# Bipartite powers of k-chordal graphs

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# *Bipartite Powers of $k$-chordal Graphs*

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Let $k$ be an integer and $k \geq 3$. A graph $G$ is *k-chordal* if $G$ does not have an induced cycle of length greater than $k$. From the definition it is clear that 3-chordal graphs are precisely the class of chordal graphs. Duchet proved that, for every positive integer $m$, if $G^m$ is chordal then so is $G^{m+2}$. Brandstädt et al. in [Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of HHD-free graphs. *Discrete Mathematics*, 177(1-3):9-16, 1997.] showed that if $G^m$ is $k$-chordal, then so is $G^{m+2}$.

Powering a bipartite graph does not preserve its bipartitedness. In order to preserve the bipartitedness of a bipartite graph while powering Chandran et al. introduced the notion of *bipartite powering*. This notion was introduced to aid their study of boxicity of chordal bipartite graphs. The $m$-th bipartite power $G^{[m]}$ of a bipartite graph $G$ is the bipartite graph obtained from $G$ by adding edges $(u, v)$ where $d_G(u, v)$ is odd and less than or equal to $m$. Note that $G^{[m]} = G^{[m+1]}$ for each odd $m$.

In this paper we show that, given a bipartite graph $G$, if $G$ is $k$-chordal then so is $G^{[m]}$, where $k$, $m$ are positive integers with $k \geq 4$.

**Keywords:** $k$-chordal graph, hole, chordality, graph power, bipartite power.

## 1 Introduction

A *hole* is a chordless (or an induced) cycle in a graph. The *chordality* of a graph $G$, denoted by $\mathcal{C}(G)$, is defined to be the size of a largest hole in $G$, if there exists a cycle in $G$. If $G$ is acyclic, then its chordality is taken as 0. A graph $G$ is *k-chordal* if $\mathcal{C}(G) \leq k$. In other words, a graph is $k$-chordal if it has no holes with more than $k$ vertices in it. Chordal graphs are exactly the class of 3-chordal graphs and chordal bipartite graphs are bipartite, 4-chordal graphs. $k$-chordal graphs have been studied in the literature in [2], [5], [6], [8], [9] and [16]. For example, Chandran and Ram [5] proved that the number of minimum cuts in a $k$-chordal graph is at most $\frac{(k+1)n}{2} - k$. Spinrad[16] showed that $(k-1)$-chordal graphs can be recognized in $O(n^{k-3}M)$ time, where $M$ is the time required to multiply two $n$ by $n$ matrices.

Powering and its effects on the chordality of a graph has been a topic of interest. The $m$-th power of a graph $G$, denoted by $G^m$, is a graph with vertex set $V(G^m) = V(G)$ and edge set $E(G^m) = \{(u, v) \mid u \neq v, d_G(u, v) \leq m\}$, where $d_G(u, v)$ represents the distance between $u$ and $v$ in $G$. Balakrishnan and

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Paulraja [1] proved that odd powers of chordal graphs are chordal. Chang and Nemhauser [7] showed that if $G$ and $G^2$ are chordal then so are all powers of $G$. Duchet [10] proved a stronger result which says that if $G^m$ is chordal then so is $G^{m+2}$. Brandstädt et al. in [3] showed that if $G^m$ is $k$-chordal then so is $G^{m+2}$, where $k \geq 3$ is an integer. Studies on families of graphs that are closed under powering can also be seen in the literature. For instance, it is known that interval graphs, proper interval graphs [14], strongly chordal graphs [13], circular-arc graphs [15][12], cocomparability graphs [11] etc. are closed under taking powers.

Subclasses of bipartite graphs, like chordal bipartite graphs, are not closed under powering since the $m$-th power of a bipartite graph need not be even bipartite. Chandran et al. in [4] introduced the notion of *bipartite powering* to retain the bipartitedness of a bipartite graph while taking power. The *$m$-th bipartite power* $G^{[m]}$ of a bipartite graph $G$ is the bipartite graph obtained from $G$ by adding edges $(u, v)$ where $d_G(u, v)$ is odd and less than or equal to $m$. Note that $G^{[m]} = G^{[m+1]}$ for each odd $m$. It was shown in [4] that the $m$-th bipartite power of a tree is chordal bipartite. The intention there was to construct chordal bipartite graphs of high boxicity. The fact that the chordal bipartite graph under consideration was obtained as a bipartite power of a tree was crucial for proving that its boxicity was high. Since trees are a subclass of chordal bipartite graphs, a natural question that came up was the following: is it true that the $m$-th bipartite power of every chordal bipartite graph is chordal bipartite? In this paper we answer this question in the affirmative. In fact, we prove a more general result.

### Our Result

Let $m$, $k$ be positive integers with $k \geq 4$. Let $G$ be a bipartite graph. If $G$ is $k$-chordal, then so is $G^{[m]}$. Note that the special case when $k = 4$ gives us the following result: chordal bipartite graphs are closed under bipartite powering.

## 2 Graph Preliminaries

Throughout this paper we consider only finite, simple, undirected graphs. For a graph $G$, we use $V(G)$ to denote the set of vertices of $G$. Let $E(G)$ denote its edge set. For every $x, y \in V(G)$, $d_G(x, y)$ represents the distance between $x$ and $y$ in $G$. For every $u \in V(G)$, $N_G(u)$ denotes its *open neighborhood* in $G$, i.e. $N_G(u) = \{v \mid (u, v) \in E(G)\}$. A path $P$ on the vertex set $V(P) = \{v_1, v_2, \ldots, v_n\}$ (where $n \geq 2$) has its edge set $E(P) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$. Such a path is denoted by $v_1 v_2 \ldots v_n$. If $v_i, v_j \in V(P)$, $v_i P v_j$ is the path $v_i v_{i+1} \ldots v_j$. The length of a path $P$ is the number of edges in it and is denoted by $||P||$. A cycle $C$ with vertex set $V(C) = \{v_1, v_2, \ldots, v_n\}$, and edge set $E(C) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\} \cup \{(v_n, v_1)\}$ is denoted as $C = v_1 v_2 \ldots v_n v_1$. We use $||C||$ to denote the length of cycle $C$.

## 3 Holes in Bipartite Powers

Let $H$ be a bipartite graph. Let $\mathcal{B}(H)$ be a family of graphs constructed from $H$ in the following manner: $H' \in \mathcal{B}(H)$ if corresponding to each vertex $v \in V(H)$ there exists a nonempty bag of vertices, say $B_v$, in $H'$ such that (a) for every $x \in B_u$, $y \in B_v$, $(x, y) \in E(H')$ if and only if $(u, v) \in E(H)$, and (b) vertices within each bag in $H'$ are pairwise non-adjacent. Below we list a few observations about $H$ and every $H'$ (, where $H' \in \mathcal{B}(H)$):

**Observation 1.** *$H'$ is bipartite.*

**Observation 2.** *$H$ is an induced subgraph of $H'$.*

**Observation 3.** *Let $k$ be an integer such that $k \geq 4$. If $H$ is $k$-chordal, then so is $H'$.*

**Proof:** Any hole of size greater than $4$ in $H'$ cannot have more than one vertex from the same bag, say $B_v$, as such vertices have the same neighborhood. Hence, the vertices of a hole (of size greater than 4) in $H'$ belong to different bags and thus there is a corresponding hole of the same size in $H$. □

**Theorem 4.** *Let $m, k$ be positive integers with $k \geq 4$. Let $G$ be a bipartite graph. If $G$ is $k$-chordal, then so is $G^{[m]}$.*

**Proof:** We prove this by contradiction. Let $p$ denote the size of a largest induced cycle, say $C = u_0u_1 \ldots u_{p-1}u_0$, in $G^{[m]}$. Assume $p > k$. Then, $p \geq 6$ (since $k \geq 4$ and $G^{[m]}$ is bipartite). Between each $u_{i-1}$ and $u_i$, where $i \in \{0, \ldots, p-1\}$, there exists a shortest path of length not more than $m$ in $G$ [(i)]. Let $P_i$ be one such shortest path between $u_{i-1}$ and $u_i$ in $G$.

Let $H$ be the subgraph induced on the vertex set $\bigcup_{i=0}^{p-1} V(P_i)$ in $G$. As mentioned in the beginning of this section, construct a graph $H'$ from $H$, where $H' \in \mathcal{B}(H)$, in the following manner: for each $v \in V(H)$, let $|B_v| = |\{P_i \mid 0 \leq i \leq p-1, v \in V(P_i)\}|$ i.e., let $B_v$ have as many vertices as the number of paths in $\{P_0 \ldots P_{p-1}\}$ that share vertex $v$ in $H$. For each $i \in \{0, \ldots, p-1\}$, let $Q_i' = u_{i-1}Q_i$ be a shortest path between $u_{i-1}$ and $u_i$ in $H'$ such that no two paths $Q_i$ and $Q_j$ (where $i \neq j$) share a vertex [(i)]. From our construction of $H'$ from $H$ it is easy to see that such paths exist. Let $Q_i = v_{i,1}v_{i,2} \ldots v_{i,r_i}u_i$, where $r_i = ||Q_i|| \geq 0$. Thus, $Q_i' = u_{i-1}v_{i,1}v_{i,2} \ldots v_{i,r_i}u_i$. Clearly, $||Q_i'|| = ||P_i|| \leq m$. The reader may also note that the cycle $C$ ($= u_0u_1 \ldots u_{p-1}u_0$) which is present in $G^{[m]}$ will be present in $H^{[m]}$ and thereby in $H'^{[m]}$ too.

In order to prove the theorem, it is enough to show that there exists an induced cycle of size at least $p$ in $H'$. Then by combining Observation 3 and the fact that $H$ is an induced subgraph of $G$, we get $k \geq \mathcal{C}(G) \geq \mathcal{C}(H) \geq \mathcal{C}(H') \geq p$ contradicting our assumption that $p > k$. Hence, in the rest of the proof we show that $\mathcal{C}(H') \geq p$.

Consider the following drawing of the graph $H'$. Arrange the vertices $u_0, u_1, \ldots, u_{p-1}$ in that order on a circle in clockwise order. Between each $u_{i-1}$ and $u_i$ on the circle arrange the vertices $v_{i,1}, v_{i,2}, \ldots, v_{i,r_i}$ in that order in clockwise order. Recall that these vertices are the internal vertices of path $Q_i'$.

*Claim* 4.1. In this circular arrangement of vertices of $H'$, each vertex has an edge (in $H'$) with both its left neighbor and right neighbor in the arrangement.

Let $x_1, x_2 \in V(H')$, where $x_1 \in V(Q_i)$, $x_2 \in V(Q_j)$. We define the *clockwise distance from $x_1$ to $x_2$*, denoted by $clock\_dist(x_1, x_2)$, as the minimum non-negative integer $s$ such that $j = i + s$. Similarly, the *clockwise distance from $x_2$ to $x_1$*, denoted by $clock\_dist(x_2, x_1)$, is the minimum non-negative integer $s'$ such that $i = j + s'$. Let $x, y, z \in V(H')$. We say $y <_x z$ if scanning the vertices of $H'$ in clockwise direction along the circle starting from $x$, vertex $y$ is encountered before $z$. Let $x \in V(Q_i)$. Vertex $y$ is called the *farthest neighbor of $x$ before $z$* if $y \in N_{H'}(x)$, $y \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$, $y <_x z$, and for every other $w \in N_{H'}(x)$ either $z <_x w$ or $w \notin V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$ or both.

*Claim* 4.2. There always exists a vertex which is the farthest neighbor of $x$ before $z$, unless $(x, z) \in E(H')$ and $z \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$.

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(i) throughout this proof expressions involving subscripts of $u$, $P$, $Q$, and $Q'$ are to be taken modulo $p$. Every such expression should be evaluated to a value in $\{0, \ldots, p-1\}$. For example, consider a vertex $u_i$, where $i < p$ Then, $p + i = i$.

Let $\{A, B\}$ be the bipartition of the bipartite graph $H'$. We categorize the edges of $H'$ as follows: an edge $(x, y) \in E(H')$ is called an *l-edge*, if $l = \min(clock\_dist(x, y), clock\_dist(y, x))$.

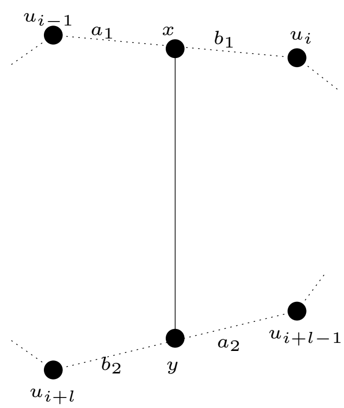


**Fig. 1:** $x \in V(Q_i), y \in V(Q_{i+l})$ and let $(x, y) \in E(H')$ be an $l$-edge, where $l > 2$. The dotted line between $u_{i-1}$ and $u_i$ indicate the path $Q_i$. Similarly, the dotted line between $u_{i+l-1}$ and $u_{i+l}$ indicate the path $Q_{i+l}$.

*Claim* 4.3. $H'$ cannot have an $l$-edge, where $l > 2$.

**Proof:** Suppose $H'$ has an $l$-edge, where $l > 2$, between $x \in Q_i$ and $y \in Q_{i+l}$ (see Fig. 1). Let $a_1 = ||u_{i-1}Q'_i x||$, $b_1 = ||xQ'_i u_i||$, $a_2 = ||u_{i+l-1}Q'_{i+l}y||$ and $b_2 = ||yQ'_{i+l}u_{i+l}||$. We consider the following two cases:
*Case* 1*:* $l$ is even
In this case $u_{i-1}$ and $u_{i+l-1}$ will be on the same side of the bipartite graph $H'$. Without loss of generality, let $u_{i-1}, u_{i+l-1} \in A$. Then, $u_i, u_{i+l} \in B$. We know that, for every $w_1, w_2 \in V(H'^{[m]})$ with $w_1 \in A$ and $w_2 \in B$, if $(w_1, w_2) \notin E(H'^{[m]})$ then $d_{H'}(w_1, w_2) \geq m + 2$ (recalling $m$ and $d_{H'}(w_1, w_2)$ are odd integers). Therefore, we have $a_1 + 1 + b_2 \geq d_{H'}(u_{i-1}, u_{i+l}) \geq m + 2$. Similarly, $b_1 + 1 + a_2 \geq d_{H'}(u_i, u_{i+l-1}) \geq m + 2$. Summing up the two inequalities we get, $(a_1 + b_1) + (a_2 + b_2) \geq 2m + 2$. This implies that either $||Q'_i||$ or $||Q'_{i+l}||$ is greater than $m$ which is a contradiction.
*Case* 2*:* $l$ is odd
(proof is similar to the above case and hence omitted).

Hence we prove the claim. □

We find a cycle $C' = z_0 z_1 \dots z_q z_0$ in $H'$ using Algorithm 3.1 [(i)]. Please read the algorithm before proceeding further. .

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[(i)] throughout this proof expressions involving subscripts of $z$ are to be taken modulo $q + 1$. Every such expression should be evaluated to a value in $\{0, \dots, q\}$. For example, consider a vertex $z_a$, where $a < q + 1$. Then, $q + 1 + a = a$.

**Algorithm 3.1** Finding Cycle $C'$ in $H'$ such that $||C'|| \geq ||C||$

1. $l \leftarrow \max_{l'}(H'$has an $l'$-edge$)$. Without loss of generality assume that this $l$-edge is between a vertex in $Q_0$ and a vertex in $Q_l$
2. Scan the vertices of $Q_0$ in clockwise direction to find the first vertex $z_0$, where $z_0 \in V(Q_0)$, which has an $l$-edge to a vertex in $Q_l$.
3. Scan the vertices of $Q_l$ in clockwise direction to find the last vertex in $Q_l$ which is a neighbor of $z_0$ in $H'$. Call it $z_1$.
4. Find the farthest neighbor of $z_1$ before $z_0$. Call it $z_2$. /* refer proof of Claim 4.4 for a proof of existence of such a $z_2$*/
5. $s \leftarrow 2$.

**while** $(z_s, z_0) \notin E(H')$ **do**
  6. Find the farthest neighbor of $z_s$ before $z_0$. Call it $z_{s+1}$. /* such a neighbor exists by Claim 4.2*/
  7. $s \leftarrow s + 1$.

**end while**

8. $q \leftarrow s$.
9. Return cycle $C' = z_0 z_1 \ldots z_q z_0$.

*Claim* 4.4. There always exists a farthest neighbor of $z_1$ before $z_0$.

**Proof:** Note that $z_0 \in Q_0$ and $z_1 \in Q_l$, where $l \leq 2$ (by Claim 4.3). Recalling that $||C|| = p \geq 6$, we have $z_0 \notin V(Q_l) \cup V(Q_{l+1}) \cup V(Q_{l+2})$. Hence by Claim 4.2, the claim is true. □

*Claim* 4.5. The while loop in Algorithm 3.1 terminates after a finite number of iterations.

**Proof:** From Claim 4.1, we know that each vertex has an edge (in $H'$) with both its left neighbor and right neighbor in the circular arrangement. Each time when Step 6 of Algorithm 3.1 is executed, a vertex $z_{s+1}$ is chosen such that $z_{s+1}$ is the farthest neighbor of $z_s$ before $z_0$. Since $H'$ is a finite graph, there will be a point of time in the execution of the algorithm when in Step 6 it picks a $z_{s+1}$ such that $(z_{s+1}, z_0) \in E(H')$. □

From Claim 4.5, we can infer that $C'$ is a cycle.

*Claim* 4.6. $C'$ is an induced cycle in $H'$.

**Proof:** Suppose $C'$ is not an induced cycle. Then there exists a chord $(z_a, z_b)$ in $C'$. Since $(z_a, z_b)$ is a chord, we have $b \neq a - 1$ or $b \neq a + 1$. Let $l = \max_{l'}(H'$ has an $l'$-edge$)$. Let $z_a \in V(Q_i)$, $z_b \in V(Q_j)$. We know that $\min(clock\_dist(z_a, z_b), clock\_dist(z_b, z_a)) \leq l$. Without loss of generality, assume $clock\_dist(z_a, z_b) \leq l \leq 2$ (from Claim 4.3). That is, $j - i \leq l \leq 2$ and $(z_a, z_b)$ is a $(j - i)$-edge. If $z_a = z_0$, then $z_b \neq z_1$ and the algorithm exits from the while loop, when $q = b$, thus returning a cycle $z_0 \ldots z_b z_0$. But in such a cycle $(z_b, z_0)$ is not a chord. Therefore, $z_a \neq z_0$. Similarly, $z_b \neq z_0$. We know that $z_{a+1} \neq z_b$, $z_{a+1} <_{z_a} z_b$, and $z_{a+1} \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$. Since $j - i \leq 2$, $z_b \in V(Q_i) \cup V(Q_{i+1}) \cup V(Q_{i+2})$. If $z_b <_{z_a} z_0$, then it contradicts the fact that $z_{a+1}$ is the farthest neighbor of $z_a$ before $z_0$. Therefore, $z_0 <_{z_a} z_b$. Then, either $z_b = z_1$ or $z_1 <_{z_a} z_b$. Recall that $l = \max_{l'}(H'$has an $l'$-edge$)$, and $(z_0, z_1)$ is an $l$-edge with $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$. Since (i) $(z_a, z_b)$ is a $(j - i)$-edge, where $j - i \leq l$, (ii) $z_0 <_{z_a} z_b$, and (iii) $z_b = z_1$ or $z_1 <_{z_a} z_b$, we have $l \geq j - i = clock\_dist(z_a, z_b) \geq clock\_dist(z_0, z_b) \geq clock\_dist(z_0, z_1) = l$. Hence, $j - i = l$ and

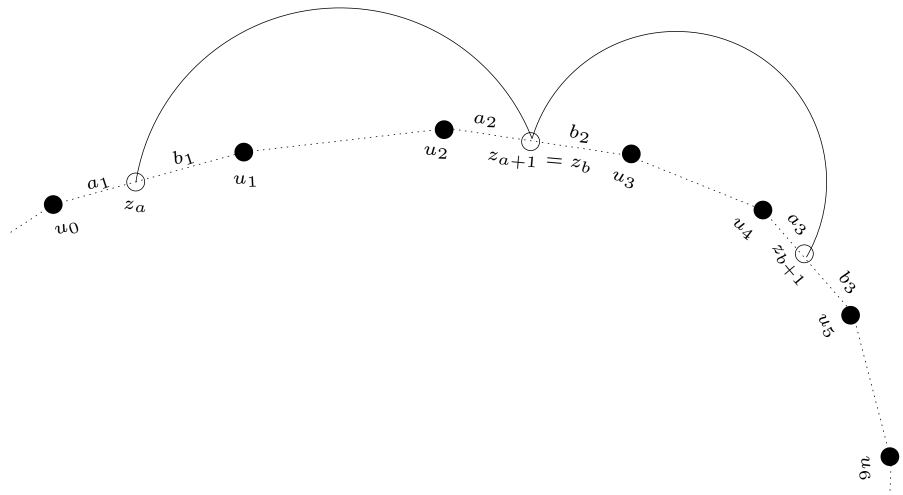


**Fig. 2:** Figure illustrates the case when path $P$ defined in Claim 4.8 is a trivial path. The dotted lines between each $u_{i-1}$ and $u_i$ indicate the path $Q'_i$. Each continuous arc corresponds to an edge in the cycle $C' = z_0 \ldots z_q z_0$.

$(z_a, z_b)$ is an $l$-edge. We know that $(z_0, z_1)$ is also an $l$-edge with $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$. Since $z_0 <_{z_a} z_b$ and $z_b = z_1$ or $z_1 <_{z_0} z_b$, we get $z_a \in V(Q_0)$ and $z_b \in V(Q_l)$. From Step 2 of the algorithm we know that $z_0$ is the first vertex (in a clockwise scan) in $Q_0$ which has an $l$-edge to a vertex in $Q_l$. This implies that, since $z_0 <_{z_a} z_b$, $z_a = z_0$ which is a contradiction. Hence we prove the claim. □

What is left now is to show that $q + 1 \geq p$, i.e., $||C'|| \geq ||C||$, where $C' = z_0 \ldots z_q z_0$ and $C = u_0 \ldots u_{p-1} u_0$. In order to show this, we state and prove the following claims.

*Claim* 4.7. For every $j \in \{0, \ldots, p-1\}$, $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') \neq \emptyset$.

**Proof:** Suppose the claim is not true. Find the minimum $j$ that violates the claim. Clearly, $j \neq 0$ as $z_0 \in V(Q_0)$. We claim that $z_q \in V(Q_{j-1})$. Suppose $z_q \notin V(Q_{j-1})$. Let $a = \max\{i \mid z_i \in V(Q_{j-1})\}$ (note that, since $j \neq 0$, by the minimality of $j$, $(V(Q_{j-1}) \cup V(Q_j)) \cap V(C') \neq \emptyset$ and therefore $V(Q_{j-1}) \cap V(C') \neq \emptyset$). Since $z_a \neq z_q$, by the maximality of $a$, we have $z_{a+1} \notin V(Q_{j-1})$. From our assumption, $(V(Q_j) \cup V(Q_{j+1})) \cap V(C') = \emptyset$ and therefore $z_{a+1} \notin V(Q_{j-1}) \cup V(Q_j) \cup V(Q_{j+1})$. Thus $z_a \neq z_q$ and $z_{a+1}$ is not the farthest neighbor of $z_a$ before $z_0$. This is a contradiction to the way $z_{a+1}$ is chosen by Algorithm 3.1. Hence, $z_q \in V(Q_{j-1})$. We know that $(z_q, z_0) \in E(H')$ with $z_q \in V(Q_{j-1})$ and $z_0 \in V(Q_0)$. Since $l = \max_{l'}(H'$ has an $l'$-edge), we have $\min(clock\_dist(z_q, z_0), clock\_dist(z_0, z_q)) \leq l$ . That is, $j \geq p + 1 - l$ or $j \leq 1 + l$. As $l \leq 2$ (by Claim 4.3), we have $j = p - 1$ or $j \leq 1 + l$. Since $z_0 \in V(Q_0)$, $(V(Q_{p-1}) \cup V(Q_0)) \cap V(C') \neq \emptyset$ and hence $j \neq p - 1$. Therefore, $j \leq 1 + l$. Since $z_0 \in V(Q_0)$ and $z_1 \in V(Q_l)$ (recall $l \leq 2$), we get $j = 1 + l$. We know that, for every $z_a, z_b \in V(C')$, if $a < b$ then $z_a <_{z_0} z_b$. Therefore, $z_1 <_{z_0} z_q$. We have $z_1 \in V(Q_l)$. Since $j = 1 + l$, we also have $z_q \in V(Q_l)$. Thus, we have $z_1, z_q \in V(Q_l)$ and $z_1 <_{z_0} z_q$. But this contradicts the fact that $z_1$ is the last vertex in $Q_l$ encountered in a clockwise scan that has $z_0$ as its neighbor. □

*Claim* 4.8. Let $(z_a, z_{a+1}), (z_b, z_{b+1}) \in E(C')$ be two 2-edges, where $a < b$. Let $P$, $P'$ denote the clockwise $z_{a+1} - z_b$, $z_{b+1} - z_a$ paths respectively in $C'$. Both $P$ and $P'$ contain at least one 0-edge.

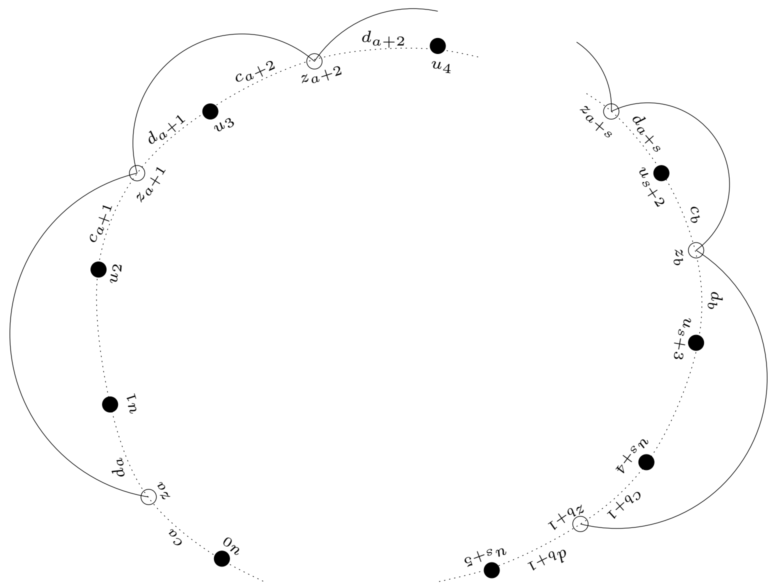

**Fig. 3:** Figure illustrates the case when path $P$ defined in Claim 4.8 is $P = z_{a+1}z_{a+2}\dots z_{a+1+s}$, where $s \geq 1$ and $z_{a+1+s} = z_b$. The dotted lines between each $u_{i-1}$ and $u_i$ indicate the path $Q'_i$. Each continuous arc corresponds to an edge in the cycle $C' = z_0 \dots z_q z_0$.

**Proof:** Consider the path $P$ (proof is similar in the case of path $P'$). Path $P$ is a non-trivial path only if $z_{a+1} \neq z_b$. Suppose $z_{a+1} = z_b$ (see Fig. 2). Let $z_a \in V(Q_f)$. For the sake of ease of notation, assume $f = 1$ (the same proof works for any value of $f$). Let $a_1 = ||u_0Q'_1z_a||$, $b_1 = ||z_aQ'_1u_1||$, $a_2 = ||u_2Q'_3z_b||$, $b_2 = ||z_bQ'_3u_3||$, $a_3 = ||u_4Q'_5z_{b+1}||$, and $b_3 = ||z_{b+1}Q'_5u_5||$. We know that, for every $w_1, w_2 \in V(H'^{[m]})$ with $w_1 \in A$ and $w_2 \in B$, if $(w_1, w_2) \notin E(H'^{[m]})$ then $d_{H'}(w_1, w_2) \geq m + 2$. Since $(u_0, u_3) \notin E(H'^{[m]})$, $(u_1, u_4) \notin E(H'^{[m]})$ and $(u_2, u_5) \notin E(H'^{[m]})$, we have $a_1 + b_2 \geq m + 1$, $b_1 + a_3 \geq m$, and $a_2 + b_3 \geq m + 1$. Adding the three inequalities and by applying an easy averaging argument we can infer that either $a_1 + b_1 = ||Q_1|| > m$, $a_2 + b_2 = ||Q_3|| > m$, or $a_3 + b_3 = ||Q_5|| > m$ which is a contradiction. Therefore $P$ is a non-trivial path i.e., $z_{a+1} \neq z_b$. Assume $P$ does not contain any 0-edge. Let $P = z_{a+1}z_{a+2}\dots z_{a+1+s}$, where $s \geq 1$, $a + 1 + s = b$, and $(z_{a+1}, z_{a+2})\dots(z_{a+s}, z_{a+1+s})$ are 1-edges (see Fig. 3). Since $(u_0, u_3) \notin E(H'^{[m]})$, $(u_1, u_4) \notin E(H'^{[m]})$, we have $c_a + d_{a+1} \geq m + 1$ and $d_a + d_{a+2} \geq m$ (please refer Fig. 3 for knowing what $c_a, d_a, \dots, c_{b+1}, d_{b+1}$ are). Summing up the two inequalities, we get $d_{a+1} + d_{a+2} \geq 2m + 1 - (c_a + d_a)$. We know that, for each $i \in \{0, \dots p - 1\}$, $||Q'_i|| \leq m$. Therefore, we have $c_a + d_a \leq m$. Hence, $d_{a+1} + d_{a+2} \geq m + 1$. Since $(c_{a+1} + d_{a+1}) + (c_{a+2} + d_{a+2}) \leq 2m$, we get

$$c_{a+1} + c_{a+2} \leq m - 1 \tag{1}$$

Since $(u_{s+2}, u_{s+5}) \notin E(H'^{[m]})$, $(u_{s+1}, u_{s+4}) \notin E(H'^{[m]})$, we have,

$$\begin{aligned} c_b + d_{b+1} &\geq m + 1 \\ c_{a+s} + c_{b+1} &\geq m \end{aligned}$$

Summing up the two inequalities, we get

$$c_b + c_{a+s} \geq 2m + 1 - (c_{b+1} + d_{b+1})$$

Since $b = a + s + 1$ and $c_{b+1} + d_{b+1} \leq m$, we get

$$c_{a+s+1} + c_{a+s} \geq m + 1 \qquad (2)$$

Substituting for $s = 1$ in Inequality 2, we get $c_{a+2} + c_{a+1} \geq m + 1$. But this contradicts Inequality 1. Hence $s > 1$. Suppose $s = 2$. Since $(u_2, u_5) \notin E(H'^{[m]}))$, we have $c_{a+1} + d_{a+3} \geq m$. Adding this with Inequality 2, we get $c_{a+1} + c_{a+2} \geq (2m + 1) - (c_{a+3} + d_{a+3}) \geq m + 1$. But this contradicts Inequality 1. Hence $s > 2$. Since $(u_s, u_{s+3}) \notin E(H'^{[m]})), \ldots, (u_2, u_5) \notin E(H'^{[m]}))$, we have the following inequalities:-

$$\begin{aligned} c_{a+s-1} + d_{a+s+1} &\geq m \\ \vdots \quad &\vdots \quad \vdots \\ c_{a+1} + d_{a+3} &\geq m \end{aligned}$$

Adding the above set of inequalities and applying the fact that $c_i + d_i \leq m$, $\forall i \in \{0, \ldots q\}$, we get $c_{a+1} + c_{a+2} + d_{a+s} + d_{a+s+1} \geq 2m$. Adding this with Inequality 2, we get $c_{a+1} + c_{a+2} \geq (3m + 1) - (c_{a+s+1} + d_{a+s+1}) - (c_{a+s} + d_{a+s}) \geq m + 1$. But this contradicts Inequality 1. Hence we prove the claim. □

*Claim* 4.9. For every $j, j' \in \{0, \ldots, p - 1\}$, where $j < j'$ and $(V(Q_j) \cup V(Q_{j'})) \cap V(C') = \emptyset$, there exist $i, i' \in \{0, \ldots, p - 1\}$, where only $i$ satisfies $j < i < j'$, such that $|V(Q_i) \cap V(C')| \geq 2$ and $|V(Q_{i'}) \cap V(C')| \geq 2$.

**Proof:** By Claim 4.7, (i) $j' \neq j + 1$ or $j' \neq j - 1$, and (ii) there exist $r, r' \in \{0, \ldots, q\}$ such that $(z_r, z_{r+1})$ is a 2-edge with its endpoints on $Q_{j-1}$ and $Q_{j+1}$ and $(z_{r'}, z_{r'+1})$ is a 2-edge with its endpoints on $Q_{j'-1}$ and $Q_{j'+1}$. By Claim 4.8, we know that if $P$, $P'$ denote the clockwise $z_{r+1} - z_{r'}$, $z_{r'+1} - z_r$ paths respectively in $C'$, then both $P$ and $P'$ contains at least one 0-edge. This proves the claim. □

In order to show that the size of cycle $C'$ $(= z_0 \ldots z_q z_0)$ is at least $p$, we consider the following three cases:-

*Case* $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 0$: In this case, for every $j \in \{0, \ldots p - 1\}$, $Q_j$ contributes to $V(C')$ and therefore $||C'|| \geq p = ||C||$.

*Case* $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| = 1$: Let $Q_j$ be that only path (among $Q_0 \ldots Q_{p-1}$) that does not contribute to $V(C')$. Then we claim that there exists a $Q_{j'}$, where $j' \neq j$, such that $V(C') \cap V(Q_{j'}) \geq 2$. Suppose the claim is not true then it is easy to see that $||C'|| = p - 1$ which is an odd number thus contradicting the bipartitedness of $H'$. Hence the claim is true. Now, by applying the claim it is easy to see that $||C'|| = \sum_j |V(C') \cap V(Q_j)| \geq p = ||C||$.

*Case* $|\{Q_j \in \{Q_0 \ldots Q_{p-1}\} \mid V(Q_j) \cap V(C') = \emptyset\}| > 1$: Scan vertices of $H'$ starting from any vertex in clockwise direction. Claim 4.9 ensures that between every $Q_j$ and $Q_{j'}$, which do not contribute to $V(C')$, encountered there exists a $Q_i$ which compensates by contributing at least two vertices to $V(C')$. Therefore, $||C'|| \geq p = ||C||$. ■

# 4 Discussion

An interesting open question that naturally follows from our result is the following: given a graph $G$ and positive integers $k, m$ where $k \geq 4$, if $G^{[m]}$ is $k$-chordal, then is $G^{[m+2]}$ also $k$-chordal? As mentioned earlier, Brandstädt et al. in [3] showed a similar result in the context of *ordinary* graph powering. They showed that, for every graph $G$, if $G^m$ is $k$-chordal, then so is $G^{m+2}$, where $k$, $m$ are positive integers with $k \geq 3$. A straightforward extension of their proof technique doesn't seem to work in our context due to the bipartite nature of the powering that we consider.

# References

[1] R. Balakrishnan and P. Paulraja. Powers of chordal graphs. *J. Aust. Math. Soc. Ser. A*, 35:211–217, 1983.

[2] H. L. Bodlaender and D. M. Thilikos. Treewidth for graphs with small chordality. *Discrete Applied Mathematics*, 79:45–61, 1997.

[3] Andreas Brandstädt, Van Bang Le, and Thomas Szymczak. Duchet-type theorems for powers of hhd-free graphs. *Discrete Mathematics*, 177(1-3):9–16, 1997.

[4] L. Sunil Chandran, Mathew C. Francis, and Rogers Mathew. Chordal bipartite graphs with high boxicity. *Graphs and Combinatorics*, 27(3):353–362, 2011.

[5] L. Sunil Chandran and L. Shankar Ram. On the number of minimum cuts in a graph. In *Proceedings of the 8th International computing and combinatorics conference, LNCS 2387*, pages 220–230, 2002.

[6] L. Sunil Chandran, C.R. Subramanian, and Vadin V. Lozin. Graphs of low chordality. To appear in Dicrete Mathematics and Theoretical Computer Science, 2005.

[7] Gerard J. Chang and George L. Nemhauser. The $k$-domination and $k$-stability problems on sun-free chordal graphs. *SIAM Journal on Algebraic and Discrete Methods*, 5(3):332–345, 1984.

[8] Yon Dourisboure. Compact routing schemes for generalised chordal graphs. *Journal of Graph Algorithms and Applications*, 9:277–297, 2005.

[9] Feodor F. Dragan. Estimating all pairs shortest paths in restricted graph families: a unified approach. *Journal of Algorithms*, 57(1):1 – 21, 2005.

[10] P. Duchet. Classical perfect graphs. *Ann. Discrete Math.*, 21:67–96, 1984.

[11] Carsten Flotow. On powers of m-trapezoid graphs. *Discrete Appl. Math.*, 63(2):187–192, 1995.

[12] Carsten Flotow. On powers of circular arc graphs and proper circular arc graphs. *Discrete Appl. Math.*, 69(3):199–207, 1996.

[13] A Lubiw. $\gamma$-free matrices. Master's thesis, Dept. of combinatorics and Optimization, University of Waterloo, 1982.

[14] A. Raychaudhuri. On powers of interval and unit interval graphs. *Congr.Numerantium*, 59:235–242, 1987.

[15] A. Raychaudhuri. On powers of strongly chordal and circular graphs. *Ars Combinatoria*, 34:147–160, 1992.

[16] Jeremy P. Spinrad. Finding large holes. *Information Processing Letters*, 39(4):227 – 229, 1991.