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PENALIZATION OF A STOCHASTIC VARIATIONAL INEQUALITY MODELING AN ELASTO-PLASTIC PROBLEM WITH NOISE.

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Abstract. In a recent work of A.Bensoussan and J.Turi *Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators, AMO, 2008*, it has been shown that the solution of a stochastic variational inequality modeling an elasto-plastic oscillator excited by a white noise has a unique invariant probability measure. The latter is useful for engineering in order to evaluate statistics of plastic deformations for large times of a certain type of mechanical structure. However, in terms of mathematics, not much is known about its regularity properties. From then on, an interesting mathematical question is to determine them. Therefore, in order to investigate this question, we introduce in this paper approximate solutions of the stochastic variational inequality by a penalization method. The idea is simple: the inequality is replaced by an equation with a nonlinear additional term depending on a parameter n penalizing the solution whenever it goes beyond a prespecified area. In this context, the dynamics is smoother. In a first part, we show that the penalized process converges towards the original solution of the aforementioned inequality on any finite time interval as n goes to ∞. Then, in a second part, we justify that for each n it has at least one invariant probability measure. We conjecture that it is unique, but unfortunately we are not able to prove it. Finally, we provide numerical experiments in support of our conjecture. Moreover, we give an empirical convergence rate of the sequence of measures related to the penalized process.

Résumé. Dans un travail récent de A.Bensoussan et J.Turi *Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators, AMO, 2008*, il a été montré que la solution d'une inéquation variationnelle stochastique modélisant un oscillateur élasto-plastique excité par un bruit blanc admet une unique mesure de probabilité invariante. Cette dernière est utile en science de l'ingénieur pour estimer les statistiques des déformations plastiques en temps grands d'un certain type de structure mécanique. Dès lors, un problème mathématique intéressant est de déterminer la régularité de cette mesure. Afin d'étudier ce problème, nous introduisons ici des solutions approchées de l'inéquation par une méthode de pénalisation. Ainsi, l'inéquation est remplacée par une équation avec un terme nonlinéaire additionnel dépendant d'un certain paramètre n pénalisant la solution en dehors d'un domaine admissible. Dans ce contexte, la dynamique stochastique est plus régulière. Dans un premier temps, nous montrons la convergence lorsque n tend vers ∞ du processus pénalisé vers la solution de l'inéquation sur tout intervalle de temps fini. Puis dans un second temps, nous montrons que pour chaque n le processus pénalisé est dissipatif et qu'ainsi il admet au moins une mesure invariante. Malheureusement, bien qu'ayant quelques pistes, nous ne sommes pas (encore) capables de démontrer son unicité, que nous conjecturons. En contrepartie, nous l'étudions numériquement et donnons un taux de convergence empirique de la suite des mesures du processus pénalisé.

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1. Background and Motivations

Phenomena with memory occur naturally in random mechanics. In particular, behaviors of a certain class of mechanical structures, which experience elastic deformations and plastic (permanent) deformations under a random forcing (e.g earthquake), can be represented using an elasto-perfectly-plastic (EPP) oscillator with noise as presented in (1.1) (see [F08]). Recently, A. Bensoussan and J. Turi [BT08] have shown that the dynamics of this oscillator can be described in the mathematical framework of a *stochastic variational inequality* (SVI) as presented in (1.2). The underlying stochastic process related to this inequality belongs to the class of non-linear and degenerate dynamical systems that constitutes an important research theme in stochastic processes and their applications. In [BT08, BM12], it has been shown that the aforementioned SVI has an appropriate structure for the study in large time of an EPP oscillator in the sense that it allows to show existence and uniqueness of an invariant probability for the solution (see (1.3)). In terms of engineering, this probability measure describes the probabilities that (1.1) experiences an elastic or a plastic state for large times, hence it is relevant for engineering purposes. Therefore, a numerical algorithm solving this probability measure has been proposed in [BMPT09] and then it has been applied to the estimation of the frequency of plastic deformation of (1.1) in [FM12].

However, from a mathematical point of view, we do not know much about the regularity of this measure. Therefore, in order to investigate this issue, a natural approach in the context of a SVI is to proceed by penalization. The idea consists in replacing the inequality by an equation with an additional nonlinear term. In this way, the solution is not constrained anymore, but when it goes beyond a given prespecified area then the nonlinear term becomes very large such that the solution is forced to come back inside that area. In this context, the dynamical system, though still degenerate, is smoother and its dynamics is described below in Equation (1.5).

The goal of this work, done at CEMRACS2013, is to investigate the properties of solutions of (1.5) that are approximate solutions of (1.2) by a penalization approach. In Theorem 2.1, we prove the convergence of the penalized process toward the solution of the SVI on any finite time interval. Then in Theorem 2.2 , we show that the penalized process is dissipative so that there exists at least one invariant probability measure $m_n(y, z)$. Our conjecture is that there exists a unique and regular invariant probability measure which solves this equation. Unfortunately, we are not able to prove uniqueness but we provide numerical experiment in support of our conjecture. This approximation would be very relevant to understand the properties of the invariant measure related to the SVI.

1.1. Settings of an elasto-perfectly-plastic oscillator

In the engineering literature, the dynamics of an elasto-perfectly-plastic oscillator is formulated in terms of a stochastic process $x(t)$, which stands for the total deformation of the oscillator that evolves with hysteresis; and the evolution of $x(t)$ is described formally by the equation

$$\ddot{x} + c_0\dot{x} + \mathbf{F} = \dot{w}, \tag{1.1}$$

with the initial displacement and velocity $x(0) = 0$ and $\dot{x}(0) = 0$ respectively. Here $c_0 > 0$ is the viscous damping coefficient, w is a Wiener process. The restoring force $\mathbf{F}$ is a nonlinear functional that depends on the entire trajectory $\{x(s), 0 \leq s \leq t\}$ up to time t; its nonlinearity comes from the switching of regimes from a linear phase (called elastic) to a nonlinear one (called plastic), or vice versa. Precisely, the restoring force $\mathbf{F}$ is expressed as follows:

$$\mathbf{F}(t) = \begin{cases} kY, & \text{if} \quad x(t) - \Delta(t) = Y, \\ k(x(t) - \Delta(t)), & \text{if} \quad -Y < x(t) - \Delta(t) < Y, \\ -kY, & \text{if} \quad x(t) - \Delta(t) = -Y, \end{cases}$$

where k is a stiffness coefficient, $\Delta(t)$ is the permanent (or plastic) deformation in $x(t)$ and Y is an elasto-plastic bound.

1.2. **Stochastic variational inequality for** (1.1)

From [BT08], we know that the relationship between the velocity $y(t) := \dot{x}(t)$ and $z(t) := x(t) - \Delta(t)$ is governed by a (stochastic) variational inequality as follows: there exists exactly one process $(y(t), z(t)) \in \mathbb{R} \times [-Y, Y]$ that satisfies

$$\begin{cases} \mathrm{d}y(t) = -(c_0 y(t) + kz(t))\,\mathrm{d}t + \mathrm{d}w(t), \\ (\mathrm{d}z(t) - y(t)\mathrm{d}t)(\phi - z(t)) \geq 0, \quad \forall\ |\phi| \leq Y, \quad |z(t)| \leq Y. \end{cases} \tag{1.2}$$

A general framework dealing with this class of inequalities can be found in [BL82]. In these settings, the plastic deformation is given by:

$$\Delta(t) = \int_0^t y(s)\mathbf{1}_{\{|z(s)|=Y\}}\mathrm{d}s.$$

Then, it has been shown that there exists a unique limiting probability measure ν for $(y(t), z(t))$ as t goes to ∞ in the following sense: for all bounded measurable functions f,

$$\lim_{t\to\infty} \mathbb{E}[f(y(t), z(t))] = \nu(f).$$

It is also known from the theory of Markov processes, that the invariant probability measure ν is characterized by a an ultra-weak variational formulation: for all smooth functions f

$$\int_D Af(y,z)\nu(\mathrm{d}y\mathrm{d}z) + \int_{D^+} B_+ f(y,Y)\nu(\mathrm{d}y) + \int_{D^-} B_- f(y,-Y)\nu(\mathrm{d}y) = 0. \tag{1.3}$$

with

$$A\varphi := -\frac{1}{2}\frac{\partial^2\varphi}{\partial y^2} + (c_0 y + kz)\frac{\partial\varphi}{\partial y} - y\frac{\partial\varphi}{\partial z},$$

$$B_+\varphi := -\frac{1}{2}\frac{\partial^2\varphi}{\partial y^2} + (c_0 y + kY)\frac{\partial\varphi}{\partial y},$$

$$B_-\varphi := -\frac{1}{2}\frac{\partial^2\varphi}{\partial y^2} + (c_0 y - kY)\frac{\partial\varphi}{\partial y}.$$

and also $D := \mathbb{R} \times (-Y, Y)$ is the elastic domain, $D^+ := (0, \infty) \times \{Y\}$ is the positive plastic domain and $D^- := (-\infty, 0) \times \{-Y\}$ is the negative plastic domain. Moreover, the measure ν has a probability density function (pdf) m composed of three L^1 functions

(1) an elastic part: $m(y, z)$ on D,
(2) a positive plastic part: $m(y, Y)$ on D^+,
(3) a negative plastic part: $m(y, -Y)$ on D^-

with the condition $m(y, z)$, $m(y, Y)$, $m(y, -Y) \geq 0$ satisfying

$$\int_D m(y,z)\mathrm{d}y\mathrm{d}z + \int_{D^+} m(y,Y)\mathrm{d}y + \int_{D^-} m(y,-Y)\mathrm{d}y = 1. \tag{1.4}$$

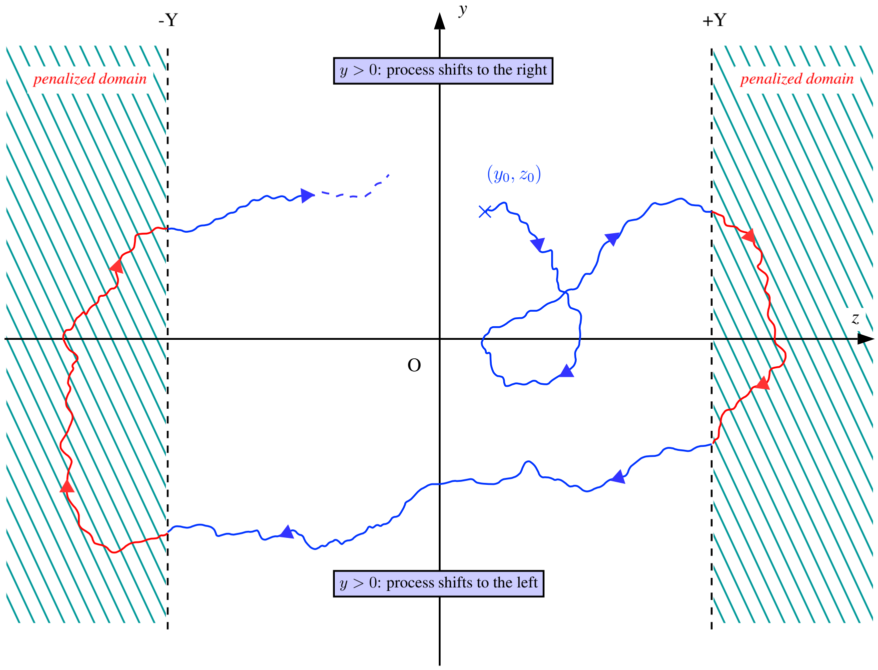

FIGURE 1. We observe $(z_n(t), y_n(t))$: the process is not constrained, but when it goes beyond a given prespecified area, then the nonlinear term becomes very large such that the solution is forced to come back inside that area.

1.3. **Settings of an approximation of** (1.2) **by a penalization technique**

In this work, we study approximate solutions of (1.2) by a penalization technique. For $n \geq 1$, we introduce in the second component $z(t)$ a penalization term depending on a parameter : n (for the magnitude of the penalization). Thus we use the notation $(y_n(t), z_n(t))$ for this approximate process. The evolution of the system is described by the stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}y_n(t) = -\big(c_0 y_n(t) + k z_n(t)\big)\mathrm{d}t + \mathrm{d}w(t), \\ \mathrm{d}z_n(t) = y_n(t)\mathrm{d}t - n\big(z_n(t) - \pi(z_n(t))\big)\mathrm{d}t \\ \text{with the initial condition } (y_n(0), z_n(0)) = (y_0, z_0) \end{cases} \tag{1.5}$$

where

$$\pi(z) \text{ is the projection of } z \text{ on the convex set (interval) } K := [-Y, Y].$$

Remark 1.1. *Note that the second equation in* (1.5) *can be written without using explicitly the projection as follows:*

$$dz_n(t) = y_n(t)\,dt - n\,\mathrm{sign}(z_n(t)) \cdot (|z_n(t)| - Y)^+\,dt.$$

Let us first mention an important property of the projection $\pi(.)$

$$\text{For any two points } x \text{ and } x' \text{ in } \mathbb{R}, \quad (x' - x)(x - \pi(x)) \leq (x' - \pi(x'))(x - \pi(x)). \tag{1.6}$$

2. Main results

Our first result concerns the convergence of the solution (y_n, z_n) related to (1.5) toward the solution (y, z) of (1.2) in the following sense:

Theorem 2.1. *Fix $T > 0$ and consider the processes $(y_n(t), z_n(t))$ and $(y(t), z(t))$ satisfying* (1.5) *and* (1.2) *respectively. Then the following convergence property holds*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left\{|y_n(t) - y(t)|^2 + |z_n(t) - z(t)|^2\right\}\right] = 0. \tag{2.1}$$

Moreover, for any $\lambda > 0$,

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty \exp(-\lambda t) f(y_n(t), z_n(t))\, dt\right] = \mathbb{E}\left[\int_0^\infty \exp(-\lambda t) f(y(t), z(t))\, dt\right]. \tag{2.2}$$

Our second result concerns the existence of an invariant probability for (y_n, z_n), for n fixed.

Theorem 2.2. *Fix n, then the process (y_n, z_n) admits at least one invariant probability measure, denoted by ν_n, whose density $m_n(y, z)$ with respect to Lebesgue measure must solve (at least in the sense of the distributions) the following Fokker-Planck equation:*

$$\frac{\partial}{\partial z}\Big(m_n(y,z)[-y + n(z - \pi(z))]\Big) + \frac{\partial}{\partial y}\Big(m_n(y,z)(c_0 y + kz)\Big) + \frac{1}{2}\frac{\partial^2}{\partial y^2} m_n(y,z) = 0, \quad (y,z) \in \mathbb{R}^2. \tag{2.3}$$

2.1. Preliminary lemmas and proof of the main results

In this section, we give preliminary lemmas and the proofs of Theorem 2.1 and Theorem 2.2. For the convenience of the reader, the proofs of the preliminary lemmas are given in Section 3.

Let us first present three useful Lemmas for Theorem 2.1. Fix $T > 0$.

Lemma 2.3. *There exists $C(T)$ such that*

$$\forall\, (m,n) \in (\mathbb{N}^*)^2 \qquad \mathbb{E}\left[\int_0^T |z_m(s) - \pi(z_m(s))||(z_n(s) - \pi(z_n(s))|\, ds\right] \le \frac{1}{mn} C(T).$$

Lemma 2.4. *The sequence (y_n, z_n) satisfies the following Cauchy property:*

$$\forall\, \epsilon > 0 \quad \exists\, N \in \mathbb{N} \quad \forall n, m > N \qquad \mathbb{E}\left[\sup_{t\in[0,T]} \left\{|y_n(t) - y_m(t)|^2 + |z_n(t) - z_m(t)|^2\right\}\right] < \epsilon.$$

Next, we introduce the following notations:

$$\tilde{y}(t) := \lim_{n\to\infty} y_n(t), \quad \tilde{z}(t) := \lim_{n\to\infty} z_n(t) \quad \text{and} \quad \tilde{\Delta}(t) := \lim_{n\to\infty} n\big[z_n(t) - \pi(z_n(t))\big].$$

Then

Lemma 2.5. *The process $\big(\tilde{y}(t), \tilde{z}(t), \tilde{\Delta}(t)\big)$ satisfies the following properties:*

(a) $\tilde{y}(0) = \tilde{y}_0\,, \qquad \tilde{z}(0) = \tilde{z}_0,$

- ***(b)*** *$\tilde{y}, \tilde{z}$ and $\tilde{\Delta}$ are adapted and continuous,*
- ***(c)*** $|\tilde{z}(t)| \leq Y \qquad \forall\, t \qquad$ *a.s.,*

and $\tilde{\Delta}$ satisfies the following properties:

- ***(d)*** *it is of bounded variation,*
- ***(e)*** $\displaystyle\int_{t_1}^{t_2} \mathbf{1}_{\{\tilde{z}(t) \in]-Y,Y[\}} d\tilde{\Delta}(t) = 0 \qquad \forall\, t_1 \leq t_2 \qquad$ *a.s.,*
- ***(f)*** $d\tilde{z}(t) = \tilde{y}(t)dt - \mathbf{1}_{\{\tilde{z}(t) = \pm Y\}} d\tilde{\Delta}(t)$*.*

We will also formulate the following Theorem from [BL82] (page 49) in our context as follows:

Theorem 2.6 ([BL82])**.** *There exists a unique process $\big(\tilde{y}(t), \tilde{z}(t), \tilde{\Delta}(t)\big)$ taking values in $\mathbb{R}^2$ and satisfying properties **(a)** $\sim$ **(f)**. Moreover this solution is characterized by the two following properties:*

- ***(i)*** *$\big(\tilde{y}, \tilde{z}\big)$ is continuous, adapted, a.s. for each t we have: $|\tilde{z}(t)| \leq Y$ and:*

$$\begin{aligned} & \tilde{y}(t) - y_0 + \int_0^t \big(c_0\tilde{y}(s) + k\tilde{z}(s)\big)ds - w(t) \\ \text{and} \qquad & \tilde{z}(t) - z_0 - \int_0^t \tilde{y}(s)ds \end{aligned}$$

are of bounded variation, and are zero for $t = 0$.

- ***(ii)*** *For any $(\varphi_1, \varphi_2) \in \mathbb{R} \times [-Y, Y]$,*

$$\big(\varphi_1 - \tilde{y}(t)\big) \cdot \big(d\tilde{y}(t) + (c_0\tilde{y}(t) + k\tilde{z}(t))ds - dw(t)\big) + \big(\varphi_2 - \tilde{z}(t)\big) \cdot \big(d\tilde{z}(t) - \tilde{y}(t)dt\big) \geq 0.$$

Lemma 2.3 is employed in the proof of Lemma 2.4 and then Lemmas 2.4, 2.5 and Theorem 2.6 are employed directly in the proof of Theorem 2.1 as shown below.

Proof of Theorem 2.1. First, we proceed with the convergence: by Lemma 2.4, there exists a limit $\{(\tilde{y}(t), \tilde{z}(t)), t \geq 0\}$ in the sense of the norm $\mathbb{E}\left[\sup_{t\in[0,T]} \left\{|\tilde{y}(t)|^2 + |\tilde{z}(t)|^2\right\}\right]$ for $\{(y_n(t), z_n(t)), t \geq 0\}$ as n goes to ∞. Then, we identify the limit: by Lemma 2.5, we can apply Theorem 2.6 to $\big(\tilde{y}(t), \tilde{z}(t)\big)$. Indeed, point **(ii)** of the characterization Theorem 2.6 rewrites:

$$\begin{aligned} & \mathrm{d}y(t) = -(c_0y(t) + kz(t))ds + \mathrm{d}w(t) \\ \text{and} \qquad & \big(\varphi - z(t)\big) \cdot \big(\mathrm{d}z(t) - y(t)dt\big) \geq 0, \qquad \forall \varphi \in [-Y, Y], \end{aligned}$$

which matches, together with points **(a)** and **(c)** of Lemma 2.5, the SVI described by (1.2). Hence $\big(\tilde{y}(t), \tilde{z}(t)\big)$ satisfies the SVI (1.2) and then: $\big(\tilde{y}(t), \tilde{z}(t)\big) = \big(y(t), z(t)\big)$ (by uniqueness of the solution). Finally, as f is bounded, for all $\epsilon > 0$, there exists $T_\epsilon > 0$ such that

$$\mathbb{E}\left[\int_0^\infty \exp(-\lambda t)\big(f(y_n(t), z_n(t)) - f(y(t), z(t))\big)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^{T_\epsilon} \exp(-\lambda t)\big(f(y_n(t), z_n(t)) - f(y(t), z(t))\big)\mathrm{d}t\right] + \frac{\epsilon}{2}.$$

Hence, relying on the decomposition above, it is clear that (2.1) implies (2.2). ◻

Next, we present two useful Lemmas for Theorem 2.2.

Lemma 2.7. *Fix $n > c_0$,*

$$\forall\, t > 0 \qquad \mathbb{E}\left[y_n^2(t) + kz_n^2(t)\right] \leq ce^{-c_0 t} + c_n$$

with $c := y_0^2 + kz_0^2$ and $c_n := \frac{kn}{2\left(1-\frac{c_0}{n}\right)} Y^2 + 2$.

Let us introduce some notations. In the following $\mathcal{C}_b$ denotes the space of continuous and bounded functions on $\mathbb{R}$, and the operator $P(t)$ (depending on n) is defined by: $P(t)\phi(y,z) = \mathbb{E}[\phi(y_n(t), z_n(t))|(y_n(0), z_n(0)) = (y,z)]$ where $\phi \in \mathcal{C}_b$. We denote by $\mu_n(t)$ the probability law of $(y_n(t), z_n(t))$ on $\left(\mathbb{R}^2, \mathcal{B}\right)$, where $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}^2$, that is:

$$\mu_n(t)\phi := \mathbb{E}[\phi\big(y_n(t), z_n(t)\big)] = \mu_n(0)(P(t)\phi) \qquad \text{for } \phi \in \mathcal{C}_b.$$

We also define, for $T > 0$, the probability law μ_n^T on $\left(\mathbb{R}^2, \mathcal{B}\right)$ by:

$$\mu_n^T \phi := \frac{1}{T}\int_0^T \mathbb{E}[\phi\big(y_n(t), z_n(t)\big)]dt = \frac{1}{T}\int_0^T \mu_n^T(0)(P(t)\phi)dt \qquad \text{for } \phi \in \mathcal{C}_b.$$

Lemma 2.8. *For any sequence $T_i \uparrow \infty$, the sequence $\{\mu_n^{T_i}\}_{i\geq 1}$ is tight.*

Lemma 2.7 is used in the proof of Lemma 2.8 which, in turn, is employed directly in the proof of Theorem 2.2 as shown below.

Proof of Theorem 2.2. We will exhibit a tight sequence of measures so that we can extract a subsequence which converges to an invariant measure for (y_n, z_n). Consider a sequence $T_i \uparrow \infty$. For any $\mu_n(0)$, $\left\{\mu_n^{T_i}\right\}$ is tight by Lemma 2.8, hence there exists a subsequence $\left\{\mu_n^{T_{i_j}}\right\}_{j\geq 1}$ that converges weakly to a certain measure μ_n, *i.e*:

$$\forall \phi \in \mathcal{C}_b, \quad \mu_n^{T_{i_j}}(\phi) \to \mu_n(\phi).$$

Then μ_n is invariant since:

$$\begin{aligned}
\mu_n(P(t)\phi) &= \lim_j \mu_n^{T_{i_j}}\,(P(t)\phi) \\
&= \lim_j \frac{1}{T_{i_j}} \int_0^{T_{i_j}} \mu_n(0)\,(P(s)P(t)\phi)\,ds \\
&= \lim_j \frac{1}{T_{i_j}} \int_0^{T_{i_j}} \mu_n(0)\,(P(s+t)\phi)\,ds \\
&= \lim_j \frac{1}{T_{i_j}} \int_t^{t+T_{i_j}} \mu_n(0)\,(P(s)\phi)\,ds \\
&= \lim_j \frac{1}{T_{i_j}} \int_0^{T_{i_j}} \mu_n(0)\,(P(s)\phi)\,ds = \mu_n(\phi).
\end{aligned}$$

□

3. Proof of Lemmas

Proof of Lemma 2.3. The proof is composed of three steps. In the first step, we show that

$$\mathbb{E}\int_0^T (y_n(s))^2 \mathrm{d}s \leq C(T), \quad C(T) := \frac{y_0^2 + kz_0^2 + 2T}{2c_0}. \tag{3.1}$$

Indeed, from (1.5) we deduce by Itô's formula:

$$\begin{cases} \mathrm{d}\left(y_n^2(t)\right) = -2c_0 y_n^2(t)\mathrm{d}t - 2k y_n(t) z_n(t)\mathrm{d}t + 2y_n(t)\mathrm{d}w(t) + 2\mathrm{d}t \\ k\mathrm{d}\left(z_n^2(t)\right) = 2k y_n(t) z_n(t)\mathrm{d}t - 2kn z_n^2(t)\mathrm{d}t + 2kn z_n(t)\pi(z_n(t))\mathrm{d}t. \end{cases}$$

Combining these two equations we have:

$$\mathrm{d}\left(y_n^2(t)\right) + k\mathrm{d}\left(z_n^2(t)\right) + 2c_0 y_n^2(t)\mathrm{d}t = 2kn z_n(t)\big(\pi(z_n(t)) - z_n(t)\big)\mathrm{d}t + 2y_n(t)\mathrm{d}w(t) + 2\mathrm{d}t.$$

Integrating over $[0,t]$ and considering then the expectation, we have:

$$\begin{aligned} &\mathbb{E}\left[y_n^2(t)\right] + k\,\mathbb{E}\left[z_n^2(t)\right] + 2c_0\,\mathbb{E}\left[\int_0^t y_n^2(s)ds\right] \\ &= y_0^2 + kz_0^2 + 2kn\,\mathbb{E}\left[\int_0^t z_n(s)\big(\pi(z_n(s)) - z_n(s)\big)ds\right] + 2t \\ &\leq y_0^2 + kz_0^2 + 2t \end{aligned}$$

so that

$$\mathbb{E}\left[\int_0^T y_n^2(s)\mathrm{d}s\right] \leq \frac{c_0^2 + kz_0^2 + 2T}{2c_0}$$

since $x(\pi(x) - x) \leq 0$ for any $x \in \mathbb{R}$. Next, in the second step, we show that

$$\int_0^T (z_n(s) - \pi(z_n(s)))^2 \mathrm{d}s \leq \frac{1}{n^2}\int_0^T (y_n(s))^2 \mathrm{d}s \tag{3.2}$$

Indeed, let φ be the function defined on $\mathbb{R}$ by $\varphi(x) := (x - \pi(x))^2$. Then:

$$\begin{aligned} \mathrm{d}\varphi(z_n(s)) &= 2\left(z_n(s) - \pi(z_n(s))\right)\left[y_n(s) - n\left(z_n(s) - \pi(z_n(s))\right)\right]\mathrm{d}s \\ &= 2y_n(s)\left[z_n(s) - \pi(z_n(s))\right]\mathrm{d}s - 2n\varphi(z_n(s))\mathrm{d}s. \end{aligned}$$

Hence, integrating over $[0,t]$ and noticing that $\varphi(z_n(0)) = 0$ (since $|z_n(0)| = |z| < Y$):

$$\varphi(z_n(t)) = 2\int_0^t y_n(s)\left[z_n(s) - \pi(z_n(s))\right]ds - 2n\int_0^t \varphi(z_n(s))ds,$$

which can be rewritten as:

$$\begin{aligned} \int_0^t \varphi(z_n(s))ds &= \frac{1}{n}\int_0^t y_n(s)\left[z_n(s) - \pi(z_n(s))\right]ds - \frac{1}{2n}\varphi(z_n(t)) \\ &\leq \frac{1}{n}\sqrt{\int_0^t y_n^2(s)ds}\sqrt{\int_0^t \varphi(z_n(s))ds} \end{aligned}$$

yielding (3.1). Finally, we conclude in the last step by using (3.1) and taking the expectation in (3.2) to deduce that

$$\mathbb{E}\int_0^T (z_n(s) - \pi(z_n(s)))^2 \mathrm{d}s \leq \frac{C(T)}{n^2}. \tag{3.3}$$

Then we apply Cauchy-Schwarz inequality on $\mathbb{E}\int_0^T (z_n(s) - \pi(z_n(s)))(z_m(s) - \pi(z_m(s)))\mathrm{d}s$ to get the result. $\square$

Proof of Lemma 2.4. Let $n, m \in N$, by Equation (1.5) for $(y_n(t), z_n(t))$ and $(y_m(t), z_m(t))$, we have:

$$\begin{cases} \mathrm{d}(y_n(t) - y_m(t)) = -\big[c_0(y_n(t) - y_m(t)) + k(z_n(t) - z_m(t))\big]\mathrm{d}t \\ \mathrm{d}(z_n(t) - z_m(t)) = (y_n(t) - y_m(t))\mathrm{d}t - n\big[z_n(t) - \pi(z_n(t))\big]\mathrm{d}t + m\big[z_m(t) - \pi(z_m(t))\big]\mathrm{d}t \end{cases}$$

Hence:

$$\begin{cases} \frac{1}{2}\mathrm{d}\big[(y_n(t) - y_m(t))^2\big] = -c_0(y_n(t) - y_m(t))^2\mathrm{d}t - k(y_n(t) - y_m(t))(z_n(t) - z_m(t))\mathrm{d}t \\ \frac{k}{2}\mathrm{d}\big[(z_n(t) - z_m(t))^2\big] = k(y_n(t) - y_m(t))(z_n(t) - z_m(t))\mathrm{d}t \\ \qquad + k(z_n(t) - z_m(t)) \underbrace{\Big[- n\big(z_n(t) - \pi(z_n(t))\big) + m\big(z_m(t) - \pi(z_m(t))\big)\Big]}_{R_{m,n}(t)} \mathrm{d}t \end{cases}$$

where $R_{m,n}(t) := -n\big(z_n(t) - \pi(z_n(t))\big) + m\big(z_m(t) - \pi(z_m(t))\big)$. Combining these two equations and integrating over $[0, t]$, we have:

$$\begin{aligned} &\frac{1}{2}(y_n(t) - y_m(t))^2 + \frac{k}{2}(z_n(t) - z_m(t))^2 + c_0 \int_0^t (y_n(s) - y_m(s))^2\mathrm{d}s \\ &= k \int_0^t (z_n(s) - z_m(s))R_{m,n}(s)\mathrm{d}s \\ &\leq (m + n) \int_0^t \big[z_n(s) - \pi(z_n(s))\big]\big[z_m(s) - \pi(z_m(s))\big]\mathrm{d}s \end{aligned}$$

where we use the property (1.6) of the projection $\pi(.)$. And then we apply Lemma 2.3 to get

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\{|y_n(t) - y_m(t)|^2 + |z_n(t) - z_m(t)|^2\right\}\right] \leq C(T)\left(\frac{1}{m} + \frac{1}{n}\right).$$

That gives the Cauchy property. □

Proof of Lemma 2.5. Proof of **(a)** and **(b)**: Notice that $\big(\tilde{y}(0), \tilde{z}(0)\big) = (y_0, z_0)$ by definition of $(\tilde{y}, \tilde{z})$, and $(\tilde{y}, \tilde{z})$ is adapted and continuous a.s., as a consequence of uniform convergence, which stems from Lemma 2.4.

Proof of **(c)**: For each t, we a.s. have $|\tilde{z}(t) - \pi(\tilde{z}(t))| = 0$, *i.e.* $|\tilde{z}(t)| \leq Y$. Indeed:

$$|\tilde{z}(t) - \pi(\tilde{z}(t))| \leq |\tilde{z}(t) - z_n(t)| + |z_n(t) - \pi(z_n(t))| + |\pi(z_n(t)) - \pi(\tilde{z}(t))| \xrightarrow[n\to\infty]{} 0$$

since $z_n(t)$ converges to $\tilde{z}(t)$ and we have Equation (3.3).

Let us now prove **(d)**, **(e)** and **(f)**. We will denote by $||\cdot||_{VT}$ the total variation of a process and by $\mathcal{C}(A, B)$ the set of continuous functions from A to B (where A and B are metric sets). We also remind that $K = [-Y, Y]$.

Proof of **(d)**: Remark that $\tilde{\Delta}$ is the uniform limit of $\Delta_n(t) := \int_0^t y_n(s)ds - z_n(t) + z_n(0)$ and $\{\Delta_n, n \in \mathbb{N}^*\}$ is uniformly bounded in total variation. Hence $\tilde{\Delta}$ is of bounded variation by the following lemma of [GPP96]:

Lemma 3.1 (see Lemma 5.8 of [GPP96])**.** *Let $z_n \in \mathcal{C}\big([0, T], K\big)$ be a sequence that converges uniformly to a function $\tilde{z}$. Let $\Delta_n \in \mathcal{C}\big([0, T], K\big)$ be a sequence that converges uniformly to $\tilde{\Delta}$ and such that:*

$$a.s. \qquad \exists C \qquad ||\Delta_n||_{VT} \leq C.$$

Then:

$$a.s. \qquad ||\tilde{\Delta}||_{VT} < C \qquad and \qquad \int_0^T z_n \, d\Delta_n \xrightarrow[n\to\infty]{} \int_0^T z \, d\tilde{\Delta}.$$

Note that from this lemma, we also obtain:

$$a.s. \qquad \int_0^T z_n \, \mathrm{d}\Delta_n \xrightarrow[n\to\infty]{} \int_0^T \tilde{z} \, \mathrm{d}\tilde{\Delta}. \tag{3.4}$$

Proof of **(e)**: By (3.4), for every $x \in [-Y, Y]$: $\int_0^T \big(\tilde{z}(t) - x\big) \mathrm{d}\tilde{\Delta}(t) \geq 0$. We conclude by applying the following technical lemma from [GPP96]:

Lemma 3.2 (see Lemma 2.1 of [GPP96])**.** *Let $\tilde{z} \in \mathcal{C}\big([0,T], K\big)$ and $\tilde{\Delta} \in \mathcal{C}\big([0,T], \mathbb{R}\big)$ a function of bounded variation, such that for all $x \in K$:*

$$\int_0^T \big(\tilde{z}(t) - x\big) \, d\tilde{\Delta}(t) \geq 0.$$

Then: $\int_0^T \mathbf{1}_{\mathring{K}}\big(\tilde{z}(t)\big) \, d\tilde{\Delta}(t) = 0.$

Proof of **(f)**: Let $n \to \infty$ in the second equation of the penalized problem (1.5):

$$\mathrm{d}z_n(t) = y_n(t)\mathrm{d}t \, - n \cdot \big[z_n(t) - \pi(z_n(t))\big]\mathrm{d}t = y_n(t)\mathrm{d}t \, - \mathrm{d}\Delta_n(t).$$

In the limit we have: $\mathrm{d}\tilde{z}(t) = \tilde{y}(t)\mathrm{d}t \, - \mathrm{d}\tilde{\Delta}(t)$. Moreover by **(e)**, $\mathrm{d}\tilde{\Delta}(t) = \mathbf{1}_{\{\tilde{z}(t) = \pm Y\}} \mathrm{d}\tilde{\Delta}(t)$ which concludes the proof of **(f)**. □

Proof of Lemma 2.7. From (1.5) we deduce by Itô's formula:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(y_n^2(t)\big) = -2c_0 y_n^2(t) - 2k y_n(t) z_n(t) + 2y_n(t) \dfrac{\mathrm{d}}{\mathrm{d}t} W(t) + 2 \\ k\dfrac{\mathrm{d}}{\mathrm{d}t}\big(z_n^2(t)\big) = 2k y_n(t) z_n(t) - 2kn z_n^2(t) + 2kn z_n(t) \pi(z_n(t)). \end{cases}$$

Combining these two equations and taking the expectations gives:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\left[y_n^2(t)\right] + k\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\left[z_n^2(t)\right] + 2c_0 \, \mathbb{E}\left[y_n^2(t)\right] = 2kn \, \mathbb{E}\left[z_n(t)\pi(z_n(t))\right] - 2kn \, \mathbb{E}\left[z_n^2(t)\right] + 2.$$

Hence:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\left[y_n^2(t)\right] + k\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\left[z_n^2(t)\right] + 2c_0 \, \mathbb{E}\left[y_n^2(t)\right] \\ &\leq kn\epsilon \, \mathbb{E}\left[z_n^2(t)\right] + \frac{kn}{\epsilon} Y^2 - 2kn \, \mathbb{E}\left[z_n^2(t)\right] + 2 \end{aligned}$$

for any $\epsilon > 0$. Taking $\epsilon = 2\left(1 - \frac{c_0}{n}\right)$, this inequality becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E}\left[y_n^2(t) + k z_n^2(t)\right] + 2c_0 \, \mathbb{E}\left[y_n^2(t) + k z_n^2(t)\right] \leq c_n.$$

with $c_n := \frac{kn}{2\left(1 - \frac{c_0}{n}\right)} Y^2 + 2$. The conclusion follows by Grönwall's lemma. □

Proof of Lemma 2.8. We want to prove that for any $\epsilon > 0$, there exists a compact set $K_\epsilon \in \mathcal{B}$ such that:

$$\forall\, i \in \mathbb{N} \qquad \mu_n^{T_i}(\mathbf{1}_{K_\epsilon}) \geq 1 - \epsilon.$$

Fix $\epsilon > 0$ and let $\delta := \sqrt{\frac{\epsilon}{C}}$, with $C := c + c_n$ (where c and c_n is defined in Lemma 2.7). We show that $K_\epsilon := \left\{(y,z) : |y| + \sqrt{k}|z| \leq \frac{1}{\delta}\right\}$ is appropriate. Indeed for any $i \in \mathbb{N}$:

$$\begin{aligned}
1 - \mu_n^{T_i}(\mathbf{1}_{K_\epsilon}) &= 1 - \frac{1}{T_i}\int_0^{T_i} \mathbb{E}\left[\mathbf{1}_{K_\epsilon}(y_n(t), z_n(t))\right] dt \\
&= 1 - \frac{1}{T_i}\int_0^{T_i} \mathbb{P}\left[(y_n(t), z_n(t)) \in K_\epsilon\right] dt \\
&= \frac{1}{T_i}\int_0^{T_i} \mathbb{P}\left[(y_n(t), z_n(t)) \notin K_\epsilon\right] dt \\
&\leq \frac{1}{T_i}\int_0^{T_i} \left\{\mathbb{P}\left[|y_n(t)| > \frac{1}{\delta}\right] + \mathbb{P}\left[\sqrt{k}|z_n(t)| > \frac{1}{\delta}\right]\right\} dt \\
&\leq \frac{1}{T_i}\int_0^{T_i} \delta^2 \left\{\mathbb{E}\left[y_n(t)^2 + k z_n(t)^2\right]\right\} dt && \text{(by Lemma 2.7)} \\
&\leq \epsilon && \text{(by definition of } \delta\text{)}
\end{aligned}$$

□

4. Numerical experiments

In this section, we present our numerical tools for dealing with experiments on the invariant measures of (1.5) and their convergence rate. First, we present a probabilistic algorithm to simulate the trajectories. Then we give a PDE framework to study (2.2). Finally, we give an empirical rate of convergence.

4.1. **Experiments on the invariant measure using probabilistic simulations**

We conjecture the uniqueness of an invariant measure of (y_n, z_n), solution of (1.5), for each n. Then, based on the above probabilistic algorithm, we can approximate its density. To do so, we use probabilistic simulations, as explained bellow. In a similar manner to what was done in [BMPT09] and [FM12] to solve (1.2), here the solution $(y_n(t), z_n(t))$ of (1.5) has explicit formulae in each phase: either $|z_n(t)| \leq Y$, $z_n(t) > Y$ or $z_n(t) < -Y$.

4.1.1. *Explicit formulae*

For the case $|z_n(t)| \leq Y$, the process $(y_n(t), z_n(t))$ behaves like a linear oscillator:

$$\begin{cases}
\mathrm{d}y_n(t) = -\big(c_0 y_n(t) + k z_n(t)\big)\mathrm{d}t + \mathrm{d}w(t) \\
\mathrm{d}z_n(t) = y_n(t)\mathrm{d}t \\
y_n(0) = y\,, \qquad z_n(0) = z.
\end{cases}$$

Therefore, we have:

$$\begin{cases} y_n(t) = -\dfrac{c_0}{2}z(t) + e^{-\frac{c_0 t}{2}}\left\{-\omega z\sin(\omega t) + \left(y+\dfrac{c_0}{2}z\right)\cos(\omega t)\right\} + \displaystyle\int_0^t e^{-\frac{c_0}{2}(t-s)}\cos(\omega(t-s))\mathrm{d}w(s) \\ z_n(t) = e^{-\frac{c_0 t}{2}}\left\{z\cos(\omega t) + \dfrac{1}{\omega}\left(y+\dfrac{c_0}{2}z\right)\sin(\omega t)\right\} + \dfrac{1}{\omega}\displaystyle\int_0^t e^{-\frac{c_0}{2}(t-s)}\sin(\omega(t-s))\mathrm{d}w(s) \end{cases}$$

where $\omega := \frac{\sqrt{|4k-c_0^2|}}{2}$. Hence $y_n(t)$ (resp. $z_n(t)$) is Gaussian variable of mean $e_y(t,y_n,z_n)$ and variance $\sigma^2_{y_n}(t)$ (resp. of mean $e_z(t,y_n,z_n)$ and variance $\sigma^2_{z_n}(t)$), where:

$$\begin{cases} e_y(t,y_n,z_n) = -\dfrac{c_0}{2}e_z(t,y_n,z_n) + e^{-\frac{c_0 t}{2}}\left\{-\omega z\sin(\omega t) + \left(y+\dfrac{c_0}{2}z\right)\cos(\omega t)\right\} \\ \qquad\qquad + \displaystyle\int_0^t e^{-\frac{c_0}{2}(t-s)}\cos(\omega(t-s))\mathrm{d}w(s) \\ \sigma^2_{y_n}(t) = \displaystyle\int_0^t e^{-c_0 s}\cos^2(\omega s)ds - \dfrac{c_0^2}{4}\sigma_z^2(t) - \dfrac{c_0}{2\omega^2}e^{-c_0 t}\sin^2(\omega t). \end{cases}$$

and:

$$\begin{cases} e_z(t,y_n,z_n) = e^{-\frac{c_0 t}{2}}\left\{z\cos(\omega t) + \dfrac{1}{\omega}\left(y+\dfrac{c_0}{2}z\right)\sin(\omega t)\right\} \\ \qquad\qquad + \displaystyle\int_0^t e^{-\frac{c_0}{2}(t-s)}\cos(\omega(t-s))\mathrm{d}w(s) \\ \sigma^2_z(t) = \dfrac{1}{\omega^2}\displaystyle\int_0^t e^{-c_0 s}\sin^2(\omega s)ds. \end{cases}$$

The covariance of $y_n(t)$ and $z_n(t)$ is given by:

$$\sigma_{yz}(t) = \frac{2}{\omega}\int_0^t e^{-c_0 t}\sin(2\omega s)ds - \frac{c_0}{2\omega^2}\int_0^t e^{-c_0 s}\sin^2(\omega s)ds.$$

For the case $|z_n(t)| > Y$, the process $(y_n(t), z_n(t))$ satisfies

$$\begin{cases} \mathrm{d}y_n(t) = -\big(c_0 y_n(t) + kz_n(t)\big)\mathrm{d}t + \mathrm{d}w(t) \\ \mathrm{d}z_n(t) = y_n(t)\mathrm{d}t - n\big(z_n(t) - Y\big)\mathrm{d}t \\ y_n(0) = y\,, \qquad z_n(0) = z \end{cases}$$

Then for any t_0, t_1 such that on $[t_0,t_1]$, $z_n > Y$ we have:

$$\begin{cases} y_n(t_1) = e^{11}(t_1-t_0)y_n(t_0) + e^{12}(t_1-t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{11}(t_1-s)\mathrm{d}w(s) + \int_{t_0}^{t_1} e^{12}(t_1-s)nY ds \\ z_n(t_1) = e^{21}(t_1-t_0)y_n(t_0) + e^{22}(t_1-t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{21}(t_1-s)\mathrm{d}w(s) + \int_{t_0}^{t_1} e^{22}(t_1-s)nY ds \end{cases}$$

with:

$$\begin{cases} e^{11}(t) := \dfrac{1}{\lambda_- - \lambda_+}\left((n+\lambda_-)e^{t\lambda_-} - (n+\lambda_+)e^{t\lambda_+}\right) \\ e^{12}(t) := \dfrac{1}{\lambda_- - \lambda_+}(n+\lambda_-)(n+\lambda_+)\left(e^{t\lambda_+} - e^{t\lambda_-}\right) \\ e^{21}(t) := e^{t\lambda_-} - e^{t\lambda_+} \\ e^{22}(t) := \dfrac{1}{\lambda_- - \lambda_+}\left((n+\lambda_-)e^{t\lambda_+} - (n+\lambda_+)e^{t\lambda_-}\right) \end{cases}$$

where:

$$\begin{cases} \lambda_- := \dfrac{-(n+c_0) - \sqrt{(n+c_0)^2 - 4(c_0 n + k)}}{2} \\ \lambda_+ := \dfrac{-(n+c_0) + \sqrt{(n+c_0)^2 - 4(c_0 n + k)}}{2} \end{cases}$$

Hence $y_n(t)$ (resp. $z_n(t)$) is Gaussian variable of mean $e_y^+(t, y_n, z_n)$ and variance $\sigma_y^+(t)^2$ (resp. of mean $e_z^+(t, y_n, z_n)$ and variance $\sigma_z^+(t)^2$), where:

$$\begin{cases} e_y^+(t_1, y_n, z_n) = e^{11}(t_1 - t_0)y_n(t_0) + e^{12}(t_1 - t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{12}(t_1 - s)nY\,ds \\ \qquad = e^{11}(t_1 - t_0)y_n(t_0) + e^{12}(t_1 - t_0)z_n(t_0) + \frac{nY(n+\lambda_-)(n+\lambda_+)}{\lambda_- - \lambda_+}\left(\frac{e^{(t_1-t_0)\lambda_+} - 1}{\lambda_+} - \frac{e^{(t_1-t_0)\lambda_+} - 1}{\lambda_-}\right) \\ \sigma_y^+(t_1)^2 = \frac{1}{(\lambda_- - \lambda_+)^2}\left[\frac{(n+\lambda_-)^2}{2\lambda_-}\left(e^{2(t_1-t_0)\lambda_-} - 1\right) - \frac{2(n+\lambda_+)(n+\lambda_-)}{\lambda_+ + \lambda_-}\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-)} - 1\right)\right. \\ \qquad\qquad \left. + \frac{(n+\lambda_+)^2}{2\lambda_+}\left(e^{2(t_1-t_0)\lambda_+} - 1\right)\right] \end{cases}$$

and:

$$\begin{cases} e_z^+(t_1, y_n, z_n) = e^{21}(t_1 - t_0)y_n(t_0) + e^{22}(t_1 - t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{22}(t_1 - s)nY\,ds \\ \qquad = e^{21}(t_1 - t_0)y_n(t_0) + e^{22}(t_1 - t_0)z_n(t_0) + \frac{nY}{\lambda_- - \lambda_+}\cdot\left[\frac{(n+\lambda_-)(e^{(t_1-t_0)\lambda_+} - 1)}{\lambda_+} - \frac{(n+\lambda_+)(e^{(t_1-t_0)\lambda_-} - 1)}{\lambda_-}\right] \\ \sigma_z^+(t_1)^2 = \frac{1}{(\lambda_- - \lambda_+)^2}\left[\frac{1}{2\lambda_+}\left(e^{2(t_1-t_0)\lambda_+} - 1\right) - \frac{2}{\lambda_+ + \lambda_-}\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-)} - 1\right)\right. \\ \qquad\qquad \left. + \frac{1}{2\lambda_-}\left(e^{2(t_1-t_0)\lambda_-} - 1\right)\right] \end{cases}$$

Finally:

$$\sigma_{yz}^+(t_1) = \frac{1}{(\lambda_- - \lambda_+)^2}\left[\frac{(n+\lambda_-)\left(e^{(t_1-t_0)2\lambda_-} - 1\right)}{2\lambda_-} + \frac{(n+\lambda_+)\left(e^{(t_1-t_0)2\lambda_+} - 1\right)}{2\lambda_+}\right.$$
$$\left. - \frac{(n+\lambda_-)\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-} - 1\right)}{\lambda_- + \lambda_+} - \frac{(n+\lambda_+)\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-)} - 1\right)}{\lambda_- + \lambda_+}\right]$$

For the case $z_n(t) < -Y$, the process $(y_n(t), z_n(t))$ satisfies

$$\begin{cases} \mathrm{d}y_n(t) = -\big(c_0 y_n(t) + k z_n(t)\big)\mathrm{d}t + \mathrm{d}w(t) \\ \mathrm{d}z_n(t) = y_n(t)\mathrm{d}t - n\big(z_n(t) + Y\big)\mathrm{d}t \\ y_n(0) = y\,, \qquad z_n(0) = z \end{cases}$$

Then for any t_0, t_1 such that on $[t_0, t_1]$, $z_n < Y$ we have:

$$\begin{cases} y_n(t_1) = e^{11}(t_1 - t_0)y_n(t_0) + e^{12}(t_1 - t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{11}(t_1 - s)\mathrm{d}w(s) - \int_{t_0}^{t_1} e^{12}(t_1 - s)nY ds \\ z_n(t_1) = e^{21}(t_1 - t_0)y_n(t_0) + e^{22}(t_1 - t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{21}(t_1 - s)\mathrm{d}w(s) - \int_{t_0}^{t_1} e^{22}(t_1 - s)nY ds \end{cases}$$

with the same notations as above. Hence $y_n(t)$ (resp. $z_n(t)$) is Gaussian variable of mean $e_y^-(t, y_n, z_n)$ and variance $\sigma_y^-(t)^2$ (resp. of mean $e_z^-(t, y_n, z_n)$ and variance $\sigma_z^-(t)^2$), where:

$$\begin{cases} e_y^-(t_1, y_n, z_n) = e^{11}(t_1 - t_0)y_n(t_0) + e^{12}(t_1 - t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{12}(t_1 - s)nY ds \\ \qquad = e^{11}(t_1 - t_0)y_n(t_0) + e^{12}(t_1 - t_0)z_n(t_0) - \frac{nY(n+\lambda_-)(n+\lambda_+)}{\lambda_- - \lambda_+}\left(\frac{e^{(t_1-t_0)\lambda_+}-1}{\lambda_+} - \frac{e^{(t_1-t_0)\lambda_+}-1}{\lambda_-}\right) \\ \sigma_y^-(t_1)^2 = \frac{1}{(\lambda_- - \lambda_+)^2}\left[\frac{(n+\lambda_-)^2}{2\lambda_-}\left(e^{2(t_1-t_0)\lambda_-} - 1\right) - \frac{2(n+\lambda_+)(n+\lambda_-)}{\lambda_+ + \lambda_-}\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-)} - 1\right)\right. \\ \qquad\qquad \left. + \frac{(n+\lambda_+)^2}{2\lambda_+}\left(e^{2(t_1-t_0)\lambda_+} - 1\right)\right] \end{cases}$$

and:

$$\begin{cases} e_z^-(t_1, y_n, z_n) = e^{21}(t_1 - t_0)y_n(t_0) + e^{22}(t_1 - t_0)z_n(t_0) + \displaystyle\int_{t_0}^{t_1} e^{22}(t_1 - s)nY ds \\ \qquad = e^{21}(t_1 - t_0)y_n(t_0) + e^{22}(t_1 - t_0)z_n(t_0) - \frac{nY}{\lambda_- - \lambda_+}\cdot\left[\frac{(n+\lambda_-)(e^{(t_1-t_0)\lambda_+}-1)}{\lambda_+} - \frac{(n+\lambda_+)(e^{(t_1-t_0)\lambda_-}-1)}{\lambda_-}\right] \\ \sigma_z^-(t_1)^2 = \frac{1}{(\lambda_- - \lambda_+)^2}\left[\frac{1}{2\lambda_+}\left(e^{2(t_1-t_0)\lambda_+} - 1\right) - \frac{2}{\lambda_+ + \lambda_-}\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-)} - 1\right)\right. \\ \qquad\qquad \left. + \frac{1}{2\lambda_-}\left(e^{2(t_1-t_0)\lambda_-} - 1\right)\right] \end{cases}$$

Finally:

$$\sigma_{yz}^-(t_1) = \frac{1}{(\lambda_- - \lambda_+)^2}\left[\frac{(n+\lambda_-)\left(e^{(t_1-t_0)2\lambda_-}-1\right)}{2\lambda_-} + \frac{(n+\lambda_+)\left(e^{(t_1-t_0)2\lambda_+}-1\right)}{2\lambda_+}\right.$$
$$\left. - \frac{(n+\lambda_-)\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-}-1\right)}{\lambda_- + \lambda_+} - \frac{(n+\lambda_+)\left(e^{(t_1-t_0)(\lambda_+ + \lambda_-)}-1\right)}{\lambda_- + \lambda_+}\right]$$

4.1.2. *Simulation algorithm with an Euler scheme*

Based on the previous explicit formulae, we have written a C code to approximate the solution of (1.5). Let $T > 0, N \in \mathbb{N}$ and $(t_n)_{n=0\ldots N}$ be a family of time which discretizes $[0, T]$, such that $t_n = n\delta t$ where $\delta t := \frac{T}{N}$. We set Σ, Σ^+, and $\Sigma^- \in \mathcal{M}_{2,2}(\mathbb{R}^2)$ such that:

$$\Sigma \cdot \Sigma^T = \begin{pmatrix} \sigma_y(\delta t)^2 & \sigma_{yz}(\delta t) \\ \sigma_{yz}(\delta t) & \sigma_z(\delta t)^2 \end{pmatrix}, \quad (\Sigma^+)\cdot(\Sigma^+)^T = \begin{pmatrix} \sigma_y^+(\delta t)^2 & \sigma_{yz}^+(\delta t) \\ \sigma_{yz}^+(\delta t) & \sigma_z^+(\delta t)^2 \end{pmatrix}, \quad (\Sigma^-)\cdot(\Sigma^-)^T = \begin{pmatrix} \sigma_y^-(\delta t)^2 & \sigma_{yz}^-(\delta t) \\ \sigma_{yz}^-(\delta t) & \sigma_z^-(\delta t)^2 \end{pmatrix}.$$

Let $\left(G_{n,m}\right)_{n=0\ldots N, m=1,2}$ be a family of independent Gaussian variables $\mathcal{N}(0,1)$. Gaussian variables are generated using Box-Muller formula and the C function `random()`. Initialize $\left(y_0^{\delta t}, z_0^{\delta t}\right) = (y_0, z_0)$. The finite difference scheme for (1.5) is written in the following manner:

We define $\theta_n^{\delta t}$ and $\tau_n^{\delta t}$ for $n = 0, 1, \dots N$ recursively by: $\theta_0^{\delta t} = \tau_0^{\delta t} = 0$ and:

$$\begin{cases} \theta_{n+1}^{\delta t} := \inf\left\{t_k > \tau_n^{\delta t} \middle| \, |z_{t_k}^{\delta t}| = Y\right\} \\ \tau_{n+1}^{\delta t} := \inf\left\{t_k > \theta_{n+1}^{\delta t} \middle| |z_{t_k}^{\delta t}| < Y\right\} \end{cases}$$

Then:

- When $t_k \in [\tau_n^{\delta t}, \theta_{n+1}^{\delta t}[$ (we have $|z_{t_k}^{\delta t}| < Y$), we set:

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} = \begin{pmatrix} e_y(\delta t, y(t_k), z(t_k)) \\ e_z(\delta t, y(t_k), z(t_k)) \end{pmatrix} + \Sigma \cdot \begin{pmatrix} G_{k,1} \\ G_{k,2} \end{pmatrix}.$$

- When $t_k \in [\theta_{n+1}^{\delta t}, \tau_{n+1}^{\delta t}[$
  - if $z_{t_k}^{\delta t} \geq Y$, we set:

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} = \begin{pmatrix} e_y^+(\delta t, y(t_k), z(t_k)) \\ e_z^+(\delta t, y(t_k), z(t_k)) \end{pmatrix} + \Sigma^+ \cdot \begin{pmatrix} G_{k,1} \\ G_{k,2} \end{pmatrix}.$$

  - if $z_{t_k}^{\delta t} \leq -Y$, we set:

$$\begin{pmatrix} y_{t_{k+1}}^{\delta t} \\ z_{t_{k+1}}^{\delta t} \end{pmatrix} = \begin{pmatrix} e_y^-(\delta t, y(t_k), z(t_k)) \\ e_z^-(\delta t, y(t_k), z(t_k)) \end{pmatrix} + \Sigma^- \cdot \begin{pmatrix} G_{k,1} \\ G_{k,2} \end{pmatrix}.$$

With use the algorithm described above in order to approximate approximate the density of the invariant measure. First, we fix a domain $D := [y_{min}, y_{max}] \times [z_{min}, z_{max}] \in \mathbb{R}^2$ and take $N_y, N_z \in \mathbb{N}^*$. This defines a mesh of $N_y \times N_z$ points with space steps $\delta_y := (y_{max} - y_{min})/N_y$ and $\delta_z := (z_{max} - z_{min})/N_z$ respectively in the y and z directions. Then we simulate the process $\left(y_{t_k}, z_{t_k}\right)_{k=0\dots N}$ and compute, for each cell c of the mesh the number of times it is visited by the process: $\left|\{k \in [|0, N|] \, s.t. \left(y_{t_k}, z_{t_k}\right) \in c\}\right|$. For the following parameters and letting n vary, we obtain Figures 2 to 7:

- $y_{min} = z_{min} = -5, y_{max} = z_{max} = 5$,
- $Y = 1$,
- $c_0 = k = 1$,
- $T = 100000, \delta t = 0.001$,
- $N_y = N_z = 50$,
- $n = 2, 4, 6, 8, 10$ according to the figure (see captions).

4.2. **Experiments on the rate of convergence using PDEs**

We conjecture the convergence of the invariant measures and we estimate empirically the convergence rate. First, for a bounded measurable function f, $n > 0$ and $\lambda > 0$, we consider the function $u_\lambda^n(y, z; f)$ solution of:

$$\begin{cases} \lambda u^n + A u^n &= f(y, z) \quad \text{on } D \\ \lambda u^n + B_+^n u^n &= f(y, z) \quad \text{on } \tilde{D}^+ \\ \lambda u^n + B_-^n u^n &= f(y, z) \quad \text{on } \tilde{D}^- \end{cases} \tag{$P_{\lambda,n}^f$}$$

Then $u_\lambda^n(y, z; f)$ satisfies: $\forall (y, z) \in \mathbb{R}^2$

$$\lim_{\lambda \to 0} \lambda u_\lambda^n(y, z; f) = \lim_{t \to \infty} \mathbb{E}[f(y_n(t), z_n(t))]$$

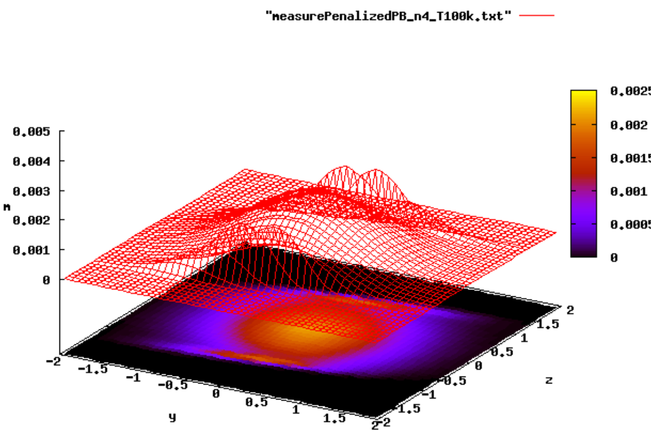

Figure 2. n=4

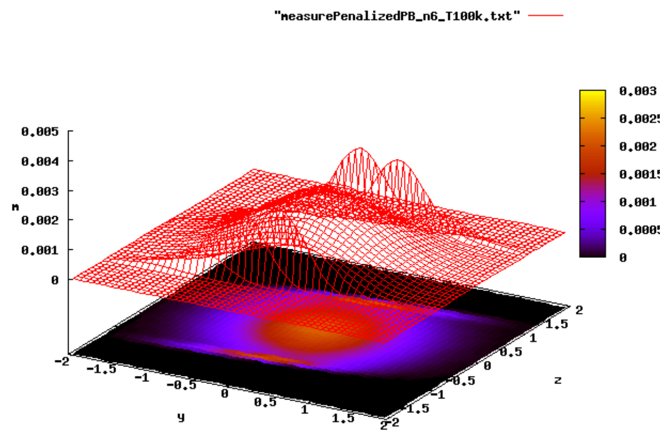

Figure 3. n=6

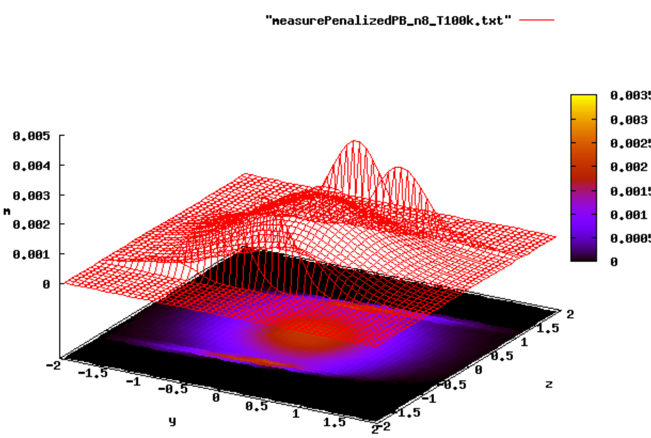

Figure 4. n=8

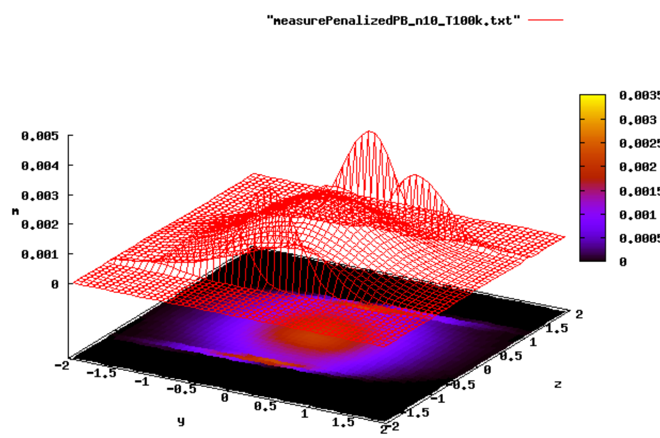

Figure 5. n=10

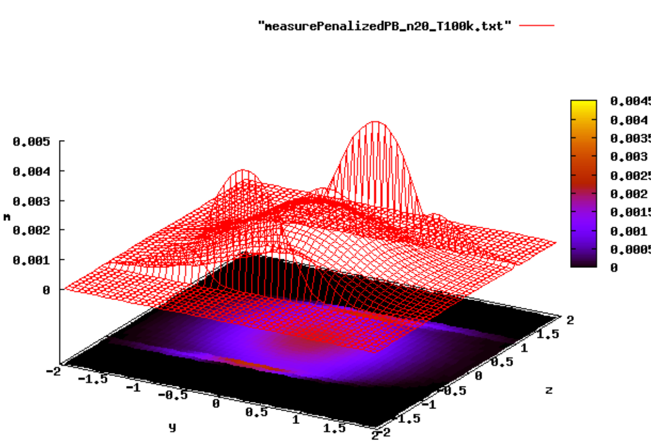

Figure 6. n=20

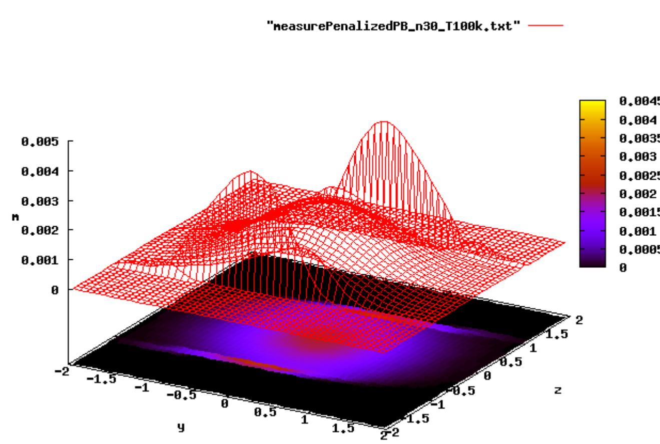

Figure 7. n=30

$$\begin{aligned} \text{where:}\quad A\varphi &:= -\frac{1}{2}\varphi_{yy} + (c_0 y + kz)\varphi_y - y\varphi_z, && D := \mathbb{R}\times]-Y,Y[\\ B_+^n\varphi &:= -\frac{1}{2}\varphi_{yy} + (c_0 y + kY)\varphi_y - y\varphi_z + n\big(z-Y\big)\varphi_z, && \tilde{D}^+ := \mathbb{R}\times[Y,+\infty[\\ B_-^n\varphi &:= -\frac{1}{2}\varphi_{yy} + (c_0 y - kY)\varphi_y - y\varphi_z + n\big(z+Y\big)\varphi_z, && \tilde{D}^- := \mathbb{R}\times]-\infty,-Y]. \end{aligned}$$

Next, define

$$\nu_n(f) := \lim_{\lambda\to 0,\lambda>0} \lambda u_\lambda^n(y,z;f).$$

The error between the measures is computed as follows: given a family $\mathcal{G}$ of numerical functions composed of gaussian kernels, we define the maximum relative error:

$$\mathbf{E}_n(\mathcal{G}) := \sup\left\{\frac{\nu_n(g)-\nu(g)}{\nu(g)} : g\in\mathcal{G}\right\}.$$

To compute $\mathbf{E}_n$ we assume that there exist a probability density $m_n \in L^1$ satisfying Equation (2.3). For the empirical approximation of this error, we take the following parameters:

- $y_{min} = z_{min} = -5, y_{max} = z_{max} = 5$,
- $Y = 1$,
- $c_0 = k = 1$.

To define the families of gaussian functions that we use, we consider the following 9×5 grids (according to axes y and z respectively), centered on $(0,0)$

(1) grid 1: with step 0.4 in the z direction and step 1.1 in the y direction.
(2) grid 2: with step 0.5 in the z direction and step 1.25 in the y direction.

Note that the second grid contains nodes on the borders $[-L,L]\times\{-Y\}$ and $[-L,L]\times\{+Y\}$ whereas the first one does not, and that none of the grids contain nodes outside the admissible domain $D=\mathbb{R}\times(-L,L)$. Then we define $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) as the family of gaussian functions centered on each node of the first (resp. second) grid, that is the set of functions $\big(g_{i,j}\big)_{i=1\ldots 9, j=1\ldots 5}$ defined by:

$$g_{i,j}(y,z) = \exp\big(-(y-y_i)^2\big)\cdot\exp\big(-(z-z_j)^2\big)$$

where (y_i,z_j) ranges over the nodes of $\mathcal{G}_1$ (resp. $\mathcal{G}_2$). The computation of the maximum relative error for the families of test functions $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) gives Figure 8 (resp. Figure 9). If we restricts ourselves to the interval before the error stagnates, we can plot the log of the relative error and see that it is well approximated by a linear function, as in Figure 10. Then the empirical rate of convergence is exponential. More precisely we obtain the following empirical estimation for the convergence rate in each case:

$$\mathbf{E}_n(\mathcal{G}_1) = 43.293\cdot\exp(-0.740\cdot n) \qquad \mathbf{E}_n(\mathcal{G}_2) = 139.91\cdot\exp(-0.728\cdot n).$$

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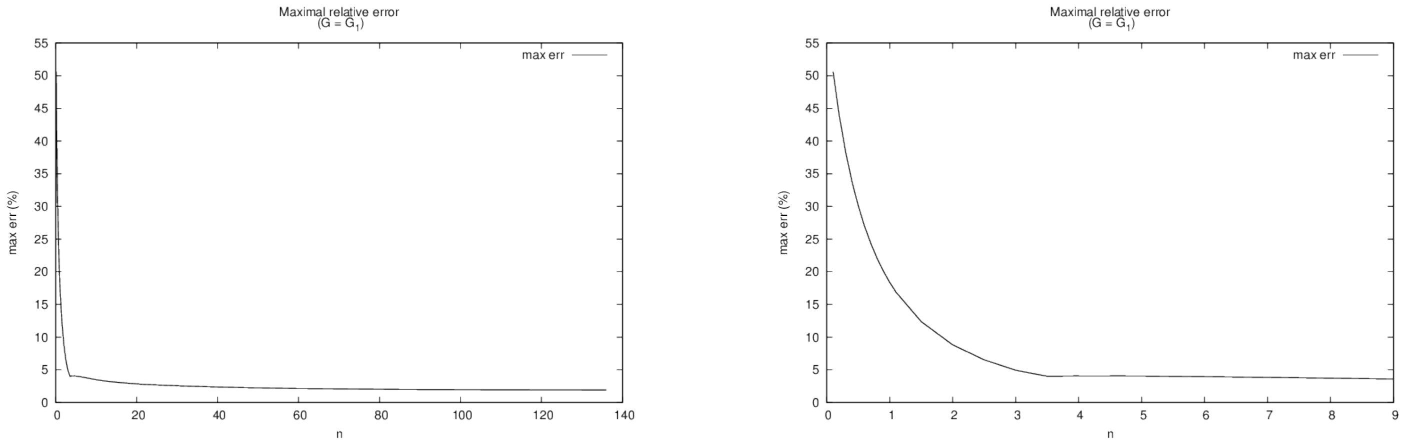
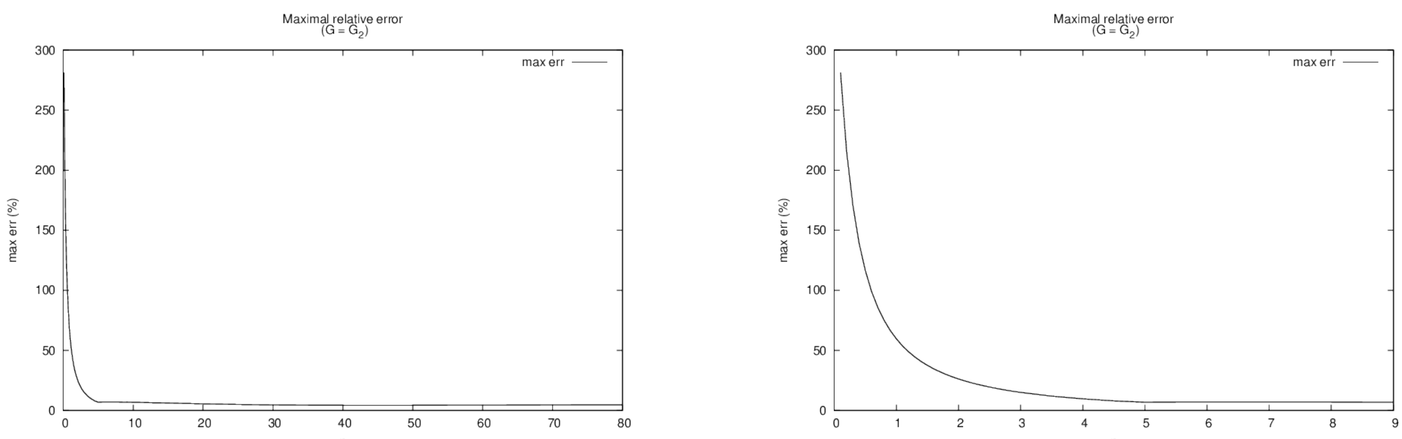
FIGURE 8. Empirical convergence rate computed for $\mathcal{G} = \mathcal{G}_1$



Error for $n \in (0, 140]$ Error for $n \in (0, 9]$

FIGURE 9. Empirical convergence rate computed for $\mathcal{G} = \mathcal{G}_2$



Error for $n \in (0, 80]$ Error for $n \in (0, 9]$

REFERENCES

[BL82] A. Bensoussan, J.L. Lions, *Impulse Control and Quasi-Variational Inequalities*, Dunod, 1982

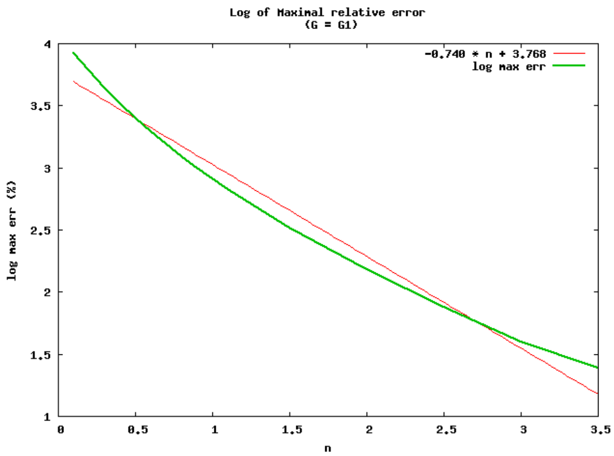
[BMPT09] A. Bensoussan, L. Mertz, O. Pironneau, J. Turi, *An Ultra Weak Finite Element Method as an Alternative to a Monte Carlo Method for anElasto-Plastic Problem with Noise*, SIAM J. Numer. Anal., **47(5)** (2009), 3374–3396.

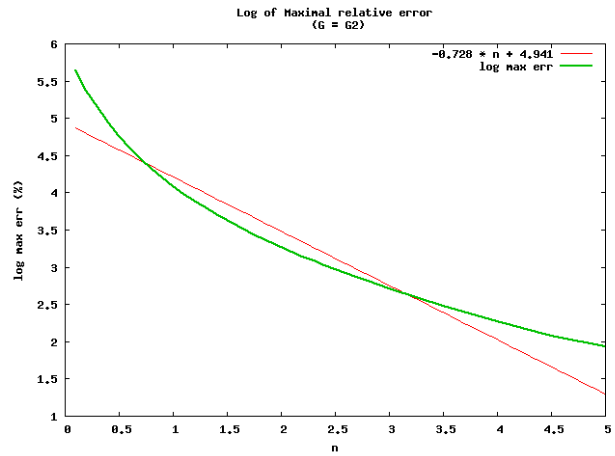
[BM12] A. Bensoussan, L. Mertz, *An analytic approach to the ergodic theory of stochastic variational inequalities*, C. R. Acad. Sci. Paris Ser. I, **350(7-8)**, (2012), 365–370.

[BT08] A. Bensoussan, J. Turi, *Degenerate Dirichlet Problems Related to the Invariant Measure of Elasto-Plastic Oscillators*, Applied Mathematics and Optimization, **58(1)** (2008), 1–27.

[F08] C. Feau, *Probabilistic response of an elastic perfectly plastic oscillator under Gaussian white noise*, Probabilistic Engineering Mechanics, **23(1)** (2008), 36–44.

Figure 10. Curve fit of the empirical convergence rate





[FM12] C. Feau, L. Mertz, *An empirical study on plastic deformations of an elasto-plastic problem with noise*, Probabilistic Engineering Mechanics, **30** (2012), 60–69.

[GPP96] A. Gegout-Petit, E. Pardoux, *Backward stochastic differential equations reflected in a convex domain*, Stochastics & Stoch. Rep. 57, no. 1-2, 111-128, 1996.