Multi-order covariance computation for estimates in stochastic subspace identification using QR decompositions

Michael Döhler, Xuan-Binh Lam, Laurent Mevel

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Michael Döhler Xuan-Binh Lam Laurent Mevel

Inria, Campus de Beaulieu, 35042 Rennes, France (e-mail: michael.doehler@inria.fr).

Abstract: For applications as Operational Modal Analysis (OMA) of vibrating structures, an output-only LTI system with state and measurement noise can be identified using subspace methods. While these identification techniques have been very suitable for the identification of such mechanical, aeronautical or civil structures, covariance expressions of the estimates of the system matrices are difficult to obtain and theoretical results from literature are hard to implement for output-only systems with unknown noise properties in practice. Moreover, the model order of the underlying system is generally unknown and due to noise and model errors, usual statistical criteria cannot be used. Instead, the system is estimated at multiple model orders and some GUI driven stabilization diagram containing the resulting modal parameters is used by the structural engineer. Then, the covariance of the estimates at these different model orders is an important information for the engineer, which, however, would be computationally expensive to obtain with the existing tools. Recently a fast multi-order version of the stochastic subspace identification approach has been proposed, which is based on the use of the QR decomposition of the observability matrix at the largest model order. In this paper, the corresponding covariance expressions for the system matrix estimates at multiple model orders are derived and successfully applied on real vibration data.

Keywords: Subspace methods, ambient noise, uncertainty in linear systems, perturbation analysis, mechanical systems

1. INTRODUCTION

The design and maintenance of mechanical or civil structures subject to noise and vibrations is an important topic in structural engineering. Laboratory and in-operation tests are performed on structures for modal analysis, where modal models are identified containing the vibration modes (frequencies, damping ratios, mode shapes) related to the poles and observed eigenvectors of a LTI system. Subspace-based linear system identification methods have been proven efficient for their identification from output-only measurements (Van Overschee and De Moor, 1996; Peeters and De Roeck, 1999; Döhler and Mevel, 2012b).

In operational modal analysis, the true system order is in general unknown. Moreover, the identified models do not only contain modes of the investigated structure, but also modes due to noise. The system order needs to be over-specified in order to retrieve all modes, especially when using data with a poor signal-to-noise ratio. This causes a number of spurious modes to appear in the identified models. Then, the identification procedure is repeated while truncating at different model orders, as spurious modes tend to vary at different model orders, while physical modes remain quite constant (Peeters and De Roeck, 1999, 2001). Like this, both kinds of modes can be distinguished using so-called *stabilization diagrams*. In (Döhler and Mevel, 2012a), an efficient algorithm has been proposed to estimate the system matrices from subspace identification at these multiple model orders, reducing the computational burden for this problem significantly. This approach is based on a pertinent use of the QR decomposition and the shift invariance property of the estimated observability matrix.

The estimated system matrices and modal parameters are afflicted with statistical uncertainty due to finite data, unknown inputs and noise properties. In-depth covariance analyses for subspace methods have been made in literature (Bauer et al., 1999; Chiuso and Picci, 2004; Bauer, 2005), which are of important theoretical value. However, the evaluation of the covariance expressions in practice is a challenge, since they require amongst others covariance expressions of the unknown state and output noise to be known, which are not simply available from output-only measurements. A different approach is pursued in (Pintelon et al., 2007), where it has been shown how covariances of estimated parameters can be computed by a sensitivity analysis, propagating a first-order perturbation from the data to the identified parameters. In (Reynders et al., 2008; Döhler and Mevel, 2013), details of this scheme are given for the covariance computation of estimates from stochastic subspace identification. This combined empirical/analytical approach has the advantage that the computed empirical covariance is easy to obtain from the data and the analytical propagation to the parameters is

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straightforward, requiring only one system identification step. In contrast, a purely empirical covariance estimate of the identified parameters would require many system identification steps and is an algorithmic challenge, since modes from different data sets would have to be matched.

In this paper, we extend the covariance expressions to the QR-based solution of the state transition matrix in subspace identification, which is the basis for the fast multi-order estimation of the system matrices in (Döhler and Mevel, 2012a). This development is based on a perturbation analysis of the QR decomposition (Chang et al., 1997; Chang and Paige, 2001). Then, an efficient computation scheme for the covariance computation of the system matrix estimates at multiple model orders is derived.

This paper is organized as follows. In Section 2, the underlying subspace algorithm is recalled. Then sensitivities for the QR-based solution of the state transition matrix are derived in Section 3. In Section 4, the computation of the system matrices at multiple model orders is detailed and an efficient covariance computation scheme at multiple model orders is derived. In Section 5, the new algorithm is applied to vibration data of a civil structure.

2. STOCHASTIC SUBSPACE IDENTIFICATION (SSI)

We consider the discrete-time state space model

$$\begin{cases} x_{k+1} = Ax_k + v_k \\ y_k = Cx_k + w_k \end{cases} \quad (1)$$

with the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^r$, the state transition matrix $A \in \mathbb{R}^{n\times n}$ and the observation matrix $C \in \mathbb{R}^{r\times n}$, where n is the model order and r the number of outputs. The state noise v and output noise w are unmeasured and assumed to be Gaussian, zero-mean, white.

From the output data, a matrix $\mathcal{H} \in \mathbb{R}^{(p+1)r\times N_c}$ is built according to a chosen SSI algorithm, see e.g. (Van Overschee and De Moor, 1996; Benveniste and Mevel, 2007) for an overview. The parameter N_c depends on the algorithm and parameter p is chosen such that $\min\{pr, N_c\} \geq n$. The matrix $\mathcal{H}$ will be called "subspace matrix" in the following, and the SSI algorithm is chosen such that the corresponding subspace matrix enjoys (asymptotically for a large number of samples) the factorization property

$$\mathcal{H} = \mathcal{O}\ \mathcal{Z}$$

into the matrix of observability

$$\mathcal{O} \stackrel{\text{def}}{=} \left[C^T\ (CA)^T \dots (CA^p)^T\right]^T$$

and a matrix $\mathcal{Z}$ depending on the selected SSI algorithm.

The observability matrix $\mathcal{O}$ is obtained from a thin Singular Value Decomposition (SVD) of the matrix $\mathcal{H}$ and its truncation at the desired model order n:

$$\mathcal{H} = U\Sigma V^T = [U_1\ U_0]\begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_0\end{bmatrix}V^T, \qquad \mathcal{O} = U_1\Sigma_1^{1/2}. \quad (2)$$

Then, the observation matrix C is found in the first block-row of the observability matrix $\mathcal{O}$. The state transition matrix A is obtained from the shift invariance property of $\mathcal{O}$, namely as the least squares solution of

$$\mathcal{O}^{\uparrow}A = \mathcal{O}^{\downarrow}, \quad (3)$$

where $\mathcal{O}^{\uparrow} = \mathcal{S}_1\mathcal{O}$, $\mathcal{O}^{\downarrow} = \mathcal{S}_2\mathcal{O}$ with the selection matrices

$$\mathcal{S}_1 \stackrel{\text{def}}{=} [0_{pr\times r}\ \ I_{pr}], \quad \mathcal{S}_2 \stackrel{\text{def}}{=} [I_{pr}\ \ 0_{pr\times r}]$$

and the identity and zero matrices I_a and $0_{a\times b}$ of size $a\times a$ and $a\times b$, respectively.

There are several ways to obtain the least squares solution of (3), e.g. by using the pseudoinverse

$$A = (\mathcal{O}^{\uparrow})^{\dagger}\mathcal{O}^{\downarrow} \quad (4)$$

or with the thin QR decomposition of $\mathcal{O}^{\uparrow}$,

$$\mathcal{O}^{\uparrow} = QR, \quad A = R^{-1}Q^T\mathcal{O}^{\downarrow}. \quad (5)$$

The latter solution is used for the fast multi-order estimation of the system matrices in (Döhler and Mevel, 2012a), which is detailed in Section 4.1.

Finally, the eigenstructure $(\lambda_i, \varphi_i)_{i=1,\dots,n}$ with the eigenvalues λ_i and observed eigenvectors φ_i yields

$$\det(A-\lambda_i I) = 0, \quad A\phi_i = \lambda_i\phi_i, \quad \varphi_i = C\phi_i, \quad (6)$$

and forms a canonical parameterization of system (1). It is the parameterization of interest for operational modal analysis, since it contains the modal parameters (frequencies f_i, damping ratios ξ_i and mode shapes φ_i) with

$$f_i = \sqrt{a_i^2+b_i^2}/(2\pi\tau), \quad \xi_i = -100b_i/\sqrt{a_i^2+b_i^2}, \quad (7)$$

where $a_i \stackrel{\text{def}}{=} |\arctan\Im(\lambda_i)/\Re(\lambda_i)|$, $b_i \stackrel{\text{def}}{=} \ln|\lambda_i|$, and $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary part of a complex number.

3. UNCERTAINTY QUANTIFICATION

In this section, the covariance computation of the system matrices and subsequently of the eigenstructure is briefly recalled from (Reynders et al., 2008), where the state transition matrix A is obtained using the pseudoinverse solution from (4). This computation is then extended to the QR-based least squares solution from (5).

3.1 Covariance Computations on Estimates from Subspace Identification

The covariance estimates are obtained through the propagation of first order perturbations. Let ΔX be a first-order perturbation of a vector-valued variable X. Then, for a function $Y = f(X)$ it holds $\Delta Y = \mathcal{J}_Y\,\Delta X$, where $\mathcal{J}_Y = \partial f(X)/\partial X$. Subsequently, covariance expressions for the estimates satisfy $\text{cov}(\widehat{Y}) \approx \widehat{\mathcal{J}}_Y\text{cov}(\widehat{X})\widehat{\mathcal{J}}_Y^T$. For simplicity of notation we dismiss the notation $\widehat{\ }$ for an estimate in the following.

In the context of the subspace identification algorithm from Section 2, a perturbation $\Delta\mathcal{H}$ of the subspace matrix $\mathcal{H}$ is propagated to perturbations on the observability matrix $\Delta\mathcal{O}$, then to perturbations on the system matrices ΔA and ΔC, and finally to perturbations on the eigenvalues $\Delta\lambda_i$ and observed eigenvectors $\Delta\varphi_i$ by a sensitivity analysis in (Reynders et al., 2008). Following from the relations (2)–(3), it holds

$$\text{vec}(\Delta\mathcal{O}) = \mathcal{J}_{\mathcal{O}}\,\text{vec}(\Delta\mathcal{H}),$$

$$\text{vec}(\Delta A) = \mathcal{J}_A\,\text{vec}(\Delta\mathcal{O}), \quad \text{vec}(\Delta C) = \mathcal{J}_C\,\text{vec}(\Delta\mathcal{O}),$$

with the sensitivities $\mathcal{J}_{\mathcal{O}}$, $\mathcal{J}_A$ (referring to the computation in (4)) and $\mathcal{J}_C$ derived in (Reynders et al., 2008), and the vectorization operator vec. Then, perturbations of the eigenstructure follow as

$$\Delta\lambda_i = \mathcal{J}_{\lambda_i}\text{vec}(\Delta A), \quad \Delta\varphi_i = \mathcal{J}_{\varphi_i}\begin{bmatrix}\text{vec}(\Delta A)\\ \text{vec}(\Delta C)\end{bmatrix},$$

where $\mathcal{J}_{\lambda_i}$ and $\mathcal{J}_{\varphi_i}$ are the respective sensitivities (Reynders et al., 2008). Finally, the perturbations of the modal parameters frequency and damping ratio follow from (7) as

$$\begin{bmatrix}\Delta f_i \\ \Delta \xi_i\end{bmatrix} = \begin{bmatrix}\mathcal{J}_{f_i} \\ \mathcal{J}_{\xi_i}\end{bmatrix} \begin{bmatrix}\Re(\Delta\lambda_i) \\ \Im(\Delta\lambda_i)\end{bmatrix}$$

with the respective sensitivities $\mathcal{J}_{f_i}$ and $\mathcal{J}_{\xi_i}$.

Let $\operatorname{cov}(\operatorname{vec}(\mathcal{H}))$ be the covariance of the vectorized subspace matrix. Its empirical estimate can be easily obtained by partitioning the sensor data into blocks on which instances of the subspace matrix $\mathcal{H}$ are computed. Then, the covariance of the vectorized system matrices can be obtained as

$$\operatorname{cov}(\operatorname{vec}(A)) = \mathcal{J}_A \; \mathcal{J}_{\mathcal{O}} \; \operatorname{cov}(\operatorname{vec}(\mathcal{H})) \; \mathcal{J}_{\mathcal{O}}^T \; \mathcal{J}_A^T, \tag{8}$$

$$\operatorname{cov}(\operatorname{vec}(C)) = \mathcal{J}_C \; \mathcal{J}_{\mathcal{O}} \; \operatorname{cov}(\operatorname{vec}(\mathcal{H})) \; \mathcal{J}_{\mathcal{O}}^T \; \mathcal{J}_C^T, \tag{9}$$

and the covariance of the modal parameters is obtained as

$$\begin{aligned}\operatorname{cov}\left(\begin{bmatrix}f_i \\ \xi_i\end{bmatrix}\right) = \begin{bmatrix}\mathcal{J}_{f_i} \\ \mathcal{J}_{\xi_i}\end{bmatrix} \begin{bmatrix}\Re(\mathcal{J}_{\lambda_i}) \\ \Im(\mathcal{J}_{\lambda_i})\end{bmatrix} \; \mathcal{J}_A \; \mathcal{J}_{\mathcal{O}} \; \operatorname{cov}(\operatorname{vec}(\mathcal{H})) \\ \cdot \; \mathcal{J}_{\mathcal{O}}^T \; \mathcal{J}_A^T \; \begin{bmatrix}\Re(\mathcal{J}_{\lambda_i}) \\ \Im(\mathcal{J}_{\lambda_i})\end{bmatrix}^T \begin{bmatrix}\mathcal{J}_{f_i} \\ \mathcal{J}_{\xi_i}\end{bmatrix}^T\end{aligned} \tag{10}$$

$$\begin{aligned}\operatorname{cov}\left(\begin{bmatrix}\Re(\varphi_i) \\ \Im(\varphi_i)\end{bmatrix}\right) = \begin{bmatrix}\Re(\mathcal{J}_{\varphi_i}) \\ \Im(\mathcal{J}_{\varphi_i})\end{bmatrix} \; \mathcal{J}_{A,C} \; \mathcal{J}_{\mathcal{O}} \; \operatorname{cov}(\operatorname{vec}(\mathcal{H})) \\ \cdot \; \mathcal{J}_{\mathcal{O}}^T \; \mathcal{J}_{A,C}^T \; \begin{bmatrix}\Re(\mathcal{J}_{\varphi_i}) \\ \Im(\mathcal{J}_{\varphi_i})\end{bmatrix}^T\end{aligned} \tag{11}$$

where $\mathcal{J}_{A,C} \stackrel{\text{def}}{=} [\mathcal{J}_A^T \;\; \mathcal{J}_C^T]^T$.

3.2 QR-Based Sensitivity Computation of A

The sensitivity $\mathcal{J}_A$ of the system matrix A is now derived for the least squares solution of A from (5) using the QR decomposition.

Define the permutation matrix

$$\mathcal{P}_{a,b} \stackrel{\text{def}}{=} \sum_{k=1}^{a}\sum_{l=1}^{b} E_{k,l}^{a,b} \otimes E_{l,k}^{b,a},$$

where $E_{k,l}^{a,b} \in \mathbb{R}^{a\times b}$ are matrices which are equal to 1 at entry (k,l) and zero elsewhere, and $\otimes$ denotes the Kronecker product. Matrix $\mathcal{P}_{a,b}$ yields the relation

$$\mathcal{P}_{a,b} \operatorname{vec}(X) = \operatorname{vec}(X^T) \tag{12}$$

for any matrix $X \in \mathbb{R}^{a\times b}$ (Pintelon et al., 2007).

Lemma 1. Let the QR decomposition $\mathcal{O}^{\uparrow} = QR$ and $S \stackrel{\text{def}}{=} Q^T\mathcal{O}^{\downarrow}$ be given, such that $A = R^{-1}S$. Then,

$$\operatorname{vec}(\Delta R) = \mathcal{J}_R \operatorname{vec}(\Delta\mathcal{O}), \quad \operatorname{vec}(\Delta S) = \mathcal{J}_S \operatorname{vec}(\Delta\mathcal{O}),$$

where

$$\begin{aligned}\mathcal{J}_R &\stackrel{\text{def}}{=} (R^T \otimes I_n)\mathcal{U}, \\ \mathcal{J}_S &\stackrel{\text{def}}{=} (\mathcal{O}^{\downarrow T} \otimes I_n)\mathcal{P}_{pr,n}\left((R^{-T}\otimes\mathcal{S}_1) - (I_n\otimes Q)\mathcal{U}\right) \\ &\quad + (I_n \otimes Q^T\mathcal{S}_2) \\ \mathcal{U} &\stackrel{\text{def}}{=} (\mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_4\mathcal{P}_{n,n})(R^{-T}\otimes Q^T\mathcal{S}_1),\end{aligned}$$

and the selection matrices $\mathcal{S}_3$ and $\mathcal{S}_4$ are defined as

$$\mathcal{S}_3 \stackrel{\text{def}}{=} \sum_{k=1}^{n} E_{(k-1)n+k,(k-1)n+k}^{n^2}, \tag{13}$$

$$\mathcal{S}_4 \stackrel{\text{def}}{=} \sum_{k_1=1}^{n-1}\sum_{k_2=1}^{k_1} E_{k_1n+k_2,k_1n+k_2}^{n^2}. \tag{14}$$

Proof. In (Chang and Paige, 2001), the perturbations for the QR decomposition $\mathcal{O}^{\uparrow} = QR$ are given as

$$\Delta R = UR, \quad \Delta Q = \Delta\mathcal{O}^{\uparrow}R^{-1} - QU, \tag{15}$$

where $U \stackrel{\text{def}}{=} \operatorname{up}(V + V^T)$ with $V \stackrel{\text{def}}{=} Q^T\Delta\mathcal{O}^{\uparrow}R^{-1}$, and the operator $\operatorname{up}(X)$ is defined for any matrix $X \in \mathbb{R}^{n\times n}$

$$X = \begin{bmatrix}X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \dots & X_{nn}\end{bmatrix}$$

such that

$$\operatorname{up}(X) = \begin{bmatrix}\frac{1}{2}X_{11} & X_{12} & \dots & X_{1n} \\ 0 & \frac{1}{2}X_{22} & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \frac{1}{2}X_{nn}\end{bmatrix}.$$

First, consider $\operatorname{vec}(U)$. The selection matrices $\mathcal{S}_3$ and $\mathcal{S}_4$ in (13) and (14) are defined such that $\mathcal{S}_3\operatorname{vec}(V)$ selects the diagonal elements of V and $\mathcal{S}_4\operatorname{vec}(V)$ selects the strictly upper triangular elements of V, while setting all other elements to 0 in both cases. Then,

$$\begin{aligned}\operatorname{vec}(U) &= \operatorname{vec}(\operatorname{up}(V + V^T)) \\ &= \mathcal{S}_3\operatorname{vec}(V) + \mathcal{S}_4\operatorname{vec}(V + V^T) \\ &= (\mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_4\mathcal{P}_{n,n})\operatorname{vec}(V),\end{aligned}$$

using (12) in the last step. Vectorizing $V = Q^T\mathcal{S}_1\Delta\mathcal{O}\,R^{-1}$ yields

$$\begin{aligned}\operatorname{vec}(U) &= (\mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_4\mathcal{P}_{n,n})(R^{-T}\otimes Q^T\mathcal{S}_1)\operatorname{vec}(\Delta\mathcal{O}) \\ &= \mathcal{U}\operatorname{vec}(\Delta\mathcal{O}).\end{aligned} \tag{16}$$

Then, $\operatorname{vec}(\Delta R) = (R^T\otimes I_n)\mathcal{U}\operatorname{vec}(\Delta\mathcal{O}) = \mathcal{J}_R\operatorname{vec}(\Delta\mathcal{O})$ follows from (15).

Now, consider $S = Q^T\mathcal{O}^{\downarrow}$. It follows

$$\begin{aligned}\Delta S &= \Delta Q^T\mathcal{O}^{\downarrow} + Q^T\,\Delta\mathcal{O}^{\downarrow} \\ &= \Delta Q^T\mathcal{O}^{\downarrow} + Q^T\mathcal{S}_2\,\Delta\mathcal{O}, \\ \operatorname{vec}(\Delta S) &= (\mathcal{O}^{\downarrow T}\otimes I_n)\operatorname{vec}(\Delta Q^T) \\ &\quad + (I_n\otimes Q^T\mathcal{S}_2)\operatorname{vec}(\Delta\mathcal{O}).\end{aligned} \tag{17}$$

From (12) follows $\operatorname{vec}(\Delta Q^T) = \mathcal{P}_{pr,n}\operatorname{vec}(\Delta Q)$ and from (15) follows

$$\operatorname{vec}(\Delta Q) = (R^{-T}\otimes\mathcal{S}_1)\operatorname{vec}(\Delta\mathcal{O}) - (I_n\otimes Q)\operatorname{vec}(U).$$

Plugging this and (16) into (17) leads to

$$\begin{aligned}\operatorname{vec}(\Delta S) &= (\mathcal{O}^{\downarrow T}\otimes I_n)\mathcal{P}_{pr,n}\left((R^{-T}\otimes\mathcal{S}_1)\operatorname{vec}(\Delta\mathcal{O})\right. \\ &\quad \left. -(I_n\otimes Q)\operatorname{vec}(U)\right) + (I_n\otimes Q^T\mathcal{S}_2)\operatorname{vec}(\Delta\mathcal{O}) \\ &= \Big((\mathcal{O}^{\downarrow T}\otimes I_n)\mathcal{P}_{pr,n}\left((R^{-T}\otimes\mathcal{S}_1) - (I_n\otimes Q)\mathcal{U}\right) \\ &\quad + (I_n\otimes Q^T\mathcal{S}_2)\Big)\operatorname{vec}(\Delta\mathcal{O})\end{aligned}$$

from where the assertion follows.

With these results, the sensitivity of A for a QR-based computation from the observability matrix can be obtained, yielding $\text{vec}(\Delta A) = \mathcal{J}_A \,\text{vec}(\Delta\mathcal{O})$.

Proposition 2. Let $\mathcal{J}_R$ and $\mathcal{J}_S$ be given in Lemma 1. Then, the sensitivity $\mathcal{J}_A$ can be written as

$$\mathcal{J}_A = -(A^T \otimes R^{-1})\,\mathcal{J}_R + (I_n \otimes R^{-1})\,\mathcal{J}_S.$$

Proof. From $A = R^{-1}S$ it follows for a perturbation of A

$$\begin{aligned}\Delta A &= -R^{-1}\Delta R\, R^{-1}S + R^{-1}\Delta S\\ &= -R^{-1}\Delta R\, A + R^{-1}\Delta S,\\ \text{vec}(\Delta A) &= -(A^T \otimes R^{-1})\,\text{vec}(\Delta R)\\ &\quad +(I_n \otimes R^{-1})\,\text{vec}(\Delta S)\end{aligned}$$

and the assertion follows from Lemma 1.

4. MULTI-ORDER SYSTEM IDENTIFICATION AND UNCERTAINTY QUANTIFICATION

In applications as modal analysis of vibrating structures (Peeters and De Roeck, 1999), vibration modes related to the eigenstructure of system (1) are extracted. The true system order is in general unknown when data are recorded under operational conditions, where the whiteness assumption of the input and output noise is often violated. Recommended techniques from statistics to estimate the best model order (Bauer, 2001) often do not help in this case, as the estimated models are combined models containing modes corresponding to the structure *and* the noise, and one is rather interested in distinguishing both kinds of modes. Due to the noise, a larger model order must be assumed in order to retrieve a desired number of physical modes of a structure. It is common practice to do system identification at different successive model orders $n = n_{\min}, \ldots, n_{\max}$ in order to distinguish the identified physical system modes from spurious noise modes, as the latter tend to vary at different orders (Peeters and De Roeck, 1999, 2001).

In this section, an efficient computation scheme of the system matrices A and C at multiple model orders is recalled from (Döhler and Mevel, 2012a), which is based on the QR decomposition (5) for the least squares problem for A. With this scheme, the main part of the computation is done only once at model order $n_{\max}$, while the system identification results at all inferior model orders n follow very fast. Then, the uncertainty computation from Section 3.2 is adapted to multiple model orders, where the computation at some lower order n is related to results obtained at order $n_{\max}$, leading to an efficient computation scheme.

The following notation is used. The index n of the matrices A_n, C_n, $\mathcal{O}_n$, $\mathcal{O}_n^\uparrow$ and $\mathcal{O}_n^\downarrow$ denotes the matrices A, C, $\mathcal{O}$, $\mathcal{O}^\uparrow$ and $\mathcal{O}^\downarrow$ at model order n, where $n \in \{n_{\min}, \ldots, n_{\max}\}$.

4.1 Computation of the System Matrices at Multiple Model Orders

Let the observability matrix $\mathcal{O}_{n_{\max}}$ at some maximal model order $n_{\max}$ be recovered from the truncation of the SVD in (2) at this order. Then, $\mathcal{O}_n$ at order n consists of the first n columns of $\mathcal{O}_{n_{\max}}$ due to the truncation in (2) with

$$\mathcal{O}_n = \mathcal{O}_{n_{\max}}\mathcal{T}_n \quad \text{where} \quad \mathcal{T}_n \overset{\text{def}}{=} \begin{bmatrix} I_n \\ 0_{n_{\max}-n,n} \end{bmatrix}. \tag{18}$$

At each system order n, the state transition matrix A_n is solution of the least squares problem

$$\mathcal{O}_n^\uparrow A_n = \mathcal{O}_n^\downarrow, \tag{19}$$

whose solution is given by (5) with the QR decomposition

$$\mathcal{O}_n^\uparrow = Q_n R_n, \quad S_n = Q_n^T \mathcal{O}_n^\downarrow, \quad A_n = R_n^{-1} S_n. \tag{20}$$

with $Q_n \in \mathbb{R}^{pr\times n}$ a matrix with orthogonal columns, $R_n \in \mathbb{R}^{n\times n}$ upper triangular and $S_n \in \mathbb{R}^{n\times n}$. R_n is assumed to be of full rank, which is reasonable as $\mathcal{O}_n$ is of full column rank.

The computation of the state transition matrices at these multiple orders is a significant computational burden. In (Döhler and Mevel, 2012a) an algorithm is presented that computes the QR decomposition for the least squares solution only once at the maximal desired model order $n_{\max}$, leading to matrices $R_{n_{\max}}$, $S_{n_{\max}}$ and $A_{n_{\max}}$. Then, instead of solving the least squares problems at all inferior orders $n < n_{\max}$, it is shown in (Döhler and Mevel, 2012a) that the state transition matrices A_n at these lower orders can be computed much more efficiently from submatrices of $R_{n_{\max}}$ and $S_{n_{\max}}$ as follows.

Theorem 3. (Döhler and Mevel (2012a)). Let $\mathcal{O}_{n_{\max}}$, $Q_{n_{\max}}$, $R_{n_{\max}}$ and $S_{n_{\max}}$ be given with

$$\mathcal{O}_{n_{\max}}^\uparrow = Q_{n_{\max}} R_{n_{\max}}, \quad S_{n_{\max}} = Q_{n_{\max}}^T \mathcal{O}_{n_{\max}}^\downarrow, \tag{21}$$

such that $A_{n_{\max}} = R_{n_{\max}}^{-1} S_{n_{\max}}$ is the least squares solution of

$$\mathcal{O}_{n_{\max}}^\uparrow A_{n_{\max}} = \mathcal{O}_{n_{\max}}^\downarrow.$$

Let $n < n_{\max}$, and let $R_{n_{\max}}$ and $S_{n_{\max}}$ be partitioned into

$$R_{n_{\max}} = \begin{bmatrix} R_n^{(11)} & R_n^{(12)} \\ 0 & R_n^{(22)} \end{bmatrix}, \quad S_{n_{\max}} = \begin{bmatrix} S_n^{(11)} & S_n^{(12)} \\ S_n^{(21)} & S_n^{(22)} \end{bmatrix},$$

where $R_n^{(11)}, S_n^{(11)} \in \mathbb{R}^{n\times n}$. Then, the state transition matrix A_n at model order n, which is the least squares solution of (19), satisfies

$$A_n = (R_n^{(11)})^{-1} S_n^{(11)}.$$

Using this theorem, the number of numerical operations is reduced significantly for the computation of A_n, as R_n and S_n do not have to be computed from the QR decomposition in (20). Instead, they can be simply selected as $R_n = R_n^{(11)}$ and $S_n = S_n^{(11)}$ from $R_{n_{\max}}$ and $S_{n_{\max}}$, respectively, once they are computed at the maximal model order.

The observation matrix C_n is obtained from the first block row of the observability matrix $\mathcal{O}_n$ and it follows from (18)

$$C_n = C_{n_{\max}}\mathcal{T}_n. \tag{22}$$

4.2 Multi-Order Uncertainty Quantification of System Matrices

The relation between the sensitivities $\mathcal{J}_{R_n}$ and $\mathcal{J}_{S_n}$ at lower model orders n and $\mathcal{J}_{R_{n_{\max}}}$ and $\mathcal{J}_{S_{n_{\max}}}$ at the maximal model order $n_{\max}$ is now obtained, based on Theorem 3.

Proposition 4. Let the sensitivities $\mathcal{J}_{R_{n_{\max}}}$ and $\mathcal{J}_{S_{n_{\max}}}$ of $R_{n_{\max}}$ and $S_{n_{\max}}$ be given in Lemma 1 at model order $n_{\max}$, and let R_n and S_n be submatrices of $R_{n_{\max}}$ and $S_{n_{\max}}$ in Theorem 3, such that $A_n = R_n^{-1}S_n$. Then, the sensitivities of R_n and S_n can be written as

$$\mathcal{J}_{R_n} = (\mathcal{T}_n^T \otimes \mathcal{T}_n^T)\mathcal{J}_{R_{n_{\max}}}, \quad \mathcal{J}_{S_n} = (\mathcal{T}_n^T \otimes \mathcal{T}_n^T)\mathcal{J}_{S_{n_{\max}}},$$

such that

$$\mathrm{vec}(\Delta R_n) = \mathcal{J}_{R_n}\mathrm{vec}(\Delta\mathcal{O}_n),\ \ \mathrm{vec}(\Delta S_n) = \mathcal{J}_{S_n}\mathrm{vec}(\Delta\mathcal{O}_n).$$

Proof. From Theorem 3 follow the relations

$$R_n = \mathcal{T}_n^T R_{n_{\max}} \mathcal{T}_n,\quad S_n = \mathcal{T}_n^T S_{n_{\max}} \mathcal{T}_n,$$

and thus

$$\mathrm{vec}(\Delta R_n) = (\mathcal{T}_n^T \otimes \mathcal{T}_n^T)\,\mathrm{vec}(\Delta R_{n_{\max}}),$$
$$\mathrm{vec}(\Delta S_n) = (\mathcal{T}_n^T \otimes \mathcal{T}_n^T)\,\mathrm{vec}(\Delta S_{n_{\max}}),$$

from where the assertion follows.

Note that the matrix $(\mathcal{T}_n^T \otimes \mathcal{T}_n^T)$ in Proposition 4 is a selection matrix. No further computations are necessary to obtain $\mathcal{J}_{R_n}$ and $\mathcal{J}_{S_n}$, once their counterparts at the maximal order $n_{\max}$ are computed.

Proposition 5. With the notation of Proposition 4, it holds

$$\mathcal{J}_{A_n} = -(A_n^T\mathcal{T}_n^T \otimes R_n^{-1}\mathcal{T}_n^T)\,\mathcal{J}_{R_{n_{\max}}} + (\mathcal{T}_n^T \otimes R_n^{-1}\mathcal{T}_n^T)\,\mathcal{J}_{S_{n_{\max}}}$$

and $\mathcal{J}_{C_n} = (\mathcal{T}_n^T \otimes I_r)\mathcal{J}_{C_{n_{\max}}}$, such that

$$\mathrm{vec}(\Delta A_n) = \mathcal{J}_{A_n}\mathrm{vec}(\Delta\mathcal{O}_n),\quad \mathrm{vec}(\Delta C_n) = \mathcal{J}_{C_n}\mathrm{vec}(\Delta\mathcal{O}_n).$$

Proof. The first part of the assertion follows from Propositions 2 and 4, and the second part from (22).

Finally, the efficient covariance computation of the system matrices and subsequently of the eigenstructure and modal parameters at multiple model orders can be summarized as follows:

- Compute $R_{n_{\max}}$ and $S_{n_{\max}}$ at maximal model order $n_{\max}$ in (21), as well as $C_{n_{\max}}$;
- Compute sensitivities $\mathcal{J}_{\mathcal{O}_{n_{\max}}}$, $\mathcal{J}_{C_{n_{\max}}}$ (both from (Reynders et al., 2008)), $\mathcal{J}_{R_{n_{\max}}}$ and $\mathcal{J}_{S_{n_{\max}}}$ (both from Lemma 1);
- Compute $\mathrm{cov}(\mathrm{vec}(\mathcal{H}))$;
- For each desired model order $n \le n_{\max}$:
  - Compute A_n from $R_{n_{\max}}$ and $S_{n_{\max}}$ in Theorem 3, C_n in (22);
  - Compute $\mathcal{J}_{A_n}$ and $\mathcal{J}_{C_n}$ in Proposition 5 and the modal parameters in (6)–(7);
  - Obtain the covariance of the vectorized system matrices in (8)–(9), or the covariance of the modal parameters in (10)–(11).

Using the same assumptions as in (Döhler and Mevel, 2013) for the evaluation of the computational complexity of the algorithm, it can be shown that the entire covariance computation of the modal parameters at the maximal model order is of order $O(n_{\max}^4)$ in an appropriate implementation. This would mean a complexity of $O(n_{\max}^5)$ for all model orders $n = 1, \ldots, n_{\max}$ in a direct implementation, while it can be shown that the efficient multi-order algorithm in this section only requires $O(n_{\max}^4)$ operations for all model orders. Hence, the complexity is the same as the comparable algorithm in (Döhler and Mevel, 2013), while the new algorithm in this paper fills the gap for the QR based least squares solution for the system matrices and is thus adapted to the covariance estimation for the fast multi-order estimation of the system matrices from (Döhler and Mevel, 2012a). Moreover, the multi-order computation with the new algorithm is far less cumbersome and more intuitive than in (Döhler and Mevel, 2013).

5. APPLICATION

An important instance of eigenstructure identification with subspace-based system identification is structural vibration analysis (Peeters and De Roeck, 1999).

5.1 Structural Vibration Analysis

The behavior of a vibrating mechanical structure is described by a continuous-time, time-invariant, linear dynamical system

$$\mathcal{M}\ddot{\mathcal{X}}(t) + \mathcal{C}\dot{\mathcal{X}}(t) + \mathcal{K}\mathcal{X}(t) = \upsilon(t) \tag{23}$$

where t denotes continuous time; $\mathcal{M}, \mathcal{C}, \mathcal{K} \in \mathbb{R}^{m\times m}$ are mass, damping, and stiffness matrices, respectively; the (high dimensional) state vector $\mathcal{X}(t) \in \mathbb{R}^m$ is the displacement vector of the m degrees of freedom of the structure; and $\upsilon(t)$ is the external unmeasured force (noise). Observing system (23) at r sensor positions (e.g. acceleration or displacement sensors) at discrete time instants $t = k\tau$ for a sampling rate $1/\tau$, it can be transformed to an equivalent discrete-time state space system (1) with model order $n = 2m$. The modal parameters corresponding to system (23) are equivalently found in (7).

In the following, the variance computation of the modal parameters is performed efficiently at multiple model orders with the algorithm from Section 4. The obtained information may be useful for the engineer when selecting physical modes and dismissing spurious noise modes, since it was empirically observed that spurious modes tend to have large variances. Note that the uncertainty computation is not aimed at model order selection and gives only information about the variance error. Unknown bias is still possible due to the choice of a wrong model order.

5.2 Numerical Example

The covariance computation presented in this paper is based on the perturbation theory in (Reynders et al., 2008), where it was validated for subspace identification with extensive Monte Carlo simulations. Since the present paper relies on the same perturbation theory, we only check that our proposed scheme produces estimates in accordance with (Reynders et al., 2008), for which we report an application on vibration data of a real structure.

The investigated structure is the S101 Bridge, which was a prestressed concrete bridge spanning over the 4-lane highway A1 in Austria. Ambient vibration data was collected in 165,000 time samples on 45 sensors with a sampling frequency of 500 Hz. The data is downsampled to 100 Hz and the first five modes are considered.

System identification results are obtained from covariance-driven stochastic subspace identification (Peeters and De Roeck, 1999), using three reference sensors and the parameters $p + 1 = q = 35$. Firstly, the modal parameters are obtained at model orders $n = 1, 2, \ldots, 100$ with the fast multi-order estimation from Section 4.1 using the QR decomposition for estimating the state transition matrix (Döhler and Mevel, 2012a). In Figure 1, the stabilization diagram of the frequencies identified at these model orders is plotted with the criteria $f_i < 18\,\mathrm{Hz}$ and $0.1 < \xi_i < 10$. The computed standard deviation $\sigma_{f_i} = \sqrt{\mathrm{cov}(f_i)}$ from

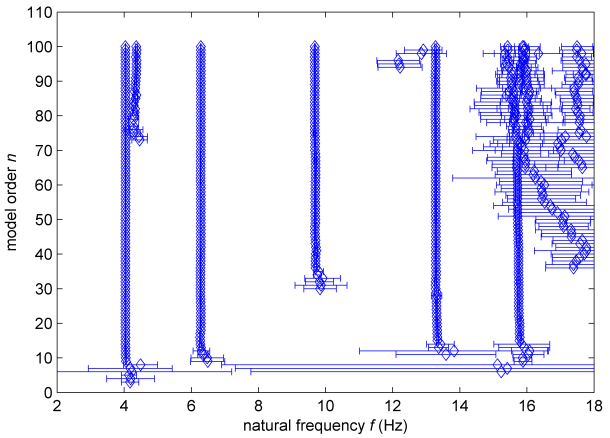

Fig. 1. Stabilization diagram: frequencies with standard deviations vs. model order.

Table 1. First 5 modes at $n = 50$ with coefficient of variation.

mode	f_i (in Hz)	$100\,\sigma_{f_i}/f_i$	ξ_i (in %)	$100\,\sigma_{\xi_i}/\xi_i$
1	4.036	0.12	0.82	13
2	6.282	0.09	0.55	20
3	9.693	0.25	1.51	17
4	13.28	0.17	1.46	16
5	15.72	0.37	1.34	17

Table 2. Discrepancies between modal parameters and their standard deviations from QR- and pseudoinverse-based computation.

$\lVert\frac{f_i^{\mathrm{QR}}-f_i^{\mathrm{pinv}}}{f_i^{\mathrm{QR}}}\rVert_\infty$	$\lVert\frac{\sigma_{f_i}^{\mathrm{QR}}-\sigma_{f_i}^{\mathrm{pinv}}}{\sigma_{f_i}^{\mathrm{QR}}}\rVert_\infty$	$\lVert\frac{\xi_i^{\mathrm{QR}}-\xi_i^{\mathrm{pinv}}}{\xi_i^{\mathrm{QR}}}\rVert_\infty$	$\lVert\frac{\sigma_{\xi_i}^{\mathrm{QR}}-\sigma_{\xi_i}^{\mathrm{pinv}}}{\sigma_{\xi_i}^{\mathrm{QR}}}\rVert_\infty$
$1.30 \cdot 10^{-12}$	$2.27 \cdot 10^{-4}$	$1.23 \cdot 10^{-10}$	$1.41 \cdot 10^{-4}$

Section 4.2 is plotted with horizontal bars. From this plot (and also from damping values and mode shapes), it is up to the user to empirically choose the final results for each mode. It can be observed that some modes show very high standard deviations, which coincide with modes that are spurious or badly estimated. However, it is not the intention of this paper to give a guideline for this choice, but to support the user with an efficient computation of the modes and their covariances at multiple model orders.

We consider the modes to be well estimated at model order 50, at which five "stable" modes (according to the above criteria) with their coefficient of variation are shown in Table 1. These results are obtained from the QR-based solution of the least squares problem for the state transition matrix. They are compared with the covariance computation using the pseudoinverse from (Reynders et al., 2008) in Table 2, where the maximal differences between the estimated frequencies, damping values and their standard deviations from both computations for all model orders are shown. These differences are found to be negligible, validating the new efficient algorithm.

6. CONCLUSIONS

In this paper, an efficient computation scheme for covariance estimates has been proposed. It is related to subspace identification at multiple model orders, based on the fast computation of the system matrices at multiple model orders in (Döhler and Mevel, 2012a) using the QR decomposition in the underlying least squares problem. First, a new algorithm for covariance estimation of the system matrices from the QR-based least squares solution was derived, and second, it was shown how results at lower model orders are efficiently obtained by the selection of submatrices from the computation at a maximal model order. This new scheme has been successfully tested on data from a European benchmark.

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