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# Bipartite Powers of $k$ -chordal Graphs

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Let  $k$  be an integer and  $k \geq 3$ . A graph  $G$  is  $k$ -chordal if  $G$  does not have an induced cycle of length greater than  $k$ . From the definition it is clear that 3-chordal graphs are precisely the class of chordal graphs. Duchet proved that, for every positive integer  $m$ , if  $G^m$  is chordal then so is  $G^{m+2}$ . Brandstädt et al. in [Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of HHD-free graphs. *Discrete Mathematics*, 177(1-3):9-16, 1997.] showed that if  $G^m$  is  $k$ -chordal, then so is  $G^{m+2}$ .

Powering a bipartite graph does not preserve its bipartitedness. In order to preserve the bipartitedness of a bipartite graph while powering Chandran et al. introduced the notion of *bipartite powering*. This notion was introduced to aid their study of boxicity of chordal bipartite graphs. The  $m$ -th bipartite power  $G^{[m]}$  of a bipartite graph  $G$  is the bipartite graph obtained from  $G$  by adding edges  $(u, v)$  where  $d_G(u, v)$  is odd and less than or equal to  $m$ . Note that  $G^{[m]} = G^{[m+1]}$  for each odd  $m$ .

In this paper we show that, given a bipartite graph  $G$ , if  $G$  is  $k$ -chordal then so is  $G^{[m]}$ , where  $k, m$  are positive integers with  $k \geq 4$ .

**Keywords:**  $k$ -chordal graph, hole, chordality, graph power, bipartite power.

## 1 Introduction

A *hole* is a chordless (or an induced) cycle in a graph. The *chordality* of a graph  $G$ , denoted by  $\mathcal{C}(G)$ , is defined to be the size of a largest hole in  $G$ , if there exists a cycle in  $G$ . If  $G$  is acyclic, then its chordality is taken as 0. A graph  $G$  is  $k$ -chordal if  $\mathcal{C}(G) \leq k$ . In other words, a graph is  $k$ -chordal if it has no holes with more than  $k$  vertices in it. Chordal graphs are exactly the class of 3-chordal graphs and chordal bipartite graphs are bipartite, 4-chordal graphs.  $k$ -chordal graphs have been studied in the literature in [2], [5], [6], [8], [9] and [16]. For example, Chandran and Ram [5] proved that the number of minimum cuts in a  $k$ -chordal graph is at most  $\frac{(k+1)n}{2} - k$ . Spinrad[16] showed that  $(k - 1)$ -chordal graphs can be recognized in  $O(n^{k-3}M)$  time, where  $M$  is the time required to multiply two  $n$  by  $n$  matrices.

Powering and its effects on the chordality of a graph has been a topic of interest. The  $m$ -th power of a graph  $G$ , denoted by  $G^m$ , is a graph with vertex set  $V(G^m) = V(G)$  and edge set  $E(G^m) = \{(u, v) \mid u \neq v, d_G(u, v) \leq m\}$ , where  $d_G(u, v)$  represents the distance between  $u$  and  $v$  in  $G$ . Balakrishnan and

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Paulraja [1] proved that odd powers of chordal graphs are chordal. Chang and Nemhauser [7] showed that if  $G$  and  $G^2$  are chordal then so are all powers of  $G$ . Duchet [10] proved a stronger result which says that if  $G^m$  is chordal then so is  $G^{m+2}$ . Brandstädt et al. in [3] showed that if  $G^m$  is  $k$ -chordal then so is  $G^{m+2}$ , where  $k \geq 3$  is an integer. Studies on families of graphs that are closed under powering can also be seen in the literature. For instance, it is known that interval graphs, proper interval graphs [14], strongly chordal graphs [13], circular-arc graphs [15][12], cocomparability graphs [11] etc. are closed under taking powers.

Subclasses of bipartite graphs, like chordal bipartite graphs, are not closed under powering since the  $m$ -th power of a bipartite graph need not be even bipartite. Chandran et al. in [4] introduced the notion of *bipartite powering* to retain the bipartiteness of a bipartite graph while taking power. The  $m$ -th bipartite power  $G^{[m]}$  of a bipartite graph  $G$  is the bipartite graph obtained from  $G$  by adding edges  $(u, v)$  where  $d_G(u, v)$  is odd and less than or equal to  $m$ . Note that  $G^{[m]} = G^{[m+1]}$  for each odd  $m$ . It was shown in [4] that the  $m$ -th bipartite power of a tree is chordal bipartite. The intention there was to construct chordal bipartite graphs of high boxicity. The fact that the chordal bipartite graph under consideration was obtained as a bipartite power of a tree was crucial for proving that its boxicity was high. Since trees are a subclass of chordal bipartite graphs, a natural question that came up was the following: is it true that the  $m$ -th bipartite power of every chordal bipartite graph is chordal bipartite? In this paper we answer this question in the affirmative. In fact, we prove a more general result.

### Our Result

Let  $m, k$  be positive integers with  $k \geq 4$ . Let  $G$  be a bipartite graph. If  $G$  is  $k$ -chordal, then so is  $G^{[m]}$ . Note that the special case when  $k = 4$  gives us the following result: chordal bipartite graphs are closed under bipartite powering.

## 2 Graph Preliminaries

Throughout this paper we consider only finite, simple, undirected graphs. For a graph  $G$ , we use  $V(G)$  to denote the set of vertices of  $G$ . Let  $E(G)$  denote its edge set. For every  $x, y \in V(G)$ ,  $d_G(x, y)$  represents the distance between  $x$  and  $y$  in  $G$ . For every  $u \in V(G)$ ,  $N_G(u)$  denotes its *open neighborhood* in  $G$ , i.e.  $N_G(u) = \{v \mid (u, v) \in E(G)\}$ . A path  $P$  on the vertex set  $V(P) = \{v_1, v_2, \dots, v_n\}$  (where  $n \geq 2$ ) has its edge set  $E(P) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$ . Such a path is denoted by  $v_1v_2 \dots v_n$ . If  $v_i, v_j \in V(P)$ ,  $v_iPv_j$  is the path  $v_iv_{i+1} \dots v_j$ . The length of a path  $P$  is the number of edges in it and is denoted by  $\|P\|$ . A cycle  $C$  with vertex set  $V(C) = \{v_1, v_2, \dots, v_n\}$ , and edge set  $E(C) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$  is denoted as  $C = v_1v_2 \dots v_nv_1$ . We use  $\|C\|$  to denote the length of cycle  $C$ .

## 3 Holes in Bipartite Powers

Let  $H$  be a bipartite graph. Let  $\mathcal{B}(H)$  be a family of graphs constructed from  $H$  in the following manner:  $H' \in \mathcal{B}(H)$  if corresponding to each vertex  $v \in V(H)$  there exists a nonempty bag of vertices, say  $B_v$ , in  $H'$  such that (a) for every  $x \in B_u, y \in B_v, (x, y) \in E(H')$  if and only if  $(u, v) \in E(H)$ , and (b) vertices within each bag in  $H'$  are pairwise non-adjacent. Below we list a few observations about  $H$  and every  $H' \in \mathcal{B}(H)$ :

**Observation 1.**  $H'$  is bipartite.

**Observation 2.**  $H$  is an induced subgraph of  $H'$ .

**Observation 3.** Let  $k$  be an integer such that  $k \geq 4$ . If  $H$  is  $k$ -chordal, then so is  $H'$ .

**Proof:** Any hole of size greater than 4 in  $H'$  cannot have more than one vertex from the same bag, say  $B_v$ , as such vertices have the same neighborhood. Hence, the vertices of a hole (of size greater than 4) in  $H'$  belong to different bags and thus there is a corresponding hole of the same size in  $H$ .  $\square$

**Theorem 4.** Let  $m, k$  be positive integers with  $k \geq 4$ . Let  $G$  be a bipartite graph. If  $G$  is  $k$ -chordal, then so is  $G^{[m]}$ .

**Proof:** We prove this by contradiction. Let  $p$  denote the size of a largest induced cycle, say  $C = u_0 u_1 \dots u_{p-1} u_0$ , in  $G^{[m]}$ . Assume  $p > k$ . Then,  $p \geq 6$  (since  $k \geq 4$  and  $G^{[m]}$  is bipartite). Between each  $u_{i-1}$  and  $u_i$ , where  $i \in \{0, \dots, p-1\}$ , there exists a shortest path of length not more than  $m$  in  $G$ <sup>(i)</sup>. Let  $P_i$  be one such shortest path between  $u_{i-1}$  and  $u_i$  in  $G$ .

Let  $H$  be the subgraph induced on the vertex set  $\bigcup_{i=0}^{p-1} V(P_i)$  in  $G$ . As mentioned in the beginning of this section, construct a graph  $H'$  from  $H$ , where  $H' \in \mathcal{B}(H)$ , in the following manner: for each  $v \in V(H)$ , let  $|B_v| = |\{P_i \mid 0 \leq i \leq p-1, v \in V(P_i)\}|$  i.e., let  $B_v$  have as many vertices as the number of paths in  $\{P_0 \dots P_{p-1}\}$  that share vertex  $v$  in  $H$ . For each  $i \in \{0, \dots, p-1\}$ , let  $Q'_i = u_{i-1} Q_i$  be a shortest path between  $u_{i-1}$  and  $u_i$  in  $H'$  such that no two paths  $Q_i$  and  $Q_j$  (where  $i \neq j$ ) share a vertex<sup>(i)</sup>. From our construction of  $H'$  from  $H$  it is easy to see that such paths exist. Let  $Q_i = v_{i,1} v_{i,2} \dots v_{i,r_i} u_i$ , where  $r_i = |Q_i| \geq 0$ . Thus,  $Q'_i = u_{i-1} v_{i,1} v_{i,2} \dots v_{i,r_i} u_i$ . Clearly,  $|Q'_i| = |P_i| \leq m$ . The reader may also note that the cycle  $C (= u_0 u_1 \dots u_{p-1} u_0)$  which is present in  $G^{[m]}$  will be present in  $H^{[m]}$  and thereby in  $H'^{[m]}$  too.

In order to prove the theorem, it is enough to show that there exists an induced cycle of size at least  $p$  in  $H'$ . Then by combining Observation 3 and the fact that  $H$  is an induced subgraph of  $G$ , we get  $k \geq \mathcal{C}(G) \geq \mathcal{C}(H) \geq \mathcal{C}(H') \geq p$  contradicting our assumption that  $p > k$ . Hence, in the rest of the proof we show that  $\mathcal{C}(H') \geq p$ .

Consider the following drawing of the graph  $H'$ . Arrange the vertices  $u_0, u_1, \dots, u_{p-1}$  in that order on a circle in clockwise order. Between each  $u_{i-1}$  and  $u_i$  on the circle arrange the vertices  $v_{i,1}, v_{i,2}, \dots, v_{i,r_i}$  in that order in clockwise order. Recall that these vertices are the internal vertices of path  $Q'_i$ .

*Claim 4.1.* In this circular arrangement of vertices of  $H'$ , each vertex has an edge (in  $H'$ ) with both its left neighbor and right neighbor in the arrangement.

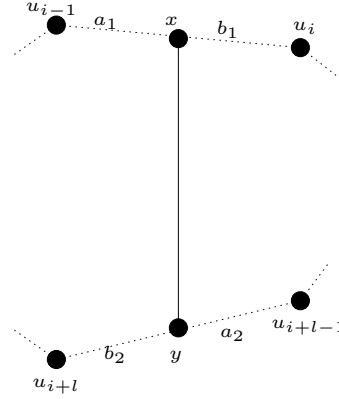
Let  $x_1, x_2 \in V(H')$ , where  $x_1 \in V(Q_i)$ ,  $x_2 \in V(Q_j)$ . We define the *clockwise distance from  $x_1$  to  $x_2$* , denoted by  $clock\_dist(x_1, x_2)$ , as the minimum non-negative integer  $s$  such that  $j = i + s$ . Similarly, the *clockwise distance from  $x_2$  to  $x_1$* , denoted by  $clock\_dist(x_2, x_1)$ , is the minimum non-negative integer  $s'$  such that  $i = j + s'$ . Let  $x, y, z \in V(H')$ . We say  $y <_x z$  if scanning the vertices of  $H'$  in clockwise direction along the circle starting from  $x$ , vertex  $y$  is encountered before  $z$ . Let  $x \in V(Q_i)$ . Vertex  $y$  is called the *farthest neighbor of  $x$  before  $z$*  if  $y \in N_{H'}(x)$ ,  $y \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ ,  $y <_x z$ , and for every other  $w \in N_{H'}(x)$  either  $z <_x w$  or  $w \notin V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$  or both.

*Claim 4.2.* There always exists a vertex which is the farthest neighbor of  $x$  before  $z$ , unless  $(x, z) \in E(H')$  and  $z \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ .

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<sup>(i)</sup> throughout this proof expressions involving subscripts of  $u$ ,  $P$ ,  $Q$ , and  $Q'$  are to be taken modulo  $p$ . Every such expression should be evaluated to a value in  $\{0, \dots, p-1\}$ . For example, consider a vertex  $u_i$ , where  $i < p$ . Then,  $p+i = i$ .

Let  $\{A, B\}$  be the bipartition of the bipartite graph  $H'$ . We categorize the edges of  $H'$  as follows: an edge  $(x, y) \in E(H')$  is called an  $l$ -edge, if  $l = \min(\text{clock\_dist}(x, y), \text{clock\_dist}(y, x))$ .



**Fig. 1:**  $x \in V(Q_i), y \in V(Q_{i+l})$  and let  $(x, y) \in E(H')$  be an  $l$ -edge, where  $l > 2$ . The dotted line between  $u_{i-1}$  and  $u_i$  indicate the path  $Q_i$ . Similarly, the dotted line between  $u_{i+l-1}$  and  $u_{i+l}$  indicate the path  $Q_{i+l}$ .

*Claim 4.3.*  $H'$  cannot have an  $l$ -edge, where  $l > 2$ .

**Proof:** Suppose  $H'$  has an  $l$ -edge, where  $l > 2$ , between  $x \in Q_i$  and  $y \in Q_{i+l}$  (see Fig. 1). Let  $a_1 = \|u_{i-1}Q'_i x\|$ ,  $b_1 = \|xQ'_i u_i\|$ ,  $a_2 = \|u_{i+l-1}Q'_{i+l} y\|$  and  $b_2 = \|yQ'_{i+l} u_{i+l}\|$ . We consider the following two cases:

*Case 1:  $l$  is even*

In this case  $u_{i-1}$  and  $u_{i+l-1}$  will be on the same side of the bipartite graph  $H'$ . Without loss of generality, let  $u_{i-1}, u_{i+l-1} \in A$ . Then,  $u_i, u_{i+l} \in B$ . We know that, for every  $w_1, w_2 \in V(H'^{[m]})$  with  $w_1 \in A$  and  $w_2 \in B$ , if  $(w_1, w_2) \notin E(H'^{[m]})$  then  $d_{H'}(w_1, w_2) \geq m + 2$  (recalling  $m$  and  $d_{H'}(w_1, w_2)$  are odd integers). Therefore, we have  $a_1 + 1 + b_2 \geq d_{H'}(u_{i-1}, u_{i+l}) \geq m + 2$ . Similarly,  $b_1 + 1 + a_2 \geq d_{H'}(u_i, u_{i+l-1}) \geq m + 2$ . Summing up the two inequalities we get,  $(a_1 + b_1) + (a_2 + b_2) \geq 2m + 2$ . This implies that either  $\|Q'_i\|$  or  $\|Q'_{i+l}\|$  is greater than  $m$  which is a contradiction.

*Case 2:  $l$  is odd*

(proof is similar to the above case and hence omitted).

Hence we prove the claim. □

We find a cycle  $C' = z_0 z_1 \dots z_q z_0$  in  $H'$  using Algorithm 3.1 <sup>(i)</sup>. Please read the algorithm before proceeding further. .

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<sup>(i)</sup> throughout this proof expressions involving subscripts of  $z$  are to be taken modulo  $q + 1$ . Every such expression should be evaluated to a value in  $\{0, \dots, q\}$ . For example, consider a vertex  $z_a$ , where  $a < q + 1$ . Then,  $q + 1 + a = a$ .

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**Algorithm 3.1** Finding Cycle  $C'$  in  $H'$  such that  $\|C'\| \geq \|C\|$

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1.  $l \leftarrow \max_{l'}(H' \text{ has an } l' \text{-edge})$ . Without loss of generality assume that this  $l$ -edge is between a vertex in  $Q_0$  and a vertex in  $Q_l$ .
  2. Scan the vertices of  $Q_0$  in clockwise direction to find the first vertex  $z_0$ , where  $z_0 \in V(Q_0)$ , which has an  $l$ -edge to a vertex in  $Q_l$ .
  3. Scan the vertices of  $Q_l$  in clockwise direction to find the last vertex in  $Q_l$  which is a neighbor of  $z_0$  in  $H'$ . Call it  $z_1$ .
  4. Find the farthest neighbor of  $z_1$  before  $z_0$ . Call it  $z_2$ . /\* refer proof of Claim 4.4 for a proof of existence of such a  $z_2$ \*/
  5.  $s \leftarrow 2$ .
  - while**  $(z_s, z_0) \notin E(H')$  **do**
    6. Find the farthest neighbor of  $z_s$  before  $z_0$ . Call it  $z_{s+1}$ . /\* such a neighbor exists by Claim 4.2\*/
    7.  $s \leftarrow s + 1$ .
  - end while**
  8.  $q \leftarrow s$ .
  9. Return cycle  $C' = z_0 z_1 \dots z_q z_0$ .
- 

*Claim 4.4.* There always exists a farthest neighbor of  $z_1$  before  $z_0$ .

**Proof:** Note that  $z_0 \in Q_0$  and  $z_1 \in Q_l$ , where  $l \leq 2$  (by Claim 4.3). Recalling that  $\|C\| = p \geq 6$ , we have  $z_0 \notin V(Q_1) \cup V(Q_{l+1}) \cup V(Q_{l+2})$ . Hence by Claim 4.2, the claim is true.  $\square$

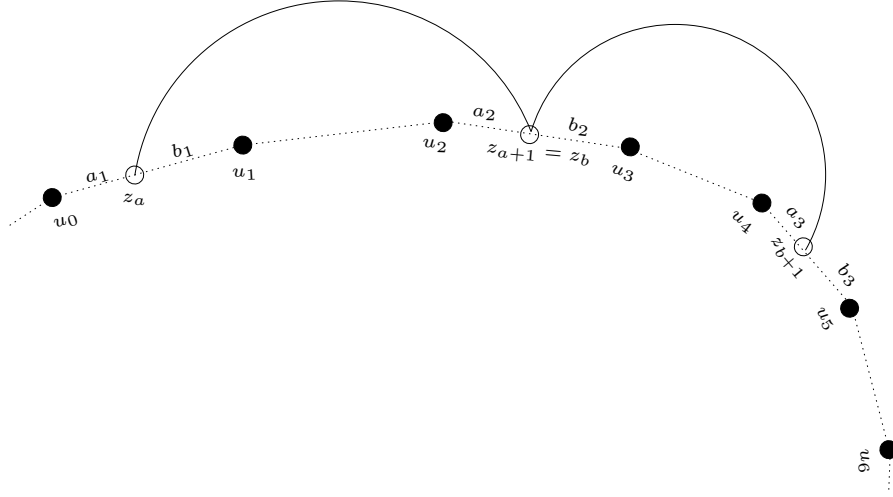
*Claim 4.5.* The while loop in Algorithm 3.1 terminates after a finite number of iterations.

**Proof:** From Claim 4.1, we know that each vertex has an edge (in  $H'$ ) with both its left neighbor and right neighbor in the circular arrangement. Each time when Step 6 of Algorithm 3.1 is executed, a vertex  $z_{s+1}$  is chosen such that  $z_{s+1}$  is the farthest neighbor of  $z_s$  before  $z_0$ . Since  $H'$  is a finite graph, there will be a point of time in the execution of the algorithm when in Step 6 it picks a  $z_{s+1}$  such that  $(z_{s+1}, z_0) \in E(H')$ .  $\square$

From Claim 4.5, we can infer that  $C'$  is a cycle.

*Claim 4.6.*  $C'$  is an induced cycle in  $H'$ .

**Proof:** Suppose  $C'$  is not an induced cycle. Then there exists a chord  $(z_a, z_b)$  in  $C'$ . Since  $(z_a, z_b)$  is a chord, we have  $b \neq a - 1$  or  $b \neq a + 1$ . Let  $l = \max_{l'}(H' \text{ has an } l' \text{-edge})$ . Let  $z_a \in V(Q_i)$ ,  $z_b \in V(Q_j)$ . We know that  $\min(\text{clock\_dist}(z_a, z_b), \text{clock\_dist}(z_b, z_a)) \leq l$ . Without loss of generality, assume  $\text{clock\_dist}(z_a, z_b) \leq l \leq 2$  (from Claim 4.3). That is,  $j - i \leq l \leq 2$  and  $(z_a, z_b)$  is a  $(j - i)$ -edge. If  $z_a = z_0$ , then  $z_b \neq z_1$  and the algorithm exits from the while loop, when  $q = b$ , thus returning a cycle  $z_0 \dots z_b z_0$ . But in such a cycle  $(z_b, z_0)$  is not a chord. Therefore,  $z_a \neq z_0$ . Similarly,  $z_b \neq z_0$ . We know that  $z_{a+1} \neq z_b$ ,  $z_{a+1} <_{z_a} z_b$ , and  $z_{a+1} \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ . Since  $j - i \leq 2$ ,  $z_b \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ . If  $z_b <_{z_a} z_0$ , then it contradicts the fact that  $z_{a+1}$  is the farthest neighbor of  $z_a$  before  $z_0$ . Therefore,  $z_0 <_{z_a} z_b$ . Then, either  $z_b = z_1$  or  $z_1 <_{z_a} z_b$ . Recall that  $l = \max_{l'}(H' \text{ has an } l' \text{-edge})$ , and  $(z_0, z_1)$  is an  $l$ -edge with  $z_0 \in V(Q_0)$  and  $z_1 \in V(Q_l)$ . Since (i)  $(z_a, z_b)$  is a  $(j - i)$ -edge, where  $j - i \leq l$ , (ii)  $z_0 <_{z_a} z_b$ , and (iii)  $z_b = z_1$  or  $z_1 <_{z_a} z_b$ , we have  $l \geq j - i = \text{clock\_dist}(z_a, z_b) \geq \text{clock\_dist}(z_0, z_b) \geq \text{clock\_dist}(z_0, z_1) = l$ . Hence,  $j - i = l$  and



**Fig. 2:** Figure illustrates the case when path  $P$  defined in Claim 4.8 is a trivial path. The dotted lines between each  $u_{i-1}$  and  $u_i$  indicate the path  $Q'_i$ . Each continuous arc corresponds to an edge in the cycle  $C' = z_0 \dots z_q z_0$ .

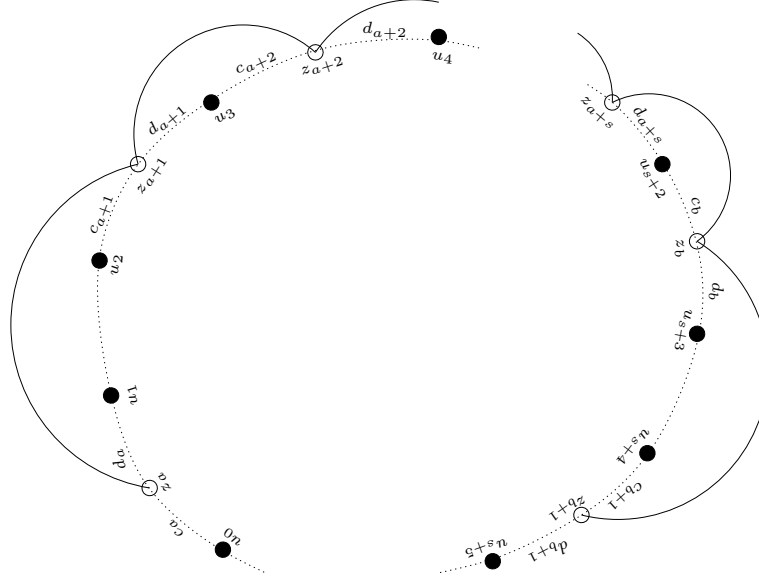
$(z_a, z_b)$  is an  $l$ -edge. We know that  $(z_0, z_1)$  is also an  $l$ -edge with  $z_0 \in V(Q_0)$  and  $z_1 \in V(Q_l)$ . Since  $z_0 <_{z_a} z_b$  and  $z_b = z_1$  or  $z_1 <_{z_0} z_b$ , we get  $z_a \in V(Q_0)$  and  $z_b \in V(Q_l)$ . From Step 2 of the algorithm we know that  $z_0$  is the first vertex (in a clockwise scan) in  $Q_0$  which has an  $l$ -edge to a vertex in  $Q_l$ . This implies that, since  $z_0 <_{z_a} z_b$ ,  $z_a = z_0$  which is a contradiction. Hence we prove the claim.  $\square$

What is left now is to show that  $q + 1 \geq p$ , i.e.,  $\|C'\| \geq \|C\|$ , where  $C' = z_0 \dots z_q z_0$  and  $C = u_0 \dots u_{p-1} u_0$ . In order to show this, we state and prove the following claims.

**Claim 4.7.** For every  $j \in \{0, \dots, p-1\}$ ,  $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') \neq \emptyset$ .

**Proof:** Suppose the claim is not true. Find the minimum  $j$  that violates the claim. Clearly,  $j \neq 0$  as  $z_0 \in V(Q_0)$ . We claim that  $z_q \in V(Q_{j-1})$ . Suppose  $z_q \notin V(Q_{j-1})$ . Let  $a = \max\{i \mid z_i \in V(Q_{j-1})\}$  (note that, since  $j \neq 0$ , by the minimality of  $j$ ,  $(V(Q_{j-1}) \cup V(Q_j)) \cap V(C') \neq \emptyset$  and therefore  $V(Q_{j-1}) \cap V(C') \neq \emptyset$ ). Since  $z_a \neq z_q$ , by the maximality of  $a$ , we have  $z_{a+1} \notin V(Q_{j-1})$ . From our assumption,  $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') = \emptyset$  and therefore  $z_{a+1} \notin V(Q_{j-1}) \cup V(Q_j) \cup V(Q_{j+1})$ . Thus  $z_a \neq z_q$  and  $z_{a+1}$  is not the farthest neighbor of  $z_a$  before  $z_0$ . This is a contradiction to the way  $z_{a+1}$  is chosen by Algorithm 3.1. Hence,  $z_q \in V(Q_{j-1})$ . We know that  $(z_q, z_0) \in E(H')$  with  $z_q \in V(Q_{j-1})$  and  $z_0 \in V(Q_0)$ . Since  $l = \max_{l'}(H'$  has an  $l'$ -edge), we have  $\min(\text{clock\_dist}(z_q, z_0), \text{clock\_dist}(z_0, z_q)) \leq l$ . That is,  $j \geq p + 1 - l$  or  $j \leq 1 + l$ . As  $l \leq 2$  (by Claim 4.3), we have  $j = p - 1$  or  $j \leq 1 + l$ . Since  $z_0 \in V(Q_0)$ ,  $(V(Q_{p-1}) \cup V(Q_0)) \cap V(C') \neq \emptyset$  and hence  $j \neq p - 1$ . Therefore,  $j \leq 1 + l$ . Since  $z_0 \in V(Q_0)$  and  $z_1 \in V(Q_l)$  (recall  $l \leq 2$ ), we get  $j = 1 + l$ . We know that, for every  $z_a, z_b \in V(C')$ , if  $a < b$  then  $z_a <_{z_0} z_b$ . Therefore,  $z_1 <_{z_0} z_q$ . We have  $z_1 \in V(Q_l)$ . Since  $j = 1 + l$ , we also have  $z_q \in V(Q_l)$ . Thus, we have  $z_1, z_q \in V(Q_l)$  and  $z_1 <_{z_0} z_q$ . But this contradicts the fact that  $z_1$  is the last vertex in  $Q_l$  encountered in a clockwise scan that has  $z_0$  as its neighbor.  $\square$

**Claim 4.8.** Let  $(z_a, z_{a+1}), (z_b, z_{b+1}) \in E(C')$  be two 2-edges, where  $a < b$ . Let  $P, P'$  denote the clockwise  $z_{a+1} - z_b, z_{b+1} - z_a$  paths respectively in  $C'$ . Both  $P$  and  $P'$  contain at least one 0-edge.



**Fig. 3:** Figure illustrates the case when path  $P$  defined in Claim 4.8 is  $P = z_{a+1}z_{a+2} \dots z_{a+1+s}$ , where  $s \geq 1$  and  $z_{a+1+s} = z_b$ . The dotted lines between each  $u_{i-1}$  and  $u_i$  indicate the path  $Q'_i$ . Each continuous arc corresponds to an edge in the cycle  $C' = z_0 \dots z_q z_0$ .

**Proof:** Consider the path  $P$  (proof is similar in the case of path  $P'$ ). Path  $P$  is a non-trivial path only if  $z_{a+1} \neq z_b$ . Suppose  $z_{a+1} = z_b$  (see Fig. 2). Let  $z_a \in V(Q_f)$ . For the sake of ease of notation, assume  $f = 1$  (the same proof works for any value of  $f$ ). Let  $a_1 = \|u_0 Q'_1 z_a\|$ ,  $b_1 = \|z_a Q'_1 u_1\|$ ,  $a_2 = \|u_2 Q'_3 z_b\|$ ,  $b_2 = \|z_b Q'_3 u_3\|$ ,  $a_3 = \|u_4 Q'_5 z_{b+1}\|$ , and  $b_3 = \|z_{b+1} Q'_5 u_5\|$ . We know that, for every  $w_1, w_2 \in V(H'^{[m]})$  with  $w_1 \in A$  and  $w_2 \in B$ , if  $(w_1, w_2) \notin E(H'^{[m]})$  then  $d_{H'}(w_1, w_2) \geq m + 2$ . Since  $(u_0, u_3) \notin E(H'^{[m]})$ ,  $(u_1, u_4) \notin E(H'^{[m]})$  and  $(u_2, u_5) \notin E(H'^{[m]})$ , we have  $a_1 + b_2 \geq m + 1$ ,  $b_1 + a_3 \geq m$ , and  $a_2 + b_3 \geq m + 1$ . Adding the three inequalities and by applying an easy averaging argument we can infer that either  $a_1 + b_1 = \|Q_1\| > m$ ,  $a_2 + b_2 = \|Q_3\| > m$ , or  $a_3 + b_3 = \|Q_5\| > m$  which is a contradiction. Therefore  $P$  is a non-trivial path i.e.,  $z_{a+1} \neq z_b$ . Assume  $P$  does not contain any 0-edge. Let  $P = z_{a+1}z_{a+2} \dots z_{a+1+s}$ , where  $s \geq 1$ ,  $a + 1 + s = b$ , and  $(z_{a+1}, z_{a+2}) \dots (z_{a+s}, z_{a+1+s})$  are 1-edges (see Fig. 3). Since  $(u_0, u_3) \notin E(H'^{[m]})$ ,  $(u_1, u_4) \notin E(H'^{[m]})$ , we have  $c_a + d_{a+1} \geq m + 1$  and  $d_a + d_{a+2} \geq m$  (please refer Fig. 3 for knowing what  $c_a, d_a, \dots, c_{b+1}, d_{b+1}$  are). Summing up the two inequalities, we get  $d_{a+1} + d_{a+2} \geq 2m + 1 - (c_a + d_a)$ . We know that, for each  $i \in \{0, \dots, p-1\}$ ,  $\|Q'_i\| \leq m$ . Therefore, we have  $c_a + d_a \leq m$ . Hence,  $d_{a+1} + d_{a+2} \geq m + 1$ . Since  $(c_{a+1} + d_{a+1}) + (c_{a+2} + d_{a+2}) \leq 2m$ , we get

$$c_{a+1} + c_{a+2} \leq m - 1 \quad (1)$$

Since  $(u_{s+2}, u_{s+5}) \notin E(H'^{[m]})$ ,  $(u_{s+1}, u_{s+4}) \notin E(H'^{[m]})$ , we have,

$$\begin{aligned} c_b + d_{b+1} &\geq m + 1 \\ c_{a+s} + c_{b+1} &\geq m \end{aligned}$$



Summing up the two inequalities, we get

$$c_b + c_{a+s} \geq 2m + 1 - (c_{b+1} + d_{b+1})$$

Since  $b = a + s + 1$  and  $c_{b+1} + d_{b+1} \leq m$ , we get

$$c_{a+s+1} + c_{a+s} \geq m + 1 \quad (2)$$

Substituting for  $s = 1$  in Inequality 2, we get  $c_{a+2} + c_{a+1} \geq m + 1$ . But this contradicts Inequality 1. Hence  $s > 1$ . Suppose  $s = 2$ . Since  $(u_2, u_5) \notin E(H'^{[m]})$ , we have  $c_{a+1} + d_{a+3} \geq m$ . Adding this with Inequality 2, we get  $c_{a+1} + c_{a+2} \geq (2m + 1) - (c_{a+3} + d_{a+3}) \geq m + 1$ . But this contradicts Inequality 1. Hence  $s > 2$ . Since  $(u_s, u_{s+3}) \notin E(H'^{[m]})$ ,  $\dots$ ,  $(u_2, u_5) \notin E(H'^{[m]})$ , we have the following inequalities:-

$$\begin{aligned} c_{a+s-1} + d_{a+s+1} &\geq m \\ &\vdots \\ &\vdots \\ c_{a+1} + d_{a+3} &\geq m \end{aligned}$$

Adding the above set of inequalities and applying the fact that  $c_i + d_i \leq m$ ,  $\forall i \in \{0, \dots, q\}$ , we get  $c_{a+1} + c_{a+2} + d_{a+s} + d_{a+s+1} \geq 2m$ . Adding this with Inequality 2, we get  $c_{a+1} + c_{a+2} \geq (3m + 1) - (c_{a+s+1} + d_{a+s+1}) - (c_{a+s} + d_{a+s}) \geq m + 1$ . But this contradicts Inequality 1. Hence we prove the claim.  $\square$

**Claim 4.9.** For every  $j, j' \in \{0, \dots, p-1\}$ , where  $j < j'$  and  $(V(Q_j) \cup V(Q_{j'})) \cap V(C') = \emptyset$ , there exist  $i, i' \in \{0, \dots, p-1\}$ , where only  $i$  satisfies  $j < i < j'$ , such that  $|V(Q_i) \cap V(C')| \geq 2$  and  $|V(Q_{i'}) \cap V(C')| \geq 2$ .

**Proof:** By Claim 4.7, (i)  $j' \neq j + 1$  or  $j' \neq j - 1$ , and (ii) there exist  $r, r' \in \{0, \dots, q\}$  such that  $(z_r, z_{r+1})$  is a 2-edge with its endpoints on  $Q_{j-1}$  and  $Q_{j+1}$  and  $(z_{r'}, z_{r'+1})$  is a 2-edge with its endpoints on  $Q_{j'-1}$  and  $Q_{j'+1}$ . By Claim 4.8, we know that if  $P, P'$  denote the clockwise  $z_{r+1} - z_{r'}, z_{r'+1} - z_r$  paths respectively in  $C'$ , then both  $P$  and  $P'$  contains at least one 0-edge. This proves the claim.  $\square$

In order to show that the size of cycle  $C' (= z_0 \dots z_q z_0)$  is at least  $p$ , we consider the following three cases:-

*Case*  $|\{Q_j \in \{Q_0 \dots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 0$ : In this case, for every  $j \in \{0, \dots, p-1\}$ ,  $Q_j$  contributes to  $V(C')$  and therefore  $\|C'\| \geq p = \|C\|$ .

*Case*  $|\{Q_j \in \{Q_0 \dots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 1$ : Let  $Q_j$  be that only path (among  $Q_0 \dots Q_{p-1}$ ) that does not contribute to  $V(C')$ . Then we claim that there exists a  $Q_{j'}$ , where  $j' \neq j$ , such that  $V(C') \cap V(Q_{j'}) \geq 2$ . Suppose the claim is not true then it is easy to see that  $\|C'\| = p - 1$  which is an odd number thus contradicting the bipartiteness of  $H'$ . Hence the claim is true. Now, by applying the claim it is easy to see that  $\|C'\| = \sum_j |V(C') \cap V(Q_j)| \geq p = \|C\|$ .

*Case*  $|\{Q_j \in \{Q_0 \dots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| > 1$ : Scan vertices of  $H'$  starting from any vertex in clockwise direction. Claim 4.9 ensures that between every  $Q_j$  and  $Q_{j'}$ , which do not contribute to  $V(C')$ , encountered there exists a  $Q_i$  which compensates by contributing at least two vertices to  $V(C')$ . Therefore,  $\|C'\| \geq p = \|C\|$ .  $\blacksquare$

## 4 Discussion

An interesting open question that naturally follows from our result is the following: given a graph  $G$  and positive integers  $k, m$  where  $k \geq 4$ , if  $G^{[m]}$  is  $k$ -chordal, then is  $G^{[m+2]}$  also  $k$ -chordal? As mentioned earlier, Brandstädt et al. in [3] showed a similar result in the context of *ordinary* graph powering. They showed that, for every graph  $G$ , if  $G^m$  is  $k$ -chordal, then so is  $G^{m+2}$ , where  $k, m$  are positive integers with  $k \geq 3$ . A straightforward extension of their proof technique doesn't seem to work in our context due to the bipartite nature of the powering that we consider.

## References

- [1] R. Balakrishnan and P. Paulraja. Powers of chordal graphs. *J. Aust. Math. Soc. Ser. A*, 35:211–217, 1983.
- [2] H. L. Bodlaender and D. M. Thilikos. Treewidth for graphs with small chordality. *Discrete Applied Mathematics*, 79:45–61, 1997.
- [3] Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of hhd-free graphs. *Discrete Mathematics*, 177(1-3):9–16, 1997.
- [4] L. Sunil Chandran, Mathew C. Francis, and Rogers Mathew. Chordal bipartite graphs with high boxicity. *Graphs and Combinatorics*, 27(3):353–362, 2011.
- [5] L. Sunil Chandran and L. Shankar Ram. On the number of minimum cuts in a graph. In *Proceedings of the 8th International computing and combinatorics conference, LNCS 2387*, pages 220–230, 2002.
- [6] L. Sunil Chandran, C.R. Subramanian, and Vadim V. Lozin. Graphs of low chordality. To appear in *Discrete Mathematics and Theoretical Computer Science*, 2005.
- [7] Gerard J. Chang and George L. Nemhauser. The  $k$ -domination and  $k$ -stability problems on sun-free chordal graphs. *SIAM Journal on Algebraic and Discrete Methods*, 5(3):332–345, 1984.
- [8] Yon Dourisboure. Compact routing schemes for generalised chordal graphs. *Journal of Graph Algorithms and Applications*, 9:277–297, 2005.
- [9] Feodor F. Dragan. Estimating all pairs shortest paths in restricted graph families: a unified approach. *Journal of Algorithms*, 57(1):1 – 21, 2005.
- [10] P. Duchet. Classical perfect graphs. *Ann. Discrete Math.*, 21:67–96, 1984.
- [11] Carsten Flotow. On powers of  $m$ -trapezoid graphs. *Discrete Appl. Math.*, 63(2):187–192, 1995.
- [12] Carsten Flotow. On powers of circular arc graphs and proper circular arc graphs. *Discrete Appl. Math.*, 69(3):199–207, 1996.
- [13] A Lubiw.  $\gamma$ -free matrices. Master's thesis, Dept. of combinatorics and Optimization, University of Waterloo, 1982.
- [14] A. Raychaudhuri. On powers of interval and unit interval graphs. *Congr.Numerantium*, 59:235–242, 1987.

- [15] A. Raychaudhuri. On powers of strongly chordal and circular graphs. *Ars Combinatoria*, 34:147–160, 1992.
- [16] Jeremy P. Spinrad. Finding large holes. *Information Processing Letters*, 39(4):227 – 229, 1991.