r -component satisfies $\gamma_t(G) \leq \frac{2n}{r+1}$.*

Proof: The proof is by induction on n and the first four claims are established for any value of $r \geq 5$. The statement is obviously true for the smallest possible order $r+1$ since then $\gamma_t(G) = 2$. Suppose the theorem to be true for graphs of order less than n and let G be a K_r -covered graph with no K_r -component of order $n \geq r + 2$. Let F be a minimally K_r -covered spanning subgraph of G . Since $\gamma_t(G) \leq \gamma_t(F)$, it is sufficient to prove $\gamma_t(F) \leq \frac{2n}{r+1}$.

If F is not connected but has no K_r -component, then applying the induction hypothesis to each component of F gives the result.

The case where F is not connected and has some K_r -components has already been considered in [1]. The second part of the proof of Theorem 6 in [1] establishes by induction on n that a graph having a certain property $\mathcal{B}(r)$ satisfies $\gamma_t(G) \leq 2n/(r + 1)$. In the case where a minimal K_r -covered spanning subgraph F of G has K_r components, the result is proved without using the induction hypothesis $\mathcal{B}(r)$. Hence the same argument holds here. As the proof is rather long, we do not repeat it and refer the reader to [1].

So we suppose now that we are working in a connected minimal K_r -covered graph F of order $n \geq r + 2$. Since F is connected, every vertex of F belongs to an edge and thus has a good neighbor by Theorem A. For every pair of adjacent vertices u and v , let

$$P(u, v) = \{x \in N(u, v) \setminus \{u, v\} \mid N(x) \subseteq N(u, v)\}.$$

Claim 1 *If $|P(u, v)| \geq r - 1$ for some pair of adjacent non-good vertices u and v , then $\gamma_t(F) \leq \frac{2n}{r+1}$.*

Proof: Let u' be any good neighbor of u . By Theorem A, the edge uu' is contained in a good clique \mathcal{C} . The $r - 2$ neighbors of u' different from u are vertices of \mathcal{C} and thus are adjacent to u . So every good neighbor of u , and similarly every good neighbor of v , belongs to $P(u, v)$. Note also that if $z \in N(u, v) \setminus (P(u, v) \cup \{u, v\})$, then z has a neighbor $z_1 \notin N(u, v)$, and so z belongs to a good clique of the graph $F' = F[V \setminus (P(u, v) \cup \{u, v\})]$. Hence F' is K_r -covered.

Let $\mathcal{C}_1, \dots, \mathcal{C}_s$ be the K_r -components of F' if any. Obviously $(N(w) \setminus V(\mathcal{C}_i)) \subseteq P(u, v) \cup \{u, v\}$ for each $w \in V(\mathcal{C}_i)$. Since F is connected, each clique \mathcal{C}_i contains at least one vertex w_i such that $(N(w_i) \setminus V(\mathcal{C}_i)) \cap (P(u, v) \cup \{u, v\}) \neq \emptyset$. From the definition of $P(u, v)$ we have $w_i \in N(u, v)$. Let $X = P(u, v) \cup \{u, v\} \cup (\cup_{i=1}^s V(\mathcal{C}_i))$. The graph $F[V \setminus X]$ is still K_r -covered and has no K_r -component. By the induction hypothesis, $\gamma_t(F[V \setminus X]) \leq \frac{2|V \setminus X|}{r+1}$. Moreover $\{u, v, w_1, w_2, \dots, w_s\}$, or $\{u, v\}$ if $s = 0$, is a total dominating set of order $s + 2$ of $F[X]$, and $|X| = |P(u, v)| + sr + 2$ with $s \geq 0$. Hence if $|P(u, v)| \geq r - 1$, then $\gamma_t(F[X]) \leq s + 2 \leq \frac{2|X|}{r+1}$ and we are done. \square

We suppose henceforth $|P(u, v)| \leq r - 2$ for every pair of adjacent non-good vertices of F . Recall that all the good neighbors of u or v belong to $P(u, v)$. If G consists of $p \geq 2$ cliques K_r sharing exactly one vertex, then $n = p(r - 1) + 1$ and $\gamma_t(G) = 2 \leq 2n/(r + 1)$. We also suppose in what follows that G has not this structure, which means that every non-good vertex has at least one non-good neighbor.

Claim 2 *Each good clique contains at most $r - 4$ good vertices.*

Proof: Suppose to the contrary that \mathcal{C} is a good clique ($\neq K_r$) with more than $r - 4$ good vertices. Let z_1, z_2, \dots, z_s with $r - 3 \leq s \leq r - 1$ be the good vertices and u a non-good vertex of \mathcal{C} . If u has a non-good neighbor v not in \mathcal{C} , let \mathcal{C}' be a good clique containing uv and z'_1, \dots, z'_t ($1 \leq t \leq r - 2$) the good vertices of \mathcal{C}' . The vertex v belongs to a second good clique $\mathcal{C}'' \neq \mathcal{C}$. Let z'' be a good vertex of \mathcal{C}'' . Then $\{z_1, \dots, z_s, z'_1, \dots, z'_t, z''\}$ is a subset of $P(u, v)$ of order at least $s + 2 \geq r - 1$, a contradiction to $|P(u, v)| \leq r - 2$. Therefore all the non-good neighbors of u belong to \mathcal{C} .

Let u, u_1, \dots, u_{r-s-1} be the non-good vertices of \mathcal{C} with $r - s \geq 2$. Let \mathcal{C}' (\mathcal{C}'_1 respectively, possibly equal to \mathcal{C}') be a second good clique containing u (u_1) and let z' (z'_1) be a good vertex of \mathcal{C}' (\mathcal{C}'_1). Then $\{z_1, \dots, z_s, z', z'_1\} \subseteq P(u, u_1)$. Since $|P(u, u_1)| \leq r - 2$, $s = r - 3$, $z' = z'_1$ and z' is the unique good vertex of the clique $\mathcal{C}' = \mathcal{C}'_1$. Among the $r - 1$ non-good vertices of \mathcal{C}' , at most three are those of \mathcal{C} and thus at least one, say u_2 , is not in \mathcal{C} . Let \mathcal{C}'' be a second good clique containing u_2 and z'' a good vertex of \mathcal{C}'' . Then $\{z_1, \dots, z_{r-3}, z', z''\} \subseteq P(u, u_2)$, a contradiction which completes the proof. \square

Claim 3 *No vertex can belong to $r - 2$ good cliques K_r .*

Proof: Assume, to the contrary, that a vertex u belongs to $r - 2$ good cliques $\mathcal{C}_1, \dots, \mathcal{C}_{r-2}$, and let x_i be a good vertex of \mathcal{C}_i for $1 \leq i \leq r - 2$. Let w and t be two non-good vertices of $\mathcal{C}_1 \setminus \{u, x_1\}$. Since $\{x_1, \dots, x_{r-2}\} \subseteq P(u, t)$ and $|P(u, t)| \leq r - 2$, we have $w \notin P(u, t)$ and w has a good neighbor w' not in $N(u, t)$ and thus distinct from x_1, \dots, x_{r-2} . But then $\{x_1, \dots, x_{r-2}, w'\} \subseteq P(u, w)$, which is a contradiction. \square

Claim 4 *$r - 3$ good cliques K_r cannot share more than one vertex.*

Proof: Assume, to the contrary, that $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{r-3}$ are $r-3$ good K_r 's all containing the (non-good) vertices u and v , and let x_i be a good vertex of \mathcal{C}_i for $1 \leq i \leq r-3$. From $|\cup_{i=1}^{r-3} V(\mathcal{C}_i)| \geq (r-1) + (r-3) = 2r-4$, we get $|\cup_{i=1}^{r-3} V(\mathcal{C}_i) \setminus \{x_1, \dots, x_{r-3}, u, v\}| \geq r-3 \geq 2$ while $|P(u, v) \setminus \{x_1, \dots, x_{r-3}\}| \leq 1$. Therefore at least one vertex z of $\cup_{i=1}^{r-3} V(\mathcal{C}_i) \setminus \{x_1, \dots, x_{r-3}, u, v\}$ is not in $P(u, v)$. Let z' be a good neighbor of z not in $N(u, v)$. Since $\{x_1, \dots, x_{r-3}, z'\} \subseteq P(u, z)$ and $|P(u, z)| \leq r-2$, v is not in $P(u, z)$ and thus belongs to a $(r-2)$ th good clique. This contradicts Claim 3. \square

End of the proof of Theorem 1 for $r=5$

By Claims 2, 3 and 4, each good clique contains exactly one good vertex and four non-good ones, each non-good vertex is contained in exactly two good K_5 's, and two good K_5 's intersect in at most one vertex. Therefore the graph F belongs to the family \mathcal{F}_5 described above and thus $\gamma_t(F) < \frac{2n}{6}$. This completes the proof for $r=5$.

End of the proof of Theorem 1 for $r=6$

Henceforth, each good clique is a K_6 containing at most two good vertices.

Claim 5 Let \mathcal{C}_1 and \mathcal{C}_2 be two good K_6 's such that $|V(\mathcal{C}_1) \cap V(\mathcal{C}_2)| \geq 2$. Then

1. $|V(\mathcal{C}) \cap V(\mathcal{C}_i)| \leq 1$ for each other good clique \mathcal{C} and $i = 1, 2$;
2. the clique \mathcal{C}_i contains exactly one good vertex for $i = 1, 2$;
3. each other good clique \mathcal{C} intersecting \mathcal{C}_1 or \mathcal{C}_2 , contains exactly one good vertex.

Proof:

- (1) Let $u, v \in V(\mathcal{C}_1) \cap V(\mathcal{C}_2)$ and let \mathcal{C} be another good clique in G . Assume, to the contrary, $|V(\mathcal{C}_1) \cap V(\mathcal{C})| \geq 2$. Let x, x_1, x_2 be good vertices of $\mathcal{C}, \mathcal{C}_1$ and \mathcal{C}_2 , respectively. Suppose $u \in \mathcal{C}$. By Claim 4, $v \notin \mathcal{C}$. Let $w \in V(\mathcal{C}) \cap V(\mathcal{C}_1)$ and $w \neq u$. Since $\{x, x_1, x_2\} \subseteq P(u, v)$ and $|P(u, v)| \leq 4$, at least one vertex t of $V(\mathcal{C}_1) \setminus \{u, v, x_1, w\}$ is not in $P(u, v)$. Let t' be a good neighbor of t not in $N(u, v)$. Now we have $\{x, x_1, x_2, t'\} \subseteq P(u, t)$. Since $|P(u, t)| \leq 4$, w is not in $P(u, t)$ and has a good neighbor w' not in $N(u, t)$. Then $\{x, x_1, x_2, w'\} \subseteq P(u, w)$. Thus v is not in $P(u, w)$ and has a good neighbor v' not in $N(u, w)$. This implies $\{x, x_1, x_2, w', v'\} \subseteq P(v, w)$ which is a contradiction. Thus $u, v \notin V(\mathcal{C})$. Let $w_1, w_2 \in V(\mathcal{C}) \cap V(\mathcal{C}_1)$. Since $\{x, x_1, x_2\} \subseteq P(u, w_1)$ and $|P(u, w_1)| \leq 4$, $v \notin P(u, w_1)$ or $w_2 \notin P(u, w_1)$.

First let $v \notin P(u, w_1)$. Let v' be a good neighbor of v not in $N(u, w_1)$. Now we have $\{x, x_1, x_2, v'\} \subseteq P(v, w_1)$. Since $|P(v, w_1)| \leq 4$, w_2 is not in $P(v, w_1)$ and has a good neighbor w'_2 not in $N(v, w_1)$. Now we have $\{x, x_1, x_2, v', w'_2\} \subseteq P(v, w_2)$ which is a contradiction.

Now let $w_2 \notin P(u, w_1)$. Let w'_2 be a good neighbor of w_2 not in $N(u, w_1)$. Now we have $\{x, x_1, x_2, w'_2\} \subseteq P(u, w_2)$. Since $|P(u, w_2)| \leq 4$, v is not in $P(u, w_2)$ and has a good neighbor v' not in $N(u, w_2)$. This implies that $|P(v, w_2)| \geq 5$ which is a contradiction.

- (2) Suppose \mathcal{C}_1 contains two good vertices x_1 and x'_1 . Since $|P(u, v)| \leq 4$, $V(\mathcal{C}_1 \cup \mathcal{C}_2) \setminus \{u, v\}$ has a non-good vertex, say w , not in $P(u, v)$. Let w' be a good neighbor of w not in $N(u, v)$. Then $\{x_1, x'_1, x_2, w'\} \subseteq P(u, w)$. Hence v is not in $P(u, w)$ and has a good neighbor $v' \notin N(u, w)$, which implies $|P(v, w)| \geq 5$, a contradiction.

- (3) Suppose \mathcal{C} contains two good vertices y and y' . If \mathcal{C} intersects $\mathcal{C}_1 \cup \mathcal{C}_2$ in u , then $\{x_1, x_2, y, y'\} \subseteq P(u, v)$ and there exists a vertex t of $V(\mathcal{C}_1 \cup \mathcal{C}_2) \setminus \{u, v\}$ with a good neighbor t' not in $N(u, v)$. Then $\{x_1, x_2, y, y', t'\} \subseteq P(u, t)$, a contradiction. If \mathcal{C} intersects $\mathcal{C}_1 \cup \mathcal{C}_2$ in w different from u and v , then, since $\{x_1, x_2, y, y'\} \subseteq P(u, w)$, v has a good neighbor v' not belonging to $N(u, w)$. Hence $\{x_1, x_2, y, y', v'\} \subseteq P(v, w)$, a contradiction.

□

Claim 6 Let three good cliques $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 share one vertex u . Then

1. $|V(\mathcal{C}) \cap V(\mathcal{C}_i)| \leq 1$ for each other good clique \mathcal{C} and $i = 1, 2, 3$;
2. for $i = 1, 2, 3$, each non-good vertex of $\mathcal{C}_i \setminus \{u\}$ belongs to exactly two good cliques;
3. for $i = 1, 2, 3$, each clique \mathcal{C}_i and each good clique \mathcal{C} intersecting one of the \mathcal{C}_i 's contains exactly one good vertex.

Proof:

- (1) Suppose that \mathcal{C} is a good clique such that $|V(\mathcal{C}) \cap V(\mathcal{C}_1)| \geq 2$. By claim 3, $u \notin V(\mathcal{C})$. Let $v, w \in V(\mathcal{C}) \cap V(\mathcal{C}_1)$. Let x, x_1, x_2, x_3 be good vertices of $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 , respectively. We have $\{x, x_1, x_2, x_3\} \subseteq P(u, v)$ and so w is not in $P(u, v)$ and has a good neighbor w' not in $N(u, v)$. Then we have $|P(u, w)| \geq 5$ which is a contradiction.
- (2) If a vertex $v \neq u$ of some \mathcal{C}_i belongs to two other good cliques, let v' and v'' two good neighbors of v respectively belonging to these two cliques. Then $\{x_1, x_2, x_3, v', v''\} \subseteq P(u, v)$, a contradiction.
- (3) If, say, \mathcal{C}_1 has a second good vertex x'_1 , then \mathcal{C}_2 and \mathcal{C}_3 have one good vertex each, for otherwise $|P(u, v)| \geq 5$ for any neighbor v of u . Hence there exists at least one non-good vertex v belonging to exactly one of the \mathcal{C}_i 's. This vertex v has a good neighbor $v' \notin \{x_1, x'_1, x_2, x_3\}$ and $|P(u, v)| \geq 5$, a contradiction. If a good clique \mathcal{C} intersecting one of the \mathcal{C}_i 's in one vertex v (necessarily different from u) contains two good neighbors x and x' , then $\{x_1, x_2, x_3, x, x'\} \subseteq P(u, v)$, a contradiction.

□

Claim 7 Let \mathcal{C} be a good clique containing two good vertices z_1, z_2 . Then

1. each good clique intersects \mathcal{C} in at most one vertex;
2. each non-good vertex of \mathcal{C} belongs to exactly two good cliques;
3. if \mathcal{C}' is a good clique intersecting \mathcal{C} in u , then \mathcal{C}' contains exactly one good vertex, each non-good vertex of \mathcal{C}' belongs to exactly two good cliques, $|V(\mathcal{C}') \cap V(\mathcal{C}_1)| \leq 1$ for each good clique \mathcal{C}_1 and if $|V(\mathcal{C}') \cap V(\mathcal{C}_1)| = 1$, then \mathcal{C}_1 contains exactly one good vertex.

Proof: (1) and (2) are consequences of Claim 5 (2) and 6 (3).

- (3) Let u' be a good vertex of \mathcal{C}' , w be a non-good vertex in $V(\mathcal{C}') \setminus \{u\}$ and w' a good neighbor of w not in $N(u)$. If w has another good neighbor w'' , which can be either a second good vertex of \mathcal{C}' or of a second clique \mathcal{C}_1 containing w , or a good vertex of a third good clique containing w , then $\{z_1, z_2, u', w', w''\} \subseteq P(u, w)$, a contradiction. If a good clique \mathcal{C}_1 intersects \mathcal{C}' in v and w (both different from u by (2)), then $v \notin P(u, w)$ since $\{z_1, z_2, u', w'\} \subseteq P(u, w)$. Therefore v has another good neighbor $v' \notin N(u, w)$ and $\{z_1, z_2, u', w', v'\} \subseteq P(u, v)$, a contradiction. \square

Let $V_i = \{u \in V(F) \mid u \text{ belongs to exactly } i \text{ good cliques}\}$, $i = 1, 2, 3$. By Claim 2, V_1, V_2, V_3 partition $V(F)$. Obviously V_1 consists of all good vertices of F . Let t be the number of good cliques that contain two good vertices. Counting the number of edges of F with one endpoint in V_1 and another in $V_2 \cup V_3$, implies by Claim 7 that

$$5|V_1| - 2t = 2|V_2| + 4t + 3|V_3|.$$

On the other hand, we have

$$2n = 2(|V_1| + |V_2| + |V_3|).$$

It follows from the last two equations

$$|V_1| = \frac{2n}{7} + \frac{|V_3| + 6t}{7}. \quad (1)$$

The following claim gives the structure of the subgraph induced by V_3 in F .

Claim 8 $F[V_3]$ is a disjoint union of s cliques with $s \geq |V_3|/5$.

Proof: Let u and v be two adjacent vertices in V_3 . If the edge uv belongs to only one good clique \mathcal{C}_z , let $\mathcal{C}_{u'}$ and $\mathcal{C}_{u''}$ (respectively $\mathcal{C}_{v'}$ and $\mathcal{C}_{v''}$) be the other two good cliques containing u (respectively v). Then $\{u', u'', v', v'', z\}$ is a set of five vertices contained in $P(u, v)$, a contradiction. Therefore every edge joining two vertices in V_3 is contained in exactly (by Claim 5) two good cliques. Let now uvw be a path of $F[V_3]$. Among the three good cliques containing v , two contain uv and two contain vw . Hence one of them, say \mathcal{C}_z , contains $\{u, v, w\}$ and u and w are adjacent. Moreover, the second good cliques respectively containing uv and vw are the same by Claim 5. This implies that $\{u, v, w\}$ is contained in exactly two good cliques \mathcal{C}_z and $\mathcal{C}_{z'}$. The preceding arguments show that $F[V_3]$ is a disjoint union of s cliques Q_i . Each Q_i is a part of the intersection of two good cliques \mathcal{C}_z and $\mathcal{C}_{z'}$, thus implying $|Q_i| \leq 5$, and each vertex u of Q_i belongs to a third clique intersecting \mathcal{C}_z and $\mathcal{C}_{z'}$ exactly in u . Finally, since $|Q_i| \leq 5$, $s \geq |V_3|/5$. \square

We define now the graph F^* with vertex set $\{z \in V(F) \mid z \text{ is a good vertex in } F\}$ and two vertices of F^* are adjacent if and only if they belong to the same clique or their corresponding good cliques have a common vertex. Since F is connected and each edge of F belongs to a good clique, the graph F^* is connected.

Three good vertices z_1, z_2, z_3 form a triangle in F^* if and only if

1. the good cliques $\mathcal{C}_{z_1}, \mathcal{C}_{z_2}$ and \mathcal{C}_{z_3} are different and share one vertex,
2. or, say, $\mathcal{C}_{z_1} = \mathcal{C}_{z_2}$ and $\mathcal{C}_{z_1} \cap \mathcal{C}_{z_3} \neq \emptyset$,

3. or the three cliques are pairwise intersecting but $\mathcal{C}_{z_1} \cap \mathcal{C}_{z_2} \cap \mathcal{C}_{z_3} = \emptyset$.

We only consider the triangles of the first two types and call them respectively 1-triangles and 2-triangles.

A 1-triangle of F^* comes from a vertex of V_3 . From Claim 6 (2), if two 1-triangles $z_1 z_2 z_3$ and $z'_1 z'_2 z'_3$ are not disjoint, then they share one edge, say, $z_1 = z'_1$ and $z_2 = z'_2$. From Claim 8, $|V(\mathcal{C}_{z_1}) \cap V(\mathcal{C}_{z_2})| \geq 2$ and each good clique \mathcal{C}_{z_3} and $\mathcal{C}_{z'_3}$ shares one vertex with \mathcal{C}_{z_1} and \mathcal{C}_{z_2} . Since $|V(\mathcal{C}_{z_1}) \cap V(\mathcal{C}_{z_2})| \leq 5$, at most five 1-triangles share a common edge. Hence the 1-triangles of F^* form multitriangles MT_i of respective order p_i with $3 \leq p_i \leq 7$. We call them multitriangles of type 1 and we associate to each of them the clique $Q_i \subseteq V(\mathcal{C}_{z_1}) \cap V(\mathcal{C}_{z_2})$ of order $p_i - 2 \leq 5$ as described in Claim 8. Therefore there are $s \geq |V_3|/5$ multitriangles of type 1 and all of them are disjoint.

A 2-triangle of F^* comes from a good clique \mathcal{C} of F with two good vertices z_1 and z_2 . By Claim 7, the four other vertices of \mathcal{C} belong to exactly one other good clique and these four good cliques are different. Hence the edge $z_1 z_2$ belongs to exactly four 2-triangles forming a multitriangle of order $p_i = 6$, called multitriangle of type 2. To each multitriangle MT_i of type 2 we associate the clique Q_i of order $p_i - 2 = 4$ of F formed by the non-good vertices of \mathcal{C} . There are t multitriangles of type 2, the number of good cliques with two good vertices. By Claim 7, they are pairwise disjoint and disjoint from the multitriangles of type 1.

Let F^{**} be a spanning subgraph of F^* containing all the edges of the multitriangles but no other cycle (the edges of F^{**} not in multitriangles form a spanning tree of the graph of order $|V_1| - \sum_{i=1}^{s+t} (p_i - 1)$ obtained from F^* by contracting each multitriangle into one vertex). We form a subset D of vertices of F as follows. For each multitriangle MT_i of order p_i , $0 \leq i \leq s + t$, put in D the $p_i - 2$ vertices of its associated clique Q_i . For each edge $z_i z_j$ of F^{**} not in a multitriangle, put in D one vertex of $\mathcal{C}_{z_i} \cap \mathcal{C}_{z_j}$. The induced subgraph $F[D]$ is connected since F^{**} is connected and can be seen as the graph representative of the 1- and 2-triangles and of the cutting edges of F^{**} . The set D contains a vertex in each good clique and thus dominates F . Hence $\gamma_t(F) \leq |D|$. Since F^{**} contains $|V_1| - \sum_{i=1}^{s+t} (p_i - 1) - 1$ cutting edges,

$$|D| = \sum_{i=1}^{s+t} (p_i - 2) + |V_1| - \sum_{i=1}^{s+t} (p_i - 1) - 1 = |V_1| - s - t - 1$$

with $s \geq |V_3|/5$. By (1) we get

$$|D| \leq \frac{2n}{7} + \frac{|V_3|}{7} + \frac{6t}{7} - \frac{|V_3|}{5} - t - 1 < \frac{2n}{7}.$$

This completes the proof of Theorem 1 for $r = 6$. \square

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