Spanning forests on the Sierpinski gasket

Shu-Chiuan Chang, Lung-Chi Chen

Discrete Mathematics and Theoretical Computer Science DMTCS vol. **10:2**, 2008, 55–76

Spanning forests on the Sierpinski gasket

Shu-Chiuan Chang[1] and Lung-Chi Chen[2]

[1] *Department of Physics, National Cheng Kung University, Tainan 70101, Taiwan*
[2] *Department of Mathematics, Fu Jen Catholic University, Taipei 24205, Taiwan*

received March 5, 2007, *revised March 19, 2008*, *accepted May 19, 2008*.

We study the number of spanning forests on the Sierpinski gasket $SG_d(n)$ at stage n with dimension d equal to two, three and four, and determine the asymptotic behaviors. The corresponding results on the generalized Sierpinski gasket $SG_{d,b}(n)$ with $d = 2$ and $b = 3, 4$ are obtained. We also derive upper bounds for the asymptotic growth constants for both SG_d and $SG_{2,b}$.

Keywords: Spanning forests, Sierpinski gasket, recursion relations, asymptotic growth constant

1 Introduction

The enumeration of the number of spanning forests $N_{SF}(G)$ on a graph G is a problem of interest in mathematics [Benjamini et al.(2001), Teranishi(2005)] and physics [Caracciolo et al.(2004)]. It is well known that the number of spanning forests is given by the Tutte polynomial $T(G, x, y)$ evaluated at $x = 2$, $y = 1$ [Welsh(1993)]. Alternatively, it corresponds to a special $q \to 0$ limit of the partition function of the q-state Potts model in statistical mechanics [Sokal(2005)]. Some recent studies on the enumeration of spanning forests and the calculation of their asymptotic growth constants on regular lattices were carried out in Refs. [Shrock(2000), Chang and Shrock(2000), Chang and Shrock(2001)(a), Chang and Shrock(2001)(b), Chang and Shrock(2001)(c), Jacobsen et al.(2005), Deng et al.(2007)]. It is of interest to consider spanning forests on self-similar fractal lattices which have scaling invariance rather than translational invariance. Fractals are geometric structures of (generally noninteger) Hausdorff dimension realized by repeated construction of an elementary shape on progressively smaller length scales [Mandelbrot(1982), Falconer(2003)]. A well-known example of a fractal is the Sierpinski gasket. We shall derive the recursion relations for the numbers of spanning forests on the Sierpinski gasket with dimension equal to two, three and four, and determine the asymptotic growth constants. We shall also consider the number of spanning forests on the generalized Sierpinski gasket with dimension equal to two.

2 Preliminaries

We first recall some relevant definitions for spanning forests and the Sierpinski gasket in this section. A connected graph (without loops) $G = (V, E)$ is defined by its vertex (site) and edge (bond) sets V and E [Biggs(1993), Harary(1969)]. Let $v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. A spanning subgraph G' is a subgraph of G with the same vertex set V and an edge set $E' \subseteq E$. While a tree is a connected graph with no cycles, a spanning forest on G is a spanning subgraph of G that is a disjoint union of trees. That is, a spanning subgraph of G without any cycles, or an acyclic graph. Here an isolated vertex is considered as a tree. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. In general, one can associate an edge weight x_{ij} to each edge connecting adjacent vertices v_i and v_j (see, for example [Alexander(1995)]). For simplicity, all edge weights are set to one throughout this paper.

When the number of spanning forests $N_{SF}(G)$ grows exponentially with $v(G)$ as $v(G) \to \infty$, there exists a constant z_G describing this exponential growth [Burton and Pemantle(1993), Lyons(2005)]:

$$z_G = \lim_{v(G)\to\infty} \frac{\ln N_{SF}(G)}{v(G)} \tag{1}$$

where G, when used as a subscript in this manner, implicitly refers to the thermodynamic limit. We will see that the limit in Eq. (1) exists for the Sierpinski gasket considered in this paper.

The construction of the two-dimensional Sierpinski gasket $SG_2(n)$ at stage n is shown in Fig. 1. At stage $n = 0$, it is an equilateral triangle; while stage $n + 1$ is obtained by the juxtaposition of three n-stage structures. In general, the Sierpinski gaskets SG_d can be built in any Euclidean dimension d with fractal dimension $D = \ln(d + 1)/\ln 2$ [Gefen and Aharony(1981)]. For the Sierpinski gasket $SG_d(n)$, the numbers of edges and vertices are given by

$$e(SG_d(n)) = \binom{d+1}{2}(d+1)^n = \frac{d}{2}(d+1)^{n+1} , \tag{2}$$

$$v(SG_d(n)) = \frac{d+1}{2}[(d+1)^n + 1] . \tag{3}$$

Except the $(d + 1)$ outmost vertices which have degree d, all other vertices of $SG_d(n)$ have degree $2d$. In the large n limit, SG_d is $2d$-regular.

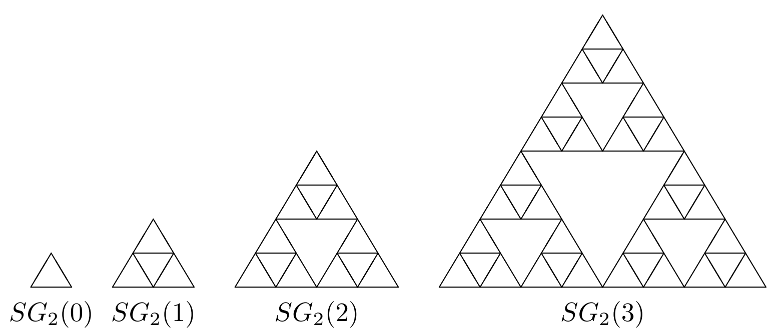

Fig. 1: The first four stages $n = 0, 1, 2, 3$ of the two-dimensional Sierpinski gasket $SG_2(n)$.

The Sierpinski gasket can be generalized, denoted by $SG_{d,b}(n)$, by introducing the side length b which is an integer larger or equal to two [Hilfer and Blumen(1984)]. The generalized Sierpinski gasket at stage $n+1$ is constructed from b layers of stage n hypertetrahedrons. The two-dimensional $SG_{2,b}(n)$ with $b=3$ at stage $n=1,2$ and $b=4$ at stage $n=1$ are illustrated in Fig. 2. The ordinary Sierpinski gasket $SG_d(n)$ corresponds to the $b=2$ case, where the index b is neglected for simplicity. The Hausdorff dimension for $SG_{d,b}$ is given by $D = \ln \binom{b+d-1}{d} / \ln b$ [Hilfer and Blumen(1984)]. Notice that $SG_{d,b}$ is not k-regular even in the thermodynamic limit.

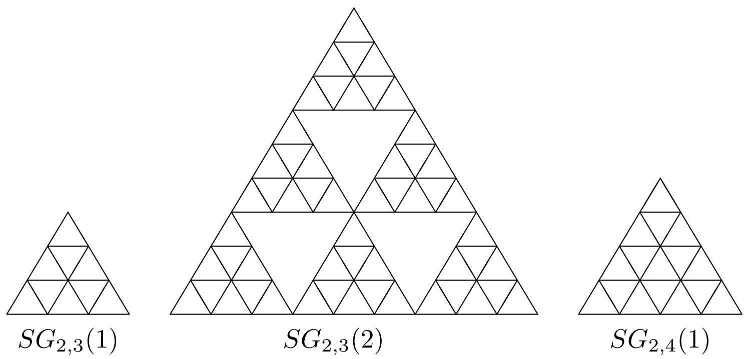

Fig. 2: The generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with $b=3$ at stage $n=1,2$ and $b=4$ at stage $n=1$.

3 The number of spanning forests on $SG_2(n)$

In this section we derive the asymptotic growth constant for the number of spanning forests on the two-dimensional Sierpinski gasket $SG_2(n)$ in detail. Let us start with the definitions of the quantities to be used.

Definition 3.1 *Consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ at stage n. (a) Define $f_{2,b}(n) \equiv N_{SF}(SG_{2,b}(n))$ as the number of spanning forests. (b) Define $t_{2,b}(n)$ as the number of spanning forests such that the three outmost vertices belong to one tree. (c) Define $ga_{2,b}(n)$, $gb_{2,b}(n)$, $gc_{2,b}(n)$ as the number of spanning forests such that one of the outmost vertices belongs to one tree and the other two outmost vertices belong to another tree. (d) Define $h_{2,b}(n)$ as the number of spanning forests such that each of the outmost vertices belongs to a different tree.*

It is clear that the values $ga_{2,b}(n)$, $gb_{2,b}(n)$, $gc_{2,b}(n)$ are the same because of rotation symmetry, and we define $g_{2,b}(n) \equiv ga_{2,b}(n) = gb_{2,b}(n) = gc_{2,b}(n)$. Since we only consider the ordinary Sierpinski gasket in this section, we use the notations $f_2(n)$, $t_2(n)$, $g_2(n)$, and $h_2(n)$ for simplicity. They are illustrated in Fig. 3, where only the outmost vertices are shown. It follows that

$$f_2(n) = t_2(n) + 3g_2(n) + h_2(n) \ . \tag{4}$$

The initial values at stage zero are $t_2(0) = 3$, $g_2(0) = 1$, $h_2(0) = 1$ and $f_2(0) = 7$. The purpose of this section is to obtain the asymptotic behavior of $f_2(n)$ as follows. The four quantities $f_2(n)$, $t_2(n)$, $g_2(n)$ and $h_2(n)$ satisfy recursion relations.

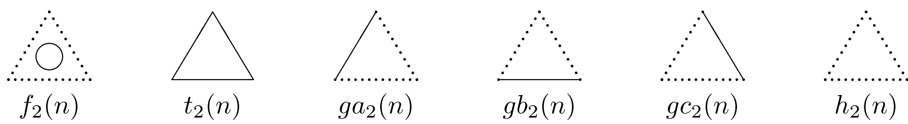

Fig. 3: Illustration for the spanning subgraphs $f_2(n)$, $t_2(n)$, $ga_2(n)$, $gb_2(n)$, $gc_2(n)$ and $h_2(n)$. The two outmost vertices at the ends of a solid line belong to one tree, while the two outmost vertices at the ends of a dot line belong to separated trees.

Lemma 3.1 *For any non-negative integer n,*

$$f_2(n+1) = f_2^3(n) - [t_2(n) + g_2(n)]^3 \ , \tag{5}$$

$$t_2(n+1) = 6t_2^2(n)g_2(n) + 3t_2(n)g_2^2(n) \ , \tag{6}$$

$$\begin{aligned} g_2(n+1) \quad = \quad & t_2^2(n)h_2(n) + 2t_2(n)g_2(n)h_2(n) + 7t_2(n)g_2^2(n) + 4g_2^3(n) \\ & + g_2^2(n)h_2(n) \ , \end{aligned} \tag{7}$$

$$\begin{aligned} h_2(n+1) \quad = \quad & 12t_2(n)g_2(n)h_2(n) + 14g_2^3(n) + 24g_2^2(n)h_2(n) + 9g_2(n)h_2^2(n) \\ & + 3t_2(n)h_2^2(n) + h_2^3(n) \ . \end{aligned} \tag{8}$$

Proof: The Sierpinski gasket $SG_2(n+1)$ is composed of three $SG_2(n)$ with three pairs of vertices identified. For the number $f_2(n+1)$, the unallowable configurations are those with a circuit, *i.e.*, the two identified vertices of each $SG_2(n)$ belong to the same tree as illustrated in Fig. 4. Therefore, we have

$$f_2(n+1) = f_2^3(n) - [t_2(n) + ga_2(n)][t_2(n) + gb_2(n)][t_2(n) + gc_2(n)] \ . \tag{9}$$

With the identity $ga_2(n) = gb_2(n) = gc_2(n) = g_2(n)$, Eq. (5) is verified.

The number $t_2(n+1)$ consists of six configurations where two of the $SG_2(n)$ belong to the class that is enumerated by $t_2(n)$ and the other one belongs to the class enumerated by $g_2(n)$, and three configurations where one of the $SG_2(n)$ belongs to the class enumerated by $t_2(n)$ and the other two belong to the class enumerated by $g_2(n)$ as illustrated in Fig. 5. Therefore, we have

$$t_2(n+1) \quad = \quad 2t_2^2(n)[ga_2(n) + gb_2(n) + gc_2(n)] + t_2(n)ga_2(n)gb_2(n)$$

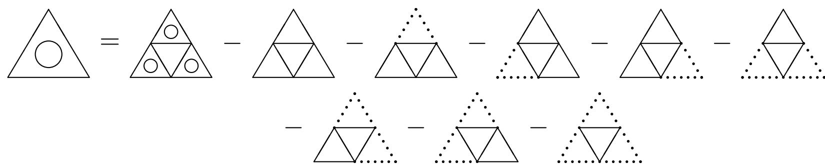

Fig. 4: Illustration for the expression of $f_2(n+1)$.

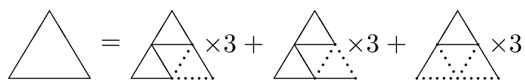

Fig. 5: Illustration for the expression of $t_2(n+1)$. The multiplication of three on the right-hand-side corresponds to the three possible orientations of $SG_2(n+1)$.

$$+t_2(n)ga_2(n)gc_2(n) + t_2(n)gb_2(n)gc_2(n)\ . \tag{10}$$

With the identity $ga_2(n) = gb_2(n) = gc_2(n) = g_2(n)$, Eq. (6) is verified.

Similarly, $ga_2(n+1)$ for $SG_2(n+1)$ can be obtained with appropriate configurations of its three constituting $SG_2(n)$ as illustrated in Fig. 6. Thus,

$$\begin{aligned} ga_2(n+1) \quad = \quad & t_2^2(n)h_2(n) + t_2(n)ga_2(n)[ga_2(n) + gc_2(n) + h_2(n)] \\ & + t_2(n)ga_2(n)[ga_2(n) + gb_2(n) + h_2(n)] + f_2(n)ga_2^2(n) \\ & + t_2(n)ga_2(n)gc_2(n) + t_2(n)ga_2(n)gb_2(n) \\ & + ga_2(n)gb_2(n)gc_2(n)\ . \end{aligned} \tag{11}$$

With the identity $ga_2(n) = gb_2(n) = gc_2(n) = g_2(n)$ and Eq. (4), Eq. (7) is verified.

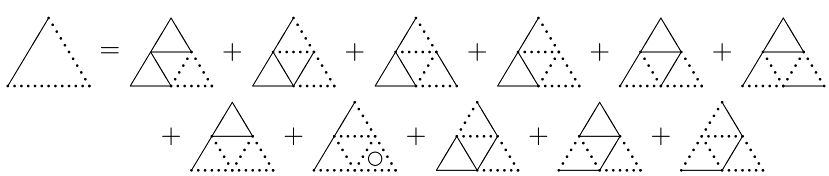

Fig. 6: Illustration for the expression of $ga_2(n+1)$.

Finally, $h_2(n+1)$ is the summation of appropriate configurations as illustrated in Fig. 7, so that

$$h_2(n+1)$$

$$
\begin{aligned}
= \; & 4t_2(n)h_2(n)[ga_2(n)+gb_2(n)+gc_2(n)] \\
& +2gc_2(n)ga_2(n)[gc_2(n)+ga_2(n)]+2ga_2(n)gb_2(n)[ga_2(n)+gb_2(n)] \\
& +2gb_2(n)gc_2(n)[gb_2(n)+gc_2(n)]+2ga_2(n)gb_2(n)gc_2(n) \\
& +3t_2(n)h_2^2(n)+h_2^3(n)+3[ga_2(n)+gb_2(n)+gc_2(n)]h_2^2(n) \\
& +\{3[ga_2(n)+gb_2(n)+gc_2(n)]^2-ga_2^2(n)-gb_2^2(n)-gc_2^2(n)\}h_2(n) \; . \qquad (12)
\end{aligned}
$$

With the identity $ga_2(n) = gb_2(n) = gc_2(n) = g_2(n)$, Eq. (8) is verified.

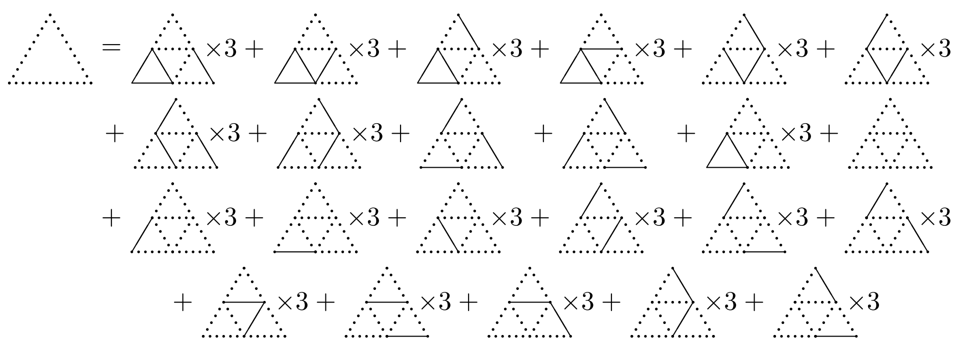

Fig. 7: Illustration for the expression of $h_2(n+1)$. The multiplication of three on the right-hand-side corresponds to the three possible orientations of $SG_2(n+1)$.

Eq. (5) can also be obtained by substituting Eqs. (6), (7) and (8) into Eq. (4). □

The values of $f_2(n)$, $t_2(n)$, $g_2(n)$, $h_2(n)$ for small n can be evaluated recursively by Eqs. (5), (6), (7), (8) as listed in Table 1. These numbers grow exponentially, and do not have simple integer factorizations, in contrast to the corresponding results for the number of spanning trees [Chang et al.(2007)]. To estimate the value of the asymptotic growth constant defined in Eq. (1), we need the following lemma.

Lemma 3.2 *The asymptotic growth constant for the number of spanning forests on $SG_2(n)$ is bounded:*

$$
\frac{2}{3^{m+1}} \ln h_2(m) < z_{SG_2} < \frac{2}{3^{m+1}} \ln f_2(m) \; , \qquad (13)
$$

where m is a positive integer.

Tab. 1: The first few values of $f_2(n)$, $t_2(n)$, $g_2(n)$, $h_2(n)$.

n	0	1	2	3
$f_2(n)$	7	279	20,592,775	8,696,126,758,781,951,722,199
$t_2(n)$	3	63	1,294,083	36,212,372,367,917,382,063
$g_2(n)$	1	41	2,022,893	215,741,040,104,979,715,185
$h_2(n)$	1	93	13,230,013	8,012,691,266,099,095,194,581

Proof: We first show that the ratio $t_2(n)/g_2(n)$ is a strictly decreasing sequence. By Eqs. (6) and (7), we have

$$\begin{aligned}
&\frac{t_2(n+1)}{g_2(n+1)} \\
&= \frac{6t_2^2(n)g_2(n) + 3t_2(n)g_2^2(n)}{t_2^2(n)h_2(n) + 2t_2(n)g_2(n)h_2(n) + 7t_2(n)g_2^2(n) + 4g_2^3(n) + g_2^2(n)h_2(n)} \\
&< \frac{t_2(n)[6t_2(n)g_2(n) + 3g_2^2(n)]}{g_2(n)[7t_2(n)g_2(n) + 4g_2^2(n)]} < \frac{6t_2(n)}{7g_2(n)} \ .
\end{aligned} \tag{14}$$

From the values in Table 1, $t_2(n)/g_2(n)$ is less than one for $n > 1$. It is clear that this ratio approaches zero as n increases. Similarly, $g_2(n)/h_2(n)$ is also a strictly decreasing sequence by Eqs. (7) and (8).

$$\begin{aligned}
&\frac{g_2(n+1)}{h_2(n+1)} \\
&< \frac{3t_2(n)g_2(n)h_2(n) + 7t_2(n)g_2^2(n) + 4g_2^3(n) + g_2^2(n)h_2(n)}{12t_2(n)g_2(n)h_2(n) + 14g_2^3(n) + 24g_2^2(n)h_2(n) + 9g_2(n)h_2^2(n) + 4t_2(n)h_2^2(n)} \\
&< \frac{g_2(n)[3t_2(n)h_2(n) + 7t_2(n)g_2(n) + 4g_2^2(n) + g_2(n)h_2(n)]}{h_2(n)[4t_2(n)h_2(n) + 12t_2(n)g_2(n) + 24g_2^2(n) + 9g_2(n)h_2(n)]} \\
&< \frac{3g_2(n)}{4h_2(n)} \qquad \text{for n} > 1 \ ,
\end{aligned} \tag{15}$$

where we have used the fact that $t_2(n) < g_2(n) < h_2(n)$ for $n > 1$. Again, $g_2(n)/h_2(n)$ approaches zero as n increases. The relation $t_2(n) \ll g_2(n) \ll h_2(n)$ for large n is expectable since it is rare to keep the three outmost vertices of $SG_2(n)$ in the same tree for $t_2(n)$ and $h_2(n)$ should dominate when n becomes large. In fact, both $f_2(n)$ and $g_2(n)$ are negligible compared with $h_2(n)$ such that $f_2(n) \sim h_2(n)$ for large n. By Eqs. (5) and (8), we have the upper and lower bounds for $f_2(n)$:

$$h_2^3(n-1) < h_2(n) < f_2(n) < f_2^3(n-1) \ , \tag{16}$$

such that

$$h_2(m)^{3^{n-m}} < f_2(n) < f_2(m)^{3^{n-m}} \ , \tag{17}$$

where m is a fixed integer. With the definition for z_{SG_2} given in Eq. (1) and the number of vertices of $SG_2(n)$ is $3(3^n+1)/2$ by Eq. (3), the proof is completed. □

Proposition 3.1 *The asymptotic growth constant for the number of spanning forests on the two-dimensional Sierpinski gasket $SG_2(n)$ in the large n limit is $z_{SG_2} = 1.24733719931...$.*

Proof: Define ratios $\alpha(n) \equiv t_2(n)/f_2(n)$ and $\beta(n) \equiv g_2(n)/f_2(n)$. By Eq. (4), it is clear that $0 \le \alpha(n) + \beta(n) < 1$. As seen in the proof of Lemma 3.2, $\alpha(n) + \beta(n)$ is a strictly decreasing sequence. By Eq. (5), let us define $r(n) \equiv f_2(n)/f_2^3(n-1) = 1 - [\alpha(n-1) + \beta(n-1)]^3$ for positive integer n. It follows that

$$\begin{aligned}
\ln f_2(n) &= 3 \ln f_2(n-1) + \ln r(n) = ... \\
&= 3^{n-m} \ln f_2(m) + \sum_{j=m+1}^{n} 3^{n-j} \ln r(j) \\
&> 3^{n-m} \ln f_2(m) + \left(\frac{3^{n-m} - 1}{2} \right) \ln r(m+1) \ .
\end{aligned} \tag{18}$$

Divide this equation by $3(3^n + 1)/2$ and take the limit $n \to \infty$, the difference between the upper bound in Eq. (13) and the asymptotic growth constant is bounded:

$$\frac{2}{3^{m+1}} \ln f_2(m) - z_{SG_2} \le \frac{-1}{3^{m+1}} \ln \left(1 - [\alpha(m) + \beta(m)]^3 \right) \ . \tag{19}$$

When m is as small as three, the right-hand-side of Eq. (19) is about 3×10^{-7} by the values given in Table 1. Similarly, it can be shown that the difference between z_{SG_2} and the lower bound (left-hand-side of Eq. (13)) quickly converges to zero as m increases. In another word, the numerical values of $\ln f_2(m)$ and $\ln h_2(m)$ are almost the same except for the first few m, and the upper and lower bounds in Eq. (13) converge to the quoted value of z_{SG_2}. In fact, one obtains the numerical value of z_{SG_2} with more than a hundred significant figures accurate when m is equal to eight. □

4 The number of spanning forests on $SG_{2,b}(n)$ with $b = 3, 4$

The method given in the previous section can be applied to the number of spanning forests on $SG_{d,b}(n)$ with larger values of d and b. The number of configurations to be considered increases as d and b increase, and the recursion relations must be derived individually for each d and b. In this section, we consider the generalized two-dimensional Sierpinski gasket $SG_{2,b}(n)$ with the number of layers b equal to three and four. For $SG_{2,3}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,3}(n)) = 3 \times 6^n \ , \tag{20}$$

$$v(SG_{2,3}(n)) = \frac{7 \times 6^n + 8}{5} \ , \tag{21}$$

where the three outmost vertices have degree two. There are $(6^n - 1)/5$ vertices of $SG_{2,3}(n)$ with degree six and $6(6^n - 1)/5$ vertices with degree four. By Definition 3.1, the number of spanning forests is $f_{2,3}(n) = t_{2,3}(n) + 3g_{2,3}(n) + h_{2,3}(n)$. The initial values are the same as for SG_2: $t_{2,3}(0) = 3$,

$g_{2,3}(0) = 1$, $h_{2,3}(0) = 1$ and $f_{2,3}(0) = 7$. By the method illustrated in the previous section, we obtain the following recursion relations for any non-negative integer n.

$$\begin{aligned}
& f_{2,3}(n+1) \\
&= f^6_{2,3}(n) - 3f^3_{2,3}(n)[t_{2,3}(n)+g_{2,3}(n)]^3 - 3f_{2,3}(n)[t_{2,3}(n)+g_{2,3}(n)]^5 - [t_{2,3}(n)+g_{2,3}(n)]^6 \\
&\quad +6t_{2,3}(n)f_{2,3}(n)[t_{2,3}(n)+g_{2,3}(n)]^4 + 6t^2_{2,3}(n)[t_{2,3}(n)+g_{2,3}(n)]^4 \\
&\quad -6t^3_{2,3}(n)[t_{2,3}(n)+g_{2,3}(n)]^3 \ , \qquad (22)
\end{aligned}$$

$$\begin{aligned}
& t_{2,3}(n+1) \\
&= 18t^4_{2,3}(n)h_{2,3}(n)g_{2,3}(n) + 45t^3_{2,3}(n)h_{2,3}(n)g^2_{2,3}(n) + \Big[142t^3_{2,3}(n) + 36t^2_{2,3}(n)h_{2,3}(n)\Big]g^3_{2,3}(n) \\
&\quad +\Big[153t^2_{2,3}(n) + 9t_{2,3}(n)h_{2,3}(n)\Big]g^4_{2,3}(n) + 45t_{2,3}(n)g^5_{2,3}(n) + 2g^6_{2,3}(n) \ , \qquad (23)
\end{aligned}$$

$$\begin{aligned}
& g_{2,3}(n+1) \\
&= \Big[22t^3_{2,3}(n)h^2_{2,3}(n) + 3t^2_{2,3}(n)h^3_{2,3}(n)\Big]g_{2,3}(n) \\
&\quad +\Big[77t^3_{2,3}(n)h_{2,3}(n) + 50t^2_{2,3}(n)h^2_{2,3}(n) + 3t_{2,3}(n)h^3_{2,3}(n)\Big]g^2_{2,3}(n) \\
&\quad +\Big[200t^2_{2,3}(n)h_{2,3}(n) + 42t_{2,3}(n)h^2_{2,3}(n) + h^3_{2,3}(n)\Big]g^3_{2,3}(n) \\
&\quad +\Big[171t^2_{2,3}(n) + 169t_{2,3}(n)h_{2,3}(n) + 12h^2_{2,3}(n)\Big]g^4_{2,3}(n) \\
&\quad +\Big[195t_{2,3}(n) + 46h_{2,3}(n)\Big]g^5_{2,3}(n) + 56g^6_{2,3}(n) + 2t^4_{2,3}(n)h^2_{2,3}(n) + t^3_{2,3}(n)h^3_{2,3}(n) \ , \qquad (24)
\end{aligned}$$

$$\begin{aligned}
& h_{2,3}(n+1) \\
&= \Big[60t^3_{2,3}(n)h^2_{2,3}(n) + 162t^2_{2,3}(n)h^3_{2,3}(n) + 90t_{2,3}(n)h^4_{2,3}(n) + 18h^5_{2,3}(n)\Big]g_{2,3}(n) \\
&\quad +\Big[552t^2_{2,3}(n)h^2_{2,3}(n) + 522t_{2,3}(n)h^3_{2,3}(n) + 135h^4_{2,3}(n)\Big]g^2_{2,3}(n) \\
&\quad +\Big[564t^2_{2,3}(n)h_{2,3}(n) + 1404t_{2,3}(n)h^2_{2,3}(n) + 534h^3_{2,3}(n)\Big]g^3_{2,3}(n) \\
&\quad +\Big[1608t_{2,3}(n)h_{2,3}(n) + 1152h^2_{2,3}(n)\Big]g^4_{2,3}(n) + \Big[468t_{2,3}(n) + 1236h_{2,3}(n)\Big]g^5_{2,3}(n) \\
&\quad +468g^6_{2,3}(n) + 14t^3_{2,3}(n)h^3_{2,3}(n) + 15t^2_{2,3}(n)h^4_{2,3}(n) + 6t_{2,3}(n)h^5_{2,3}(n) + h^6_{2,3}(n) \ . \qquad (25)
\end{aligned}$$

The figures for these configurations are too many to be shown here. Some values of $f_{2,3}(n)$, $t_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$ are listed in Table 2. These numbers grow exponentially, and do not have simple integer factorizations.

Tab. 2: The first few values of $f_{2,3}(n)$, $t_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$.

n	0	1	2
$f_{2,3}(n)$	7	61,905	53,145,523,900,850,102,434,114,604,001
$t_{2,3}(n)$	3	8,372	218,891,276,004,139,532,538,695,680
$g_{2,3}(n)$	1	8,020	1,242,664,072,161,818,527,545,741,824
$h_{2,3}(n)$	1	29,473	49,198,640,408,360,507,318,938,682,849

Lemma 4.1 *The asymptotic growth constant for the number of spanning forests on $SG_{2,3}(n)$ is bounded:*

$$\frac{5}{7\times 6^m}\ln h_{2,3}(m) < z_{SG_{2,3}} < \frac{5}{7\times 6^m}\ln f_{2,3}(m)\ , \tag{26}$$

where m is a positive integer.

Proof: We first show that $5t_{2,3}(n) < g_{2,3}(n)$ and $30g_{2,3}(n) < h_{2,3}(n)$ for integer $n > 1$ by induction. It is clear from Table 2 that $5t_{2,3}(2) < g_{2,3}(2)$ and $30g_{2,3}(2) < h_{2,3}(2)$. In order to save space, we will use t_n, g_n, h_n to denote $t_{2,3}(n)$, $g_{2,3}(n)$, $h_{2,3}(n)$ for the lengthy equations in this Lemma. From Eq. (23) and the assumption that $5t_{2,3}(n) < g_{2,3}(n)$ for $n > 1$, we have

$$t_{n+1} < \frac{18}{5}t_n^3h_ng_n^2 + 9t_n^2h_ng_n^3 + \Big(\frac{142}{5}t_n^2 + \frac{36}{5}t_nh_n\Big)g_n^4 + \Big(\frac{153}{5}t_n + \frac{9}{5}h_n\Big)g_n^5 + 11g_n^6\ , \tag{27}$$

such that $5t_{2,3}(n+1) < g_{2,3}(n+1)$ is established after Eq. (27) is compared with Eq. (24). Similarly from Eq. (24), we have

$$\begin{aligned} g_{n+1} &< \Big(\frac{22}{5}t_n^2h_n^2 + \frac{3}{5}t_nh_n^3\Big)g_n^2 + \Big(\frac{77}{5}t_n^2h_n + 10t_nh_n^2 + \frac{8}{5}h_n^3\Big)g_n^3 + \Big(40t_nh_n + \frac{102}{5}h_n^2\Big)g_n^4 \\ &\quad + \Big(\frac{171}{5}t_n + \frac{399}{5}h_n\Big)g_n^5 + 95g_n^6 + \Big(\frac{2}{5}t_n^3h_n^2 + \frac{1}{5}t_n^2h_n^3\Big)g_n \\ &< \Big(\frac{22}{5}t_n^2h_n^2 + \frac{3}{5}t_nh_n^3\Big)g_n^2 + \Big(\frac{77}{5}t_n^2h_n + 10t_nh_n^2 + \frac{8}{5}h_n^3\Big)g_n^3 + \Big(40t_nh_n + \frac{102}{5}h_n^2\Big)g_n^4 \\ &\quad + \Big(\frac{171}{5}t_n + \frac{399}{5}h_n\Big)\frac{h_n^5}{30^5} + \frac{95}{30^6}h_n^6 + \Big(\frac{2}{5}t_n^3h_n^2 + \frac{1}{5}t_n^2h_n^3\Big)g_n\ , \end{aligned} \tag{28}$$

such that $30g_{2,3}(n+1) < h_{2,3}(n+1)$ is established after Eq. (28) is compared with Eq. (25).

The next step is to show that $g_{2,3}(n)^2 < t_{2,3}(n)h_{2,3}(n)$ for integer $n > 1$ by induction. It is clear from Table 2 that $g_{2,3}(2)^2 < t_{2,3}(2)h_{2,3}(2)$. The terms of $g_{2,3}(n)$ in Eq. (24) can be rearranged to give

$$\begin{aligned} g_{n+1} &= \Big(t_n^3 + 3t_n^2g_n + 3t_ng_n^2 + g_n^3\Big)h_n^3 + \Big(2t_n^4 + 22t_n^3g_n + 50t_n^2g_n^2 + 42t_ng_n^3 + 12g_n^4\Big)h_n^2 \\ &\quad + \Big(77t_n^3g_n^2 + 200t_n^2g_n^3 + 169t_ng_n^4 + 46g_n^5\Big)h_n + 171t_n^2g_n^4 + 195t_ng_n^5 + 56g_n^6 \\ &< \frac{216}{125}g_n^3h_n^3 + \frac{14112}{625}g_n^4h_n^2 + \frac{11052}{125}g_n^5h_n + \frac{2546}{25}g_n^6 \end{aligned}$$

$$< \quad 2.6 g_n^3 h_n^3 \ , \tag{29}$$

where inequalities $5t_{2,3}(n) < g_{2,3}(n)$ and $30 g_{2,3}(n) < h_{2,3}(n)$ are used. On the other hand, we have $h_{2,3}(n+1) > h_{2,3}^6(n)$ from Eq. (25), and

$$t_{2,3}(n+1) > 9t_{2,3}(n)h_{2,3}(n)g_{2,3}^4(n) + 2g_{2,3}^6(n) > 11g_{2,3}^6(n) \tag{30}$$

from Eq. (24) with the assumption that $g_{2,3}^2(n) < t_{2,3}(n)h_{2,3}(n)$ for $n > 1$. It follows that

$$h_{2,3}(n+1)t_{2,3}(n+1) > 11g_{2,3}^6(n)h_{2,3}^6(n) > g_{2,3}^2(n+1) \ . \tag{31}$$

With above inequalities, we are ready to show that $t_{2,3}(n)/g_{2,3}(n)$ is a strictly decreasing sequence. From Eqs. (23) and (24), we have

$$\begin{aligned}
&\frac{t_{n+1}}{g_{n+1}} \\
&= \frac{t_n(18t_n^3h_ng_n + 45t_n^2h_ng_n^2 + 142t_n^2g_n^3 + 36t_nh_ng_n^3 + 153t_ng_n^4 + 9h_ng_n^4 + 45g_n^5) + 2g_n^6}{g_n(77t_n^3h_ng_n + 200t_n^2h_ng_n^2 + 171t_n^2g_n^3 + 169t_nh_ng_n^3 + 195t_ng_n^4 + 46h_ng_n^4 + 56g_n^5) + \ldots} \\
&< \frac{t_n(18t_n^3h_ng_n + 45t_n^2h_ng_n^2 + 142t_n^2g_n^3 + 36t_nh_ng_n^3 + 153t_ng_n^4 + 9h_ng_n^4 + 45g_n^5) + 2t_nh_ng_n^4}{g_n(77t_n^3h_ng_n + 200t_n^2h_ng_n^2 + 171t_n^2g_n^3 + 169t_nh_ng_n^3 + 195t_ng_n^4 + 46h_ng_n^4 + 56g_n^5)} \\
&< \frac{142t_n}{171g_n} \ .
\end{aligned} \tag{32}$$

From the values in Table 2, $t_{2,3}(n)/g_{2,3}(n)$ is less than one for $n > 1$. Therefore, this ratio approaches zero as n increases. Similarly $g_{2,3}(n)/h_{2,3}(n)$ is also a strictly decreasing sequence from Eqs. (24) and (25) as follows.

$$\begin{aligned}
g_{n+1} \quad = \quad & g_n\Big(22t_n^3h_n^2 + 3t_n^2h_n^3 + 77t_n^3h_ng_n + 50t_n^2h_n^2g_n + 3t_nh_n^3g_n + 200t_n^2h_ng_n^2 + 42t_nh_n^2g_n^2 \\
& + h_n^3g_n^2 + 171t_n^2g_n^3 + 169t_nh_ng_n^3 + 12h_n^2g_n^3 + 195t_ng_n^4 + 46h_ng_n^4 + 56g_n^5\Big) + 2t_n^4h_n^2 \\
& + t_n^3h_n^3
\end{aligned} \tag{33}$$

where $2t_n^4h_n^2 < (2/5)t_n^3g_nh_n^2$, $t_n^3h_n^3 < (1/5)t_n^2g_nh_n^3$, so that

$$\begin{aligned}
g_{n+1} \quad < \quad & g_n\Big(22.4t_n^3h_n^2 + 3.2t_n^2h_n^3 + 77t_n^3h_ng_n + 50t_n^2h_n^2g_n + 3t_nh_n^3g_n + 200t_n^2h_ng_n^2 + 42t_nh_n^2g_n^2 \\
& + h_n^3g_n^2 + 171t_n^2g_n^3 + 169t_nh_ng_n^3 + 12h_n^2g_n^3 + 195t_ng_n^4 + 46h_ng_n^4 + 56g_n^5\Big)
\end{aligned} \tag{34}$$

where $22.4t_n^3h_n^2 < (22.4/150)t_n^2h_n^3$ and $77t_n^3h_ng_n < (77/150)t_n^2h_n^2g_n$. Compared with

$$\begin{aligned}
h_{n+1} \quad > \quad & h_n\Big(15t_n^2h_n^3 + 162t_n^2h_n^2g_n + 90t_nh_n^3g_n + 552t_n^2h_ng_n^2 + 522t_nh_n^2g_n^2 + 135h_n^3g_n^2 + 564t_n^2g_n^3 \\
& + 1404t_nh_ng_n^3 + 534h_n^2g_n^3 + 1608t_ng_n^4 + 1152h_ng_n^4 + 1236g_n^5\Big) \ ,
\end{aligned} \tag{35}$$

we arrive at the inequality

$$\frac{g_{2,3}(n+1)}{h_{2,3}(n+1)} < \frac{25g_{2,3}(n)}{69h_{2,3}(n)}. \tag{36}$$

Again, $g_{2,3}(n)/h_{2,3}(n)$ approaches zero as n increases, such that $t_{2,3}(n) \ll g_{2,3}(n) \ll h_{2,3}(n)$ for large n. As for the ordinary Sierpinski gasket, both $f_{2,3}(n)$ and $g_{2,3}(n)$ are negligible compared with $h_{2,3}(n)$ such that $f_{2,3}(n) \sim h_{2,3}(n)$ for large n. By Eqs. (22) and (25), we have the upper and lower bounds for $f_{2,3}(n)$:

$$h_{2,3}^6(n-1) < h_{2,3}(n) < f_{2,3}(n) < f_{2,3}^6(n-1) \ , \tag{37}$$

such that

$$h_{2,3}(m)^{6^{n-m}} < f_{2,3}(n) < f_{2,3}(m)^{6^{n-m}} \ , \tag{38}$$

where m is a fixed integer. With the definition for $z_{SG_{2,3}}$ given in Eq. (1) and the vertex number of $SG_{2,3}(n)$ by Eq. (21), the proof is completed. □

We have the following proposition.

Proposition 4.1 *The asymptotic growth constant for the number of spanning forests on the two-dimensional Sierpinski gasket $SG_{2,3}(n)$ in the large n limit is $z_{SG_{2,3}} = 1.31235755933....$*

The convergence of the upper and lower bounds remains rapid. By the same method as given in the proof of Proposition 3.1, the difference between the upper bound in Eq. (26) and the asymptotic growth constant is bounded:

$$\frac{5}{7 \times 6^m} \ln f_{2,3}(m) - z_{SG_{2,3}} \le \frac{-1}{7 \times 6^m} \ln \left(1 - 7 \left[\frac{t_{2,3}(m)}{f_{2,3}(m)} + \frac{g_{2,3}(m)}{f_{2,3}(m)} \right]^3 \right) \ . \tag{39}$$

More than a hundred significant figures for $z_{SG_{2,3}}$ can be obtained when m is equal to five.

For $SG_{2,4}(n)$, the numbers of edges and vertices are given by

$$e(SG_{2,4}(n)) = 3 \times 10^n \ , \tag{40}$$

$$v(SG_{2,4}(n)) = \frac{4 \times 10^n + 5}{3} \ , \tag{41}$$

where again the three outmost vertices have degree two. There are $(10^n - 1)/3$ vertices of $SG_{2,4}(n)$ with degree six, and $(10^n - 1)$ vertices with degree four. By Definition 3.1, the number of spanning forests is $f_{2,4}(n) = t_{2,4}(n) + 3g_{2,4}(n) + h_{2,4}(n)$. The initial values are the same as for SG_2: $t_{2,4}(0) = 3$, $g_{2,4}(0) = 1$, $h_{2,4}(0) = 1$ and $f_{2,4}(0) = 7$. We wrote a computer program to obtain the recursion relations. Using the shorthand notation $tg_{2,4}(n) = t_{2,4}(n) + g_{2,4}(n)$, we have

$$\begin{aligned} & f_{2,4}(n+1) \\ = \quad & f_{2,4}^{10}(n) - 6f_{2,4}^7(n)tg_{2,4}^3(n) - 9f_{2,4}^5(n)tg_{2,4}^5(n) + 18f_{2,4}^5(n)tg_{2,4}^4(n)t_{2,4}(n) \\ & +2f_{2,4}^4(n)tg_{2,4}^6(n) + 18f_{2,4}^4(n)tg_{2,4}^4(n)t_{2,4}^2(n) - 18f_{2,4}^4(n)tg_{2,4}^3(n)t_{2,4}^3(n) \end{aligned}$$

$$
\begin{aligned}
&-6f_{2,4}^3(n)tg_{2,4}^7(n) + 30f_{2,4}^3(n)tg_{2,4}^6(n)t_{2,4}(n) - 30f_{2,4}^3(n)tg_{2,4}^5(n)t_{2,4}^2(n)\\
&+3f_{2,4}^2(n)tg_{2,4}^8(n) + 24f_{2,4}^2(n)tg_{2,4}^7(n)t_{2,4}(n) - 36f_{2,4}^2(n)tg_{2,4}^6(n)t_{2,4}^2(n)\\
&-54f_{2,4}^2(n)tg_{2,4}^5(n)t_{2,4}^3(n) + 60f_{2,4}^2(n)tg_{2,4}^4(n)t_{2,4}^4(n) - 5f_{2,4}(n)tg_{2,4}^9(n)\\
&+42f_{2,4}(n)tg_{2,4}^8(n)t_{2,4}(n) - 42f_{2,4}(n)tg_{2,4}^7(n)t_{2,4}^2(n)\\
&-168f_{2,4}(n)tg_{2,4}^6(n)t_{2,4}^3(n) + 330f_{2,4}(n)tg_{2,4}^5(n)t_{2,4}^4(n)\\
&-162f_{2,4}(n)tg_{2,4}^4(n)t_{2,4}^5(n) + 8f_{2,4}(n)tg_{2,4}^3(n)t_{2,4}^6(n) + 42tg_{2,4}^8(n)t_{2,4}^2(n)\\
&-162tg_{2,4}^7(n)t_{2,4}^3(n) + 102tg_{2,4}^6(n)t_{2,4}^4(n) + 288tg_{2,4}^5(n)t_{2,4}^5(n)\\
&-432tg_{2,4}^4(n)t_{2,4}^6(n) + 162tg_{2,4}^3(n)t_{2,4}^7(n)\ . \qquad (42)
\end{aligned}
$$

The other recursion relations for $SG_{2,4}(n)$ are too lengthy to be included here. They are available from the authors on request. Some values of $f_{2,4}(n)$, $t_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$ are listed in Table 3. These numbers grow exponentially, and do not have simple integer factorizations.

Tab. 3: The first few values of $f_{2,4}(n)$, $t_{2,4}(n)$, $g_{2,4}(n)$, $h_{2,4}(n)$.

n	1	2
$f_{2,4}(n)$	75,908,209	6,053,025,303,996,636,848,970,430,785,675,468,144,409,657,412,247,800,423,390,303,465,602,821,564,523,873
$t_{2,4}(n)$	6,665,475	772,069,425,849,585,011,183,346,692,712,538,703,294,972,628,973,372,161,275,424,155,207,555,217,357
$g_{2,4}(n)$	8,406,453	17,447,838,129,920,655,302,865,270,986,884,479,355,572,603,291,172,150,410,900,983,156,421,717,259,395
$h_{2,4}(n)$	44,023,375	5,999,909,720,181,025,298,050,651,626,022,102,167,639,644,629,745,310,599,996,325,091,978,348,857,528,331

By a similar argument as in Lemma 3.2, the asymptotic growth constant for the number of spanning forests on $SG_{2,4}(n)$ is bounded:

$$\frac{3}{4 \times 10^m} \ln h_{2,4}(m) < z_{SG_{2,4}} < \frac{3}{4 \times 10^m} \ln f_{2,4}(m)\ , \qquad (43)$$

with m a positive integer. We have the following proposition.

Proposition 4.2 *The asymptotic growth constant for the number of spanning forests on the two-dimensional Sierpinski gasket $SG_{2,4}(n)$ in the large n limit is $z_{SG_{2,4}} = 1.36051646575....$*

The convergence of the upper and lower bounds is rapid again. By the same method as given in the proof of Proposition 3.1, the difference between the upper bound in Eq. (43) and the asymptotic growth constant is bounded:

$$\frac{3}{4\times 10^m}\ln f_{2,4}(m) - z_{SG_{2,4}} \quad \le \quad \frac{-1}{12\times 10^m}\ln\left(1 - 15\left[\frac{tg_{2,4}(m)}{f_{2,4}(m)}\right]^3\right) . \tag{44}$$

More than a hundred significant figures for $z_{SG_{2,4}}$ can be obtained when m is equal to four.

5 The number of spanning forests on $SG_d(n)$ with $d = 3, 4$

In this section, we derive the asymptotic growth constant of spanning forests on $SG_d(n)$ with $d = 3, 4$. For the three-dimensional Sierpinski gasket $SG_3(n)$, we use the following definitions.

Definition 5.1 *Consider the three-dimensional Sierpinski gasket $SG_3(n)$ at stage n. (a) Define $f_3(n) \equiv N_{SF}(SG_3(n))$ as the number of spanning forests. (b) Define $t_3(n)$ as the number of spanning forests such that the four outmost vertices belong to one tree. (c) Define $g_3(n)$ as the number of spanning forests such that one of the outmost vertices belongs to one tree and the other three outmost vertices belong to another tree. (d) Define $h_3(n)$ as the number of spanning forests such that two of the outmost vertices belong to one tree and the other two outmost vertices belong to another tree. (e) Define $p_3(n)$ as the number of spanning forests such that two of the outmost vertices belong to one tree and the other two outmost vertices separately belong to other trees. (f) Define $q_3(n)$ as the number of spanning forests such that each of the outmost vertices belongs to a different tree.*

The quantities $f_3(n)$, $t_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$ and $q_3(n)$ are illustrated in Fig. 8, where only the outmost vertices are shown. There are four different classes of forests enumerated by $g_3(n)$, three classes enumerated by $h_3(n)$, and six classes enumerated by $p_3(n)$. By definition,

$$f_3(n) = t_3(n) + 4g_3(n) + 3h_3(n) + 6p_3(n) + q_3(n) . \tag{45}$$

The initial values at stage zero are $t_3(0) = 16$, $g_3(0) = 3$, $h_3(0) = 1$, $p_3(0) = 1$, $q_3(0) = 1$ and $f_3(0) = 38$.

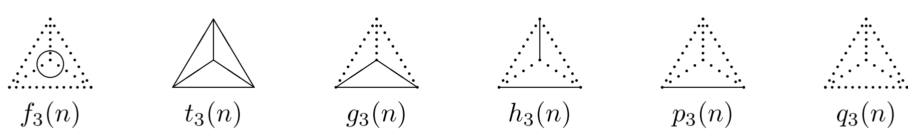

Fig. 8: Illustration for the spanning subgraphs $f_3(n)$, $t_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$ and $q_3(n)$. The two outmost vertices at the ends of a solid line belong to one tree, while the two outmost vertices at the ends of a dot line belong to separated trees.

The recursion relations are lengthy and given in the appendix. Some values of $f_3(n)$, $t_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$, $q_3(n)$ are listed in Table 4. These numbers grow exponentially, and do not have simple integer factorizations.

Tab. 4: The first few values of $f_3(n)$, $t_3(n)$, $g_3(n)$, $h_3(n)$, $p_3(n)$, $q_3(n)$.

n	0	1	2
$f_3(n)$	38	701,866	150,308,440,552,729,541,599,408
$t_3(n)$	16	173,880	14,568,001,216,879,127,537,520
$g_3(n)$	3	63,354	10,109,099,387,983,187,560,398
$h_3(n)$	1	9,059	1,150,970,295,799,746,536,513
$p_3(n)$	1	31,357	9,282,357,698,529,097,198,747
$q_3(n)$	1	59,251	36,156,984,705,343,841,018,275

By a similar argument as in Lemma 3.2, the asymptotic growth constant for the number of spanning forests on $SG_3(n)$ is bounded:

$$\frac{\ln q_3(m)}{2 \times 4^m} < z_{SG_3} < \frac{\ln f_3(m)}{2 \times 4^m} , \tag{46}$$

with m a positive integer. We have the following proposition.

Proposition 5.1 *The asymptotic growth constant for the number of spanning forests on the three-dimensional Sierpinski gasket $SG_3(n)$ in the large n limit is $z_{SG_3} = 1.66680628117...$.*

The convergence of the upper and lower bounds is not as quick as for the two-dimensional cases. By the same method as given in the proof of Proposition 3.1, the difference between the upper bound in Eq. (46) and the asymptotic growth constant is bounded:

$$\begin{aligned} & \frac{1}{2 \times 4^m} \ln f_3(m) - z_{SG_3} \\ \leq \quad & \frac{-1}{6 \times 4^m} \ln \left(1 - 7 \left[\frac{t_3(m)}{f_3(m)} + \frac{2g_3(m)}{f_3(m)} + \frac{h_3(m)}{f_3(m)} + \frac{p_3(m)}{f_3(m)} \right]^3 \right) . \end{aligned} \tag{47}$$

More than a hundred significant figures for z_{SG_3} can be obtained when m is equal to nine.

For the four-dimensional Sierpinski gasket $SG_4(n)$, we use the following definitions.

Definition 5.2 *Consider the four-dimensional Sierpinski gasket $SG_4(n)$ at stage n. (a) Define $f_4(n) \equiv N_{SF}(SG_4(n))$ as the number of spanning forests. (b) Define $t_4(n)$ as the number of spanning forests such that the five outmost vertices belong to one tree. (c) Define $g_4(n)$ as the number of spanning forests such that two of the outmost vertices belong to one tree and the other three outmost vertices belong to another tree. (d) Define $h_4(n)$ as the number of spanning forests such that one of the outmost vertices belong to one tree and the other four outmost vertices belong to another tree. (e) Define $p_4(n)$ as the number of spanning forests such that one of the outmost vertices belong to one tree, two of the other outmost vertices belong to another tree and the remaining two outmost vertices belong to a third tree. (f) Define $q_4(n)$ as the number of spanning forests such that three of the outmost vertices belong to one tree and the other two outmost vertices separately belong to other trees. (g) Define $r_4(n)$ as the number of spanning forests* *such that two of the outmost vertices belong to one tree and the other three outmost vertices separately belong to other trees. (h) Define $s_4(n)$ as the number of spanning forests such that each of the outmost vertices belongs to a different tree.*

The quantities $f_4(n)$, $t_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$ and $s_4(n)$ are illustrated in Fig. 9, where only the outmost vertices are shown. There are ten different classes of forests enumerated by $g_4(n)$, five classes enumerated by $h_4(n)$, fifteen classes enumerated by $p_4(n)$, ten classes enumerated by $q_4(n)$ and ten classes enumerated by $r_4(n)$. By definition,

$$f_4(n) = t_4(n) + 10g_4(n) + 5h_4(n) + 15p_4(n) + 10q_4(n) + 10r_4(n) + s_4(n) \; . \tag{48}$$

The initial values at stage zero are $t_4(0) = 125$, $g_4(0) = 3$, $h_4(0) = 16$, $p_4(0) = 1$, $q_4(0) = 3$, $r_4(0) = 1$, $s_4(0) = 1$ and $f_4(0) = 291$.

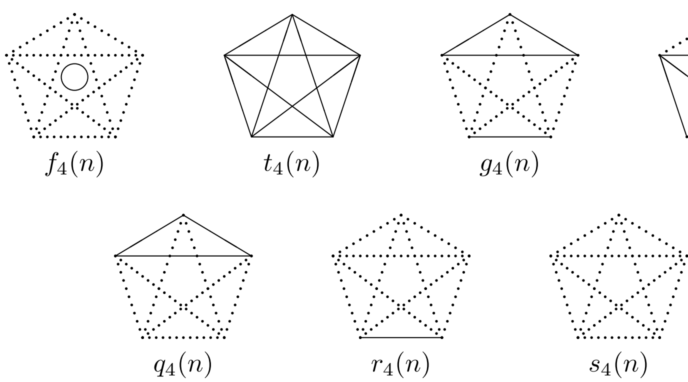

Fig. 9: Illustration for the spanning subgraphs $f_4(n)$, $t_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$ and $s_4(n)$. The two outmost vertices at the ends of a solid line belong to one tree, while the two outmost vertices at the ends of a dot line belong to separated trees.

We wrote a computer program to obtain the recursion relations. Using the shorthand notations $tr_4(n) = t_4(n) + 4g_4(n) + 3h_4(n) + 3p_4(n) + 3q_4(n) + r_4(n)$, $tq_4(n) = t_4(n) + g_4(n) + 2h_4(n) + q_4(n)$, $tp_4(n) = t_4(n) + 2g_4(n) + h_4(n) + p_4(n)$ and $th_4(n) = t_4(n) + h_4(n)$, we have

$$\begin{aligned}
& f_4(n+1) \\
= \;& f_4^5(n) - 10f_4^2(n)tr_4^3(n) - 15f_4(n)tr_4^4(n) - 30f_4(n)tq_4^4(n) - 12tr_4^5(n) \\
& +60f_4(n)tr_4^2(n)tq_4^2(n) - 15tr_4^4(n)tp_4(n) + 30tr_4^4(n)th_4(n) \\
& -30tr_4^3(n)tp_4^2(n) + 120tr_4^3(n)tp_4(n)th_4(n) + 140tr_4^3(n)tq_4^2(n)
\end{aligned}$$

$$-120tr_4^3(n)th_4^2(n) + 240tr_4^2(n)tq_4^2(n)tp_4(n) - 480tr_4^2(n)tq_4^2(n)th_4(n)$$
$$+300tr_4(n)tq_4^2(n)tp_4^2(n) - 1200tr_4(n)tq_4^2(n)tp_4(n)th_4(n)$$
$$-180tr_4(n)tq_4^4(n) + 1200tr_4(n)tq_4^2(n)th_4^2(n) - 51tp_4^5(n)$$
$$+510tp_4^4(n)th_4(n) + 260tq_4^2(n)tp_4^3(n) - 2040tp_4^3(n)th_4^2(n)$$
$$-1560tq_4^2(n)tp_4^2(n)th_4(n) + 4080tp_4^2(n)th_4^3(n) - 210tq_4^4(n)tp_4(n)$$
$$+3120tq_4^2(n)tp_4(n)th_4^2(n) - 4080tp_4(n)th_4^4(n) + 420tq_4^4(n)th_4(n)$$
$$-2080tq_4^2(n)th_4^3(n) + 1632th_4^5(n) \ . \tag{49}$$

The other recursion relations for $SG_4(n)$ are too lengthy to be included here. They are available from the authors on request. Some values of $f_4(n)$, $t_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$, $s_4(n)$ are listed in Table 5. These numbers grow exponentially, and do not have simple integer factorizations.

Tab. 5: The first few values of $f_4(n)$, $t_4(n)$, $g_4(n)$, $h_4(n)$, $p_4(n)$, $q_4(n)$, $r_4(n)$, $s_4(n)$.

n	1	2
$f_4(n)$	85,824,132,029	7,035,17,527,028,105,500,700,677,412,563,863,619,648,991,055,157,831,483
$t_4(n)$	3,412,986,435	96,263,552,482,319,683,899,326,687,304,651,572,426,360,843,549,870,965
$g_4(n)$	392,122,089	2,066,883,222,491,708,347,294,489,449,954,683,350,540,164,424,914,435
$h_4(n)$	5,923,774,096	40,841,537,587,690,687,322,887,835,686,137,425,636,710,177,922,212,520
$p_4(n)$	224,652,411	1,952,486,255,633,069,494,764,677,365,066,319,434,639,193,908,980,317
$q_4(n)$	1,740,690,487	19,621,800,909,697,266,778,177,200,667,594,598,639,201,513,851,821,683
$r_4(n)$	693,438,141	12,210,477,454,458,190,580,945,663,798,559,596,025,810,422,699,074,029
$s_4(n)$	1,159,981,779	34,767,376,906,364,680,701,267,847,191,441,347,363,970,403,604,091,693

By a similar argument as in Lemma 3.2, the asymptotic growth constant for the number of spanning forests on $SG_4(n)$ is bounded:

$$\frac{2}{5^{m+1}} \ln s_4(m) < z_{SG_4} < \frac{2}{5^{m+1}} \ln f_4(m) \ , \tag{50}$$

with m a positive integer. We have the following proposition.

Proposition 5.2 *The asymptotic growth constant for the number of spanning forests on the four-dimensional Sierpinski gasket $SG_4(n)$ in the large n limit is $z_{SG_4} = 1.98101707560...$.*

The convergence of the upper and lower bounds is even slower compared with that for SG_3. By the same method as given in the proof of Proposition 3.1, the difference between the upper bound in Eq. (50) and the asymptotic growth constant is bounded:

$$\frac{2}{5^{m+1}} \ln f_4(m) - z_{SG_4} \quad \leq \quad \frac{-1}{2 \times 5^{m+1}} \ln \left(1 - 67 \left[\frac{tr_4(m)}{f_4(m)} \right]^3 \right) . \tag{51}$$

We only have fourteen significant figures for z_{SG_4} with m calculated up to six.

6 Bounds of the asymptotic growth constants

As the spanning tree is a special case of spanning forest where there is only one component, it is clear that the number of spanning trees $N_{ST}(G)$ is always less than $N_{SF}(G)$. Define

$$\underline{z}_G = \lim_{v(G)\to\infty} \frac{\ln N_{ST}(G)}{v(G)} , \tag{52}$$

then $\underline{z}_G < z_G$. We have obtained such asymptotic growth constants for the number of spanning trees on the Sierpinski gasket SG_d for general d and $SG_{2,b}$ with $b = 3, 4$ in Ref. [Chang et al.(2007)]. They serve as the lower bounds for our current consideration for the spanning forests.

By Eq. (3) and a similar argument as in Lemma 3.2, we have the upper bound of the asymptotic growth constant for the number of spanning forests on $SG_d(n)$:

$$z_{SG_d} < \frac{2}{(d+1)^{m+1}} \ln N_{SF}(SG_d(m)) , \tag{53}$$

with m a positive integer. Although the number $N_{SF}(SG_d(m))$ for general m is difficult to obtain, it is known for $m = 0$. We first recall that $SG_d(0)$ at stage zero is a complete graph with $(d+1)$ vertices, each of which is adjacent to all of the other vertices. The number of spanning forests on the complete graph is given by sequence A001858 in Ref. [Sloane]. The first few values of $N_{SF}(SG_d(0))$ are 7, 38, 291, 2932 for d from 2 to 5 [Callan(2003)]. Define

$$\bar{z}_{SG_d} = \frac{2}{d+1} \ln N_{SF}(SG_d(0)) , \tag{54}$$

then $z_{SG_d} < \bar{z}_{SG_d}$. We list the first few values of $\underline{z}_{SG_d}$, z_{SG_d}, $\bar{z}_{SG_d}$ and their ratios in Table 6. Notice that the upper bound is closer to the exact value when d is small, while the lower bound is closer to the exact value when d is large.

For the generalized Sierpinski gasket $SG_{2,b}(n)$ with dimension equal to two, the number of vertices can be calculated to be

$$v(SG_{2,b}(n)) = \frac{b+4}{b+2} \left[\frac{b(b+1)}{2} \right]^n + \frac{2(b+1)}{b+2} . \tag{55}$$

The upper bound of the asymptotic growth constant for the number of spanning forests on $SG_{2,b}(n)$ is given by

$$z_{SG_{2,b}} < \left(\frac{b+2}{b+4} \right) \frac{\ln N_{SF}(SG_{2,b}(m))}{[b(b+1)/2]^m} , \tag{56}$$

Tab. 6: Numerical values of $\underline{z}_{SG_d}$, z_{SG_d}, $\bar{z}_{SG_d}$ and their ratios. The last digits given are rounded off.

d	D	$\underline{z}_{SG_d}$	z_{SG_d}	$\bar{z}_{SG_d}$	$\underline{z}_{SG_d}/z_{SG_d}$	$z_{SG_d}/\bar{z}_{SG_d}$
2	1.585	1.048594857	1.247337199	1.297273433	0.8406667076	0.9615067787
3	2	1.569396409	1.666806281	1.818793080	0.9415589724	0.9164353546
4	2.322	1.914853265	1.981017076	2.269329307	0.9666010902	0.8729526691
5	2.585	2.172764568	-	2.661146688	-	-

Tab. 7: Numerical values of $\underline{z}_{SG_{2,b}}$, $z_{SG_{2,b}}$, $\bar{z}_{SG_{2,b}}$ and their ratios. The last digits given are rounded off.

b	D	$\underline{z}_{SG_{2,b}}$	$z_{SG_{2,b}}$	$\bar{z}_{SG_{2,b}}$	$\underline{z}_{SG_{2,b}}/z_{SG_{2,b}}$	$z_{SG_{2,b}}/\bar{z}_{SG_{2,b}}$
3	1.631	1.133231895	1.312357559	1.389935821	0.8635084908	0.9441857241
4	1.661	1.194401491	1.360516466	1.459432612	0.8779030028	0.9322228754
∞	2	-	-	1.945910149	-	-

with m a positive integer. Although the number $N_{SF}(SG_{2,b}(m))$ for general m is difficult to obtain, it is always equal to seven for stage zero since $SG_{2,b}(0)$ is the equilateral triangle. Define

$$\bar{z}_{SG_{2,b}} = \frac{b+2}{b+4} \ln 7 \, , \tag{57}$$

then $z_{SG_{2,b}} < \bar{z}_{SG_{2,b}}$. We list the first few values of $\underline{z}_{SG_{2,b}}$, $z_{SG_{2,b}}$, $\bar{z}_{SG_{2,b}}$ and their ratios in Table 7. Notice that the upper bound is closer to the exact value when b is small, while the lower bound is closer to the exact value when b is large.

Acknowledgements

The research of S.C.C. was partially supported by the NSC grant NSC-95-2112-M-006-004. The research of L.C.C was partially supported by the NSC grant NSC-95-2115-M-030-002.

A Recursion relations for $SG_3(n)$

We give the recursion relations for the three-dimensional Sierpinski gasket $SG_3(n)$ here. Since the subscript is $d = 3$ for all the quantities throughout this appendix, we will use the simplified notation f_{n+1} to denote $f_3(n+1)$ and similar notations for other quantities. For any non-negative integer n, we have

$$\begin{aligned} f_{n+1} &= f_n^4 - 4f_n[t_n + 2g_n + h_n + p_n]^3 - 3[t_n + 2g_n + h_n + p_n]^4 \\ &\quad +12[t_n + 2g_n + h_n + p_n]^2[t_n + g_n]^2 - 6[t_n + g_n]^4 \, , \end{aligned} \tag{58}$$

$$\begin{aligned} t_{n+1} &= 72t_n^2 p_n[g_n+h_n] + 56t_n[g_n+h_n]^3 + 24t_n^2 p_n^2 + 12t_n p_n[11g_n^2 + 12g_n h_n + h_n^2] \\ &\quad + 12g_n^2[3g_n^2 + 8g_n h_n + 6h_n^2] + 12t_n p_n^2[4g_n + h_n] + 48g_n^2 p_n[g_n+h_n] + 4t_n p_n^3 + 12g_n^2 p_n^2 \ , \end{aligned} \tag{59}$$

$$\begin{aligned} g_{n+1} &= 6t_n^2 q_n[g_n+h_n] + 24t_n^2 p_n^2 + 12t_n p_n[9g_n^2 + 16g_n h_n + 7h_n^2] \\ &\quad + 4g_n[5g_n^3 + 18g_n^2 h_n + 24g_n h_n^2 + 14h_n^3] + 6t_n^2 p_n q_n + 6t_n p_n^2[21g_n + 11h_n] \\ &\quad + 3t_n q_n[5g_n^2 + 6g_n h_n + h_n^2] + 24g_n p_n[5g_n^2 + 9g_n h_n + 4h_n^2] + 6t_n p_n q_n[3g_n + h_n] \\ &\quad + 21t_n p_n^3 + 6g_n p_n^2[19g_n + 13h_n] + 3g_n q_n[3g_n^2 + 4g_n h_n + h_n^2] + 3t_n p_n^2 q_n \\ &\quad + 6g_n p_n q_n[2g_n + h_n] + 25g_n p_n^3 + 3g_n p_n^2 q_n \ , \end{aligned} \tag{60}$$

$$\begin{aligned} h_{n+1} &= 2t_n^2 p_n^2 + 12t_n p_n[g_n^2 + 4g_n h_n + 3h_n^2] + 2[g_n^4 + 8g_n^3 h_n + 18g_n^2 h_n^2 + 16g_n h_n^3 + 11h_n^4] \\ &\quad + 8t_n p_n^2[2g_n + 3h_n] + 8p_n[2g_n^3 + 9g_n^2 h_n + 9g_n h_n^2 + 2h_n^3] + 4t_n p_n^3 \\ &\quad + 2p_n^2[10g_n^2 + 24g_n h_n + 9h_n^2] + 8p_n^3[g_n + h_n] + p_n^4 \ , \end{aligned} \tag{61}$$

$$\begin{aligned} p_{n+1} &= 6t_n^2 p_n q_n + 120t_n p_n^2[g_n+h_n] + 14t_n q_n[g_n+h_n]^2 + 88p_n[g_n+h_n]^3 \\ &\quad + 4t_n p_n q_n[13g_n + 10h_n] + 78t_n p_n^3 + 6p_n^2[49g_n^2 + 78g_n h_n + 29h_n^2] \\ &\quad + 2q_n[11g_n^3 + 26g_n^2 h_n + 19g_n h_n^2 + 4h_n^3] + t_n^2 q_n^2 + 2t_n q_n^2[2g_n + h_n] + 26t_n p_n^2 q_n \\ &\quad + 2p_n q_n[38g_n^2 + 50g_n h_n + 15h_n^2] + 2p_n^3[115g_n + 76h_n] + 2t_n p_n q_n^2 \\ &\quad + q_n^2[4g_n^2 + 4g_n h_n + h_n^2] + 2p_n^2 q_n[31g_n + 18h_n] + 49p_n^4 + 2p_n q_n^2[2g_n + h_n] + 14p_n^3 q_n \\ &\quad + p_n^2 q_n^2 \ , \end{aligned} \tag{62}$$

$$\begin{aligned} q_{n+1} &= 144t_n p_n q_n[g_n+h_n] + 208t_n p_n^3 + 720p_n^2[g_n+h_n]^2 + 56q_n[g_n+h_n]^3 + 24t_n q_n^2[g_n+h_n] \\ &\quad + 252t_n p_n^2 q_n + 24p_n q_n[25g_n^2 + 44g_n h_n + 19h_n^2] + 104p_n^3[17g_n + 15h_n] + 60t_n p_n q_n^2 \\ &\quad + 24q_n^2[3g_n^2 + 5g_n h_n + 2h_n^2] + 12p_n^2 q_n[110g_n + 89h_n] + 972p_n^4 + 4t_n q_n^3 \\ &\quad + 12p_n q_n^2[22g_n + 17h_n] + 776p_n^3 q_n + 4q_n^3[4g_n + 3h_n] + 210p_n^2 q_n^2 + 24p_n q_n^3 + q_n^4 \ . \end{aligned} \tag{63}$$

References

[Benjamini et al.(2001)] I. Benjamini, R. Lyons, Y. Peres and O. Schramm, Special invited paper: uniform spanning forests, *Ann. Probab.*, 29: 1-65, 2001.

[Teranishi(2005)] Y. Teranishi, The number of spanning forests of a graph, *Discrete Math.*, 290: 259-267, 2005.

[Caracciolo et al.(2004)] S. Caracciolo, J. L. Jacobsen, H. Saleur, A. D. Sokal and A. Sportiello, Fermionic Field Theory for Trees and Forests, *Phys. Rev. Lett.*, 93: 080601, 2004.

[Welsh(1993)] D. J. A. Welsh, *Complexity: Knots, Colourings, and Counting (London Math. Soc. Lecture Notes series 186)*, Cambridge University Press, Cambridge, 1993.

[Sokal(2005)] A. D. Sokal, The multivariate Tutte polynomial (alias Potts model) for graphs and matroids, in *Surveys in Combinatorics, 2005*, B. S. Webb, ed., Cambridge University Press, Cambridge, 2005, pp. 173-226.

[Shrock(2000)] R. Shrock, Exact Potts model partition functions on ladder graphs, *Physica A*, 283: 388-446, 2000.

[Chang and Shrock(2000)] S.-C. Chang and R. Shrock, Exact Potts model partition function on strips of the triangular lattice, *Physica A*, 286: 189-238, 2000.

[Chang and Shrock(2001)(a)] S.-C. Chang and R. Shrock, Exact Potts model partition functions on strips of the honeycomb lattice, *Physica A*, 296: 183-233, 2001.

[Chang and Shrock(2001)(b)] S.-C. Chang and R. Shrock, Exact partition function for the Potts model with next-nearest neighbor couplings on arbitrary-length ladders, *Int. J. Mod. Phys. B*, 15: 443-478, 2001.

[Chang and Shrock(2001)(c)] S.-C. Chang and R. Shrock, Exact Potts model partition functions on wider arbitrary-length strips of the square lattice, *Physica A*, 296: 234-288, 2001.

[Jacobsen et al.(2005)] J. L. Jacobsen, J. Salas and A. D. Sokal, Spanning forests and the q-State Potts model in the limit $q \to 0$, *J. Stat. Phys.*, 119: 1153-1281, 2005.

[Deng et al.(2007)] Y. Deng, T. M. Garoni and A. D. Sokal, Ferromagnetic phase transition for spanning forests in three or more dimensions, *Phys. Rev. Lett.*, 98: 030602, 2007.

[Mandelbrot(1982)] B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982.

[Falconer(2003)] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, 2nd ed., Wiley, Chichester, 2003.

[Biggs(1993)] N. L. Biggs, *Algebraic Graph Theory*, 2nd ed., Cambridge University Press, Cambridge, 1993.

[Harary(1969)] F. Harary, *Graph Theory*, Addison-Wesley, New York, 1969.

[Alexander(1995)] K. S. Alexander, Percolation and minimal spanning forests in infinite graphs, *Ann. Probab.*, 23: 87-104, 1995.

[Burton and Pemantle(1993)] R. Burton and R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances, *Ann. Probab.*, 21: 1329-1371, 1993.

[Lyons(2005)] R. Lyons, Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.*, 14: 491-522, 2005.

[Gefen and Aharony(1981)] Y. Gefen and A. Aharony, Solvable fractal family, and its possible relation to the backbone at percolation, *Phys. Rev. Lett.*, 47: 1771-1774, 1981.

[Hilfer and Blumen(1984)] R. Hilfer and A. Blumen, Renormalisation on Sierpinski-type fractals, *J. Phys. A: Math. Gen.*, 17: L537-L545, 1984.

[Chang et al.(2007)] S.-C. Chang, L.-C. Chen and W.-S. Yang, Spanning trees on the Sierpinski gasket, *J. Stat. Phys.*, 126: 649-667, 2007.

[Sloane] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*. Published electronically at `http://www.research.att.com/∼njas/sequences/`.

[Callan(2003)] D. Callan, A combinatorial derivation of the number of labeled forests, *J. Integer Sequences*, 6: Article 03.4.7. 2003.