COMPUTING PERIODS OF RATIONAL INTEGRALS

by

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Abstract. — A period of a rational integral is the result of integrating, with respect to one or several variables, a rational function over a closed path. This work focuses particularly on periods depending on a parameter: in this case the period under consideration satisfies a linear differential equation, the Picard-Fuchs equation. I give a reduction algorithm that extends the Griffiths-Dwork reduction and apply it to the computation of Picard-Fuchs equations. The resulting algorithm is elementary and has been successfully applied to problems that were previously out of reach.

Contents

Introduction

This work studies periods of rational integrals, that is, the result of the integration, with respect to one or several variables, of a rational function over a closed path. I focus especially on the case where the period depends on a parameter. The fact that periods depending on a parameter of rational or algebraic integrals satisfy linear differential equations with polynomial coefficients has been a continuous discovery since Euler and his computation of a differential equation for the perimeter of an ellipse as a function of eccentricity. Since then, these differential equations, known as *Picard-Fuchs equations*, have proven to be useful in numerous domains such as combinatorics,[1]

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[1] Bousquet-Mélou and Mishna, "Walks with small steps in the quarter plane".

number theory[2] or physics.[3] Research in computer algebra has devoted great efforts to provide algorithms for computing integrals and, in particular, Picard-Fuchs equations. Nevertheless the practical efficiency of current methods is not satisfactory in many cases. One reason might be the high level of generality of most algorithms, which apply to the integration of general holonomic functions. Rational functions are certainly very specific among holonomic functions, but the numerous applications of Picard-Fuchs equations as well as the fundamental nature of rational functions make them worth developing specific methods.

Formulation of the problem. — Let R be a rational function in the variables $x_1, \dots, x_n$, denoted $\mathbf{x}$, and a parameter t, with coefficients in $\mathbb{C}$. Let γ be a n-cycle in $\mathbb{C}^n$, e.g., an embedding of the sphere $\mathbb{S}^n$ in $\mathbb{C}^n$, on which R is continuous when t ranges over some connected open set U of $\mathbb{C}$. We can form the following integral, depending on $t \in U$,

$$P(t) \overset{\text{def}}{=} \oint_\gamma R(t, \mathbf{x}) \mathrm{d}\mathbf{x}, \tag{1}$$

where $\mathrm{d}\mathbf{x}$ stands for $\mathrm{d}x_1 \cdots \mathrm{d}x_n$.

Example 1. — For $t \in \mathbb{C}$, with $|t| < 17 - 12\sqrt{2}$

$$\sum_{n \geqslant 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = \frac{1}{(2\pi i)^3} \oint_\gamma \frac{\mathrm{d}x \mathrm{d}y \mathrm{d}z}{1 - (1 - xy)z - txzy(1-x)(1-y)(1-z)},$$

where the cycle of integration γ is $\left\{ (x, y, z) \in \mathbb{C}^3 \;\middle|\; |x| = |y| = |z| = 1/2 \right\}$. This is the generating function of Apéry numbers.[4]

These integrals, for different cycles γ, are called the *periods* of the integral $\oint R$. It is well-known that $P(t)$ satisfies a linear differential equation with polynomial coefficients. Let $\mathcal{L}_{R,\gamma}$ denote the differential operator in t and ∂_t which corresponds to the minimal-order equation of $P(t)$. That is to say $\mathcal{L}_{R,\gamma}$ is the non zero operator $\sum_{k=0}^r a_k(t) \partial_t^k$ with coprime polynomial coefficients and minimal r, such that

$$\mathcal{L}_{R,\gamma}(P) \overset{\text{def}}{=} \sum_{k=0}^{r} a_k(t) P^{(k)}(t) = 0.$$

Every linear differential equation for $P(t)$ translates into an operator which is a left multiple of $\mathcal{L}_{R,\gamma}$.

It often happens that the description of the cycle γ is analytic or topological, sometimes not even explicit, and, to say the least, unsuitable to a formal algorithmic treatment. In fact there is no harm in simply discarding γ: there exists a differential equation satisfied by all the periods of $\oint R$. In other words, there exists an operator

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[2] Beukers, “Irrationality of π^2, periods of an elliptic curve and $\Gamma_1(5)$”.
[3] Morrison and Walcher, “D-branes and normal functions”.
[4] Beukers, “Irrationality of π^2, periods of an elliptic curve and $\Gamma_1(5)$”.

in t and ∂_t which is a left multiple of all $\mathcal{L}_{R,\gamma}$. Let $\mathcal{L}_R$ denote the least common left multiple of the $\mathcal{L}_{R,\gamma}$. The classical result which allows the algorithmic computation of $\mathcal{L}_R$ is that it is the minimal operator $\mathcal{L}$ such that

$$\mathcal{L}(R) = \sum_{i=1}^{n} \partial_i(B_i) \tag{2}$$

for some rational functions B_i in $\mathbb{C}(t, \mathbf{x})$ whose denominators divide a power of the denominator of R, and where ∂_i denotes $\partial/\partial x_i$. This article presents an algorithm that compute the operator $\mathcal{L}_R$, or at least a left multiple of it.

Example 2. — In the case of Example 1, the operators $\mathcal{L}_R$ and $\mathcal{L}_{R,\gamma}$ both equal

$$\mathcal{L}_R = t^2(t^2 - 34t + 1)\partial_t^3 + 3t(2t^2 - 51t + 1)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + (t - 5).$$

Note that integrals of algebraic functions are easily translated into integrals of rational functions with one variable more: if $W(t, \mathbf{x})$ is a function such that $P(t, \mathbf{x}, W) = 0$ for some polynomial P in $\mathbb{C}[t, \mathbf{x}, y]$, elementary residue calculus shows that

$$W(t, \mathbf{x}) = \frac{1}{2\pi i} \oint_\tau \frac{y\partial_y P}{P} \mathrm{d}y$$

over some adequate contour τ and where ∂_y denotes the derivation $\partial/\partial y$, so that

$$\oint_\gamma W(t, \mathbf{x}) \mathrm{d}\mathbf{x} = \frac{1}{2\pi i} \oint_{\gamma \times \tau} \frac{y\partial_y P}{P} \mathrm{d}\mathbf{x}\mathrm{d}y.$$

Contributions. — Following the principle of the reduction of the pole order, I define a family of finer and finer reductions $[\,]_r$, for $r \geqslant 1$, that given a rational function R in several variables produces another rational function $[R]_r$ that differs from R only by a sum of partial derivatives of other rational functions (Section 4). The first reduction $[\,]_1$ is the Griffiths-Dwork reduction (Section 3).

When applied to the case of periods depending on a parameter, these reductions can solve Equation (2), and hence compute Picard-Fuchs equations of rational integrals (Section 6). A major difficulty is to fix an r such that the rth reduction $[\,]_r$ will be fine enough to ensure the termination of the algorithm. It is solved by applying a theorem of Dimca (Section 5).

The new algorithm has been implemented and shows excellent performance (Section 7). For example, I applied it to compute 137 periods coming from mathematical physics that were previously out of reach[5] (Section 8).

[5] Batyrev and Kreuzer, "Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions".

Reduction of pole order. — The principle of the method originates from Hermite reduction.[6] It is a procedure for computing a normal form of a univariate function modulo derivatives. Hermite introduced his method as a way to compute the algebraic part of the primitive of a univariate rational function without computing the roots of its denominator, as opposed to the classical partial fraction decomposition method. I denote by $[R]$ the reduction of a fraction R. It is defined as follows. Let a/f^q be a rational function in $\mathbb{C}(x)$, with f a square-free polynomial and q a positive integer. Every fraction can be written in this way since a and f are not assumed to be relatively prime. If $q = 1$ then $[a/f]$ is defined to be r/f, where r is the remainder in the Euclidean division of a by f. If $q > 1$ then $[a/f^q]$ is, by induction on q, defined to be $[u/f^{q-1} + \frac{1}{q-1}v'/f^{q-1}]$, where $a = uf + vf'$, with the motive that

$$\frac{a}{f^q} = \frac{u + \frac{1}{q-1}v'}{f^{q-1}} - \left(\frac{1}{q-1}\frac{v}{f^{q-1}}\right)'.$$

Hermite reduction enjoys the following properties: it is linear; the fractions $[R]$ and R differ only by a derivative of a rational function; $[R]$ is zero if and only if R is the derivative of a rational function.

The principle of Hermite reduction gives an efficient way to compute the Picard-Fuchs equation of simple integrals.[7] Let R be a rational function in $\mathbb{C}(t, x)$. Hermite reduction can be performed without modification over the field with one parameter $\mathbb{C}(t)$. To compute $\mathcal{L}_R$, it is sufficient to compute the reductions $[\partial_t^k R]$, for $k \geqslant 0$, until finding a linear dependency relation over $\mathbb{C}(t)$

$$\sum_{k=0}^{r} a_k(t)[\partial_t^k R] = 0.$$

Then the properties of the Hermite reduction assure that $\mathcal{L}_R$ is $\sum_{k=0}^{r} a_k(t)\partial_t^k$. The computations of all the reductions $[\partial_t^k R]$ is improved significantly when noting the inductive formula

$$\left[\partial_t^{k+1} R\right] = \left[\partial_t[\partial_t^k R]\right].$$

With several variables, the construction of a normal form modulo derivatives is considerably harder than with a single variable. Nonetheless, as soon as we obtain such a normal form, it is possible to compute Picard-Fuchs equations as above, by finding linear relations between the $[\partial_t^k R]$.

Related works. — Several existing algorithms are applicable to the computation of $\mathcal{L}_R$. The reader may refer to Chyzak[8] for an extensive survey of “creative telescoping” approaches. A first family, originating in the work of Fasenmyer[9] and Verbaeten,[10] gave

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[6] Hermite, “Sur l’intégration des fractions rationnelles”.
[7] Bostan, Chen, Chyzak, and Li, “Complexity of creative telescoping for bivariate rational functions”.
[8] Chyzak, *The ABC of Creative Telescoping: Algorithms, Bounds, Complexity*.
[9] Fasenmyer, “Some generalized hypergeometric polynomials”.
[10] Verbaeten, “The automatic construction of pure recurrence relations”.

rise to an algorithm by Wilf and Zeilberger,[11] refined by Apagodu and Zeilberger,[12] applicable to proper hyperexponential terms, which includes rational functions. The idea is to transform Equation (2) into a linear system over $\mathbb{C}(t)$ by bounding *a priori* the order of a left multiple of $\mathcal{L}_R$ and the degree of the polynomials appearing in the certificate. While being an interesting method, especially because it gives *a priori* bounds, the order of the linear system to be solved is large even for moderate sizes of the input.

Zeilberger's "fast algorithm"[13] for hypergeometric summation is the origin of a different family of algorithms, whose key idea is to reduce the resolution of Equation (2) to the computation of rational solutions of systems of ordinary linear differential equations. Interestingly, Picard used this idea much earlier in a method for computing double rational integrals.[14] Chyzak's algorithm[15] and Koutschan's semi-algorithm[16]—termination is not proven—belong to this line and apply to D-finite ideals in Ore algebras. Rational functions are a very specific case.

A last family of algorithms coming from $\mathcal{D}$-module theory has given algorithms for numerous operations on $\mathcal{D}$-modules and, in particular, an algorithm by Oaku and Takayama[17] to compute the de Rham cohomology of the complement of an affine hypersurface, which would allow, in theory, to compute Picard-Fuchs equations. It is worth noticing that an algorithm to compute the integration of a holonomic $\mathcal{D}$-module does not give as such an algorithm applicable to our problem: computing the annihilator of a rational function in the Weyl algebra is far from being an easy task.[18]

The domain of application of each of these three families is much larger than just rational integrals: any comparison with the present algorithm must be done with this point in mind.

The *guessing* method, or *equation reconstruction*, a totally different method, applies to the computation of $\mathcal{L}_{R,\gamma}$. It often happens that beside the integral formula for $P(t)$ one has a way to compute a power series expansion. After computing sufficiently many terms, it is possible to recover $\mathcal{L}_{R,\gamma}$ *via* Hermite-Padé approximants. It may be difficult to prove that the operator computed is indeed correct, but not too hard to get convinced. The simplicity of this method counterbalances a certain lack of delicacy and justifies its ample use. When the power series expansion of $P(t)$ is, for some reason, easy to compute, it can find Picard-Fuchs equations which are far out of reach

[11] Wilf and Zeilberger, "An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities".
[12] Apagodu and Zeilberger, "Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf- Zeilberger theory".
[13] Zeilberger, "The method of creative telescoping".
[14] Picard, "Sur les intégrales doubles de fonctions rationnelles dont tous les résidus sont nuls".
[15] Chyzak, "An extension of Zeilberger's fast algorithm to general holonomic functions".
[16] Koutschan, "A fast approach to creative telescoping".
[17] Oaku and Takayama, "An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation".
[18] Ibid.

of any existing algorithms.[19] Most of the time, though, the power series expansion of $P(t)$ is expensive to compute. For example, I am aware of no general method allowing to compute directly the first p terms of a diagonal of a rational function in n variables in less than p^n arithmetic operations.[20]

Picard and Simart have studied the case of simple and double integrals of algebraic functions and gave methods to compute normal forms modulo derivatives extensively[21]. Chen, Kauers, and Singer[22] gave an algorithm in this direction, for double rational integrals. This algorithm is an echo, independently discovered, of one of the methods of Picard.[23] Interestingly, it has two steps: a first one based on a reduction *à la* Hermite and another one based on creative telescoping.

Well later after Picard, Griffiths resumed the search for a normal form in the setting of de Rham cohomology of smooth projective hypersurfaces, defining what is now known as the Griffiths-Dwork reduction.[24] This reduction is in many respects similar to the Hermite reduction. It can be applied to the computation of Picard-Fuchs equations in the same way as Hermite reduction applies to simple integrals. The smoothness hypothesis can be worked around with a generic deformation. This leads to an interesting complexity result about the computation of Picard-Fuchs equations[25] but to disappointing practical efficiency in singular cases. The direction that Griffiths shown has been pursued, in particular by Dimca[26] and Saito,[27] and some results are known in the case of a singular hypersurface.

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[19] See, for example, Kauers and Zeilberger, "The computational challenge of enumerating high-dimensional rook walks".
[20] However, see Metelitsyn, "How to compute the constant term of a power of a Laurent polynomial efficiently", for an improvement in the space complexity.
[21] Picard and Simart, *Théorie des fonctions algébriques de deux variables indépendantes*.
[22] Chen, Kauers, and Singer, "Telescopers for Rational and Algebraic Functions via Residues".
[23] Picard, "Sur les intégrales doubles de fonctions rationnelles dont tous les résidus sont nuls".
[24] Griffiths, "On the periods of certain rational integrals. I, II", §4.
[25] Bostan, Lairez, and Salvy, "Creative telescoping for rational functions using the Griffiths–Dwork method".
[26] Dimca, "On the de Rham cohomology of a hypersurface complement", "On the Milnor fibrations of weighted homogeneous polynomials".
[27] Dimca and Saito, "A generalization of Griffiths's theorem on rational integrals".

PART I
REDUCTION OF PERIODS

Let $\mathbb{K}$ be a field of characteristic zero, and let A be the polynomial ring $\mathbb{K}[x_0, \dots, x_n]$, for some integer n. Let f be an homogeneous element of A and let A_f be the localized ring $A[1/f]$. The degree of f is denoted N.

This section addresses the problem of finding an algorithm *à la* Hermite that computes an idempotent linear map $R \mapsto [R]$, from A_f to itself such that $[R]$ equals zero if and only if R is in the linear subspace $\sum_{i=0}^n \partial_i A_f$. This problem is solved by the Hermite reduction when n is 1 and by the Griffiths-Dwork reduction when f satisfies an additional regularity hypothesis (see Theorems 3 and 10). To this purpose, a family of maps, denoted $[\,]_r$, is constructed such that $[\,]_1$ is the Griffiths-Dwork reduction and such that $[\,]_{r+1}$ factors through $[\,]_r$. I give an efficient algorithm to compute these maps. Conjecturally, $[\,]_{n+1}$ satisfies the desired properties. Fortunately, other results allow to avoid relying on this conjecture when dealing with periods depending on a parameter.

1. Overview

1.1. Griffiths-Dwork reduction. — To achieve a normal form modulo derivatives, the guiding principle is the *reduction of pole order*. Let us first consider the decision problem: given a rational function a/f^q, decide whether it lies in $\sum_{i=0}^n \partial_i A_f$. A major actor of the study is $\operatorname{Jac} f$, the Jacobian ideal of f. It is the ideal of A generated by the partial derivatives $\partial_0 f, \dots, \partial_n f$. The basic observation is that the differentiation formula

$$\sum_{i=0}^n \partial_i \left(\frac{b_i}{f^{q-1}} \right) = \frac{\sum_{i=0}^n \partial_i b_i}{f^{q-1}} - (q-1) \frac{\sum_{i=0}^n b_i \partial_i f}{f^q} \tag{3}$$

implies, by reading it right-to-left, that if a is in $\operatorname{Jac} f$ and $q > 1$ then a/f^q equals a'/f^{q-1} modulo derivatives, for some polynomial a'. Namely, if a equals $\sum_i b_i \partial_i f$ then

$$\frac{a}{f^q} \equiv \frac{\frac{1}{q-1} \sum_{i=0}^n \partial_i b_i}{f^{q-1}} \mod \sum_{i=0}^n \partial_i A_f.$$

Griffiths[28] proved the converse property in the case when $\operatorname{Jac} f$ is zero-dimensional or, equivalently, when the projective variety defined by f is smooth. Under this hypothesis, if $q > 1$ and if a/f^q equals a'/f^{q-1}, modulo derivatives, for some polynomial a', then a is in $\operatorname{Jac} f$. This gives an algorithm to solve the decision problem, by induction on the pole order q.

[28] Griffiths, "On the periods of certain rational integrals. I, II".

1.2. Singular cases. — In presence of singularities, Griffiths' theorem always fails. For example, with f equal to $xy^2 - z^3$,

$$\frac{x^3}{f^2} = \partial_x\left(\frac{\frac{2}{7}x^4}{f^2}\right) - \partial_y\left(\frac{\frac{1}{7}x^3y}{f^2}\right), \tag{4}$$

but x^3 is not in $\operatorname{Jac} f$, which is here the ideal (xy, y^2, z^2). This identity is a consequence of the following particular case of Equation (3):

$$\sum_{i=0}^{n} b_i\partial_i f = 0 \Rightarrow \sum_{i=0}^{n} \partial_i\left(\frac{b_i}{f^q}\right) = \frac{\sum_{i=0}^{n}\partial_i b_i}{f^q}. \tag{5}$$

Tuples of polynomials $(b_0, \dots, b_n)$ such that $\sum_{i=0}^{n} b_i\partial_i f$ are called *syzygies* (of the sequence $\partial_0 f, \dots, \partial_n f$). Therefore, in order to complete the reduction of pole order strategy, we should not only consider elements of the Jacobian ideal, but also elements of the form $\sum_i \partial_i b_i$, where $(b_i)_i$ is a syzygy. Such elements are called *differentials of syzygies.*

Considering differential of syzygies is not always enough. For example, with f equal to $x_0^4x_1 - x_0^2x_1x_2^2 + x_0x_2^4$:

$$\frac{x_1^7}{f^2} = \frac{1062347}{276480}\frac{89x_0^2 + 96x_0x_1 + 712x_2^2}{f} + \sum_{i=0}^{2}\partial_i\left(\frac{b_i}{f^3}\right),$$

for some lengthy polynomials b_i, whereas x_1^7 is not a sum of a differential of a syzygy and of an element of $\operatorname{Jac} f$. Note the exponent 3 appearing in $\partial_i(b_i/f^3)$.

1.3. Higher order relations. — Let M_q be the set of rational functions of the form a/f^q. Let W_q^1 be the subset of $M_q \times M_{q-1}$ defined by

$$W_q^1 = \left\{ \left((q-1)\frac{\sum_{i=0}^{n} b_i\partial_i f}{f^q}, \frac{\sum_{i=0}^{n}\partial_i b_i}{f^{q-1}} \right) \;\middle|\; b_i \in A \right\},$$

so that for all (R, R') in W_q^1, the first element R has a pole of order at most q and is equivalent, modulo derivatives, to the second element R', which has a pole of order at most $q-1$. The following statement is a rewording of Griffiths' result:

***Theorem 3* (Griffiths).** — *Assume that $V(f)$ is smooth. For all R in M_q, homogeneous of degree $-n-1$, the following assertions are equivalent:*

1. *R is in $\sum_i \partial_i A_f$;*
2. *there exists R' in M_{q-1} such that (R, R') is in W_q^1 and such that R' is in $\sum_i \partial_i A_f$.*

The starting point of the method in the general case is to observe that W_q^1 contains ordered pairs in the form $(0, R')$. Namely, if $b_0, \dots, b_n$ is a syzygy, then $(0, \sum_i \partial_i b_i/f^{q-1})$ is in W_q^1. For all such pairs $(0, R')$, the rational function R' is in $\sum_i \partial_i A_f$, since it is equivalent to 0 modulo derivatives.

However, it is possible, as remarked above, that R' is not part of a pair (R', R'') in W^1_{q-1}. This motivates the definition of W^2_q as

$$W^2_q \stackrel{\text{def}}{=} W^1_q + \left\{(R,0) \;\middle|\; (0,R) \in W^1_{q+1}\right\}.$$

Of course, this can be iterated:

$$W^{r+1}_q \stackrel{\text{def}}{=} W^r_q + \left\{(R,0) \;\middle|\; (0,R) \in W^r_{q+1}\right\}.$$

The basic property that is preserved through this induction is that for all (R, R') in W^r_q, the first element R has a pole of order at most q and is equivalent, modulo derivatives, to the second element R', which has a pole of order at most $q-1$.

Moreover, this construction is somehow exhaustive. The main result is the following, with no assumption on $V(f)$:

Theorem 4. — *There exists an integer $r \geqslant 1$, depending only on f, such that for all q and all R in M_q, homogeneous of degree $-n-1$, the following assertions are equivalent:*

1. *R is in $\sum_i \partial_i A_f$;*
2. *there exists R' in M_{q-1} such that (R, R') is in W^r_q and such that R' is in $\sum_i \partial_i A_f$.*

The algorithm presented in this article is based on this theorem, and provides an efficient way to compute the spaces W^r_q. The two main ingredients of efficiency are, firstly, the use of Gröbner bases to reduce the dimension in which we need to perform linear algebra and, secondly, the computation of a basis of *non-trivial* syzygies.

1.4. Trivial syzygies. — The space W^2_q is made from W^1_q and elements in the form $(\sum_i \partial_i b_i / f^q, 0)$, where $b_0, \dots, b_n$ is a syzygy, that is $\sum_i b_i \partial_i f$ vanishes.

Among syzygies, the *trivial syzygies* do not bring new relations to the relations already in W^1_r. A syzygy $b_0, \dots, b_n$ is called *trivial* if there exist polynomials $c_{i,j}$, with $c_{i,j} = -c_{j,i}$, such that

$$b_i = \sum_{j=0}^n c_{i,j} \partial_j f.$$

The antisymmetry property implies that this defines a syzygy, and we check that

$$\sum_{i=0}^n \partial_i b_i = \sum_{j=0}^n \left(\sum_{i=0}^n \partial_i c_{ij} \right) \partial_j f + \underbrace{\sum_{i,j=0}^n c_{i,j} \partial_i \partial_j f}_{=0},$$

so that $\sum_{i=0}^n \partial_i b_i$ is in the Jacobian ideal. Moreover

$$\sum_j \partial_j \left(\sum_i \partial_i c_{ij} \right) = 0.$$

It follows that the ordered pair $(\sum_i \partial_i b_i / f^q, 0)$ is already in W^1_q. Thus, in order to compute W^2_q, one may discard trivial syzygies. Quantitatively, the trivial syzygies are numerous among the syzygies—see, for example, Table 2—so that discarding them is a tremendous improvement. A basis of *non-trivial* syzygies can be computed efficiently by means of Gröbner bases.

1.5. Reduction procedure. — Let $R = a/f^q$ be a fraction in M_q. The reduced form $[R]_r$ is defined by induction on q in the following way. Let S be the *minimal*—with respect to a monomial order, for example—element of M_q such that there exists a R' in M_{q-1} with $(R - S, R') \in W_q^r$. Then $[R]_r$ is defined to be $S + [R']_r$. A constraint on the homogeneity degree of R will ensure that R' is zero at some point of the induction.

2. Exponential isomorphism

The *exponential isomorphism*, due to Dimca[29], allows to manipulate polynomials rather than rational functions. We work in a homogeneous setting and we deal only with homogeneous fractions R of degree $-n-1$. (So that $R\mathrm{d}x_0 \cdots \mathrm{d}x_n$ is homogeneous of degree 0.) A fraction a/f^q is therefore represented solely by its numerator a: if a/f^q is homogeneous of degree $-n-1$, the numerator a is homogeneous of degree $q \deg f - n - 1$, so that q may be recovered from a. To the usual partial derivative ∂_i on the rational side corresponds the *twisted* derivative on the polynomial side

$$\partial_i' a \overset{\text{def}}{=} \partial_i a - (\partial_i f) a = e^f \partial_i (a e^{-f}).$$

The exponential isomorphism, Theorem 5 and Corollary 7, relates, on the one hand, homogeneous fractions a/f^q of degree $-n-1$ modulo derivatives and, on the other hand, homogeneous polynomials, with degree in $(\deg f)\mathbb{Z} - n - 1$, modulo twisted derivatives.

2.1. Differential forms. — This section is a short reminder about differential forms, or simply *forms*.[30] Let Ω^1 denote the polynomial differential 1-forms: it is the free A-module of rank $n+1$, and the basis is denoted by the symbols $\mathrm{d}x_0, \dots, \mathrm{d}x_n$. The differential map d from A to Ω^1 is defined by

$$\mathrm{d}a = \sum_{i=0}^{n} \partial_i a \, \mathrm{d}x_i.$$

The A-algebra of differential forms, denoted Ω, is the exterior algebra over Ω^1. Its multiplication is denoted $\wedge$, it is generated by the $\mathrm{d}x_i$ and is subject to the relations $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$. The A-module of p-forms, denoted Ω^p, is the submodule of Ω generated by the $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$. With the multi-index notation, this is denoted $\mathrm{d}x^I$, with $I = (i_1, \dots, i_p)$. Ω^p is a free module of rank $\binom{n}{p}$. The module of 0-forms Ω^0 is identified with A. As a module, Ω decomposes as $\oplus_{p=0}^n \Omega^p$. Specifically,

[29] Similar results have been shown by Dwork earlier.
[30] See, for example, Matsumura, *Commutative algebra*, chap. 10, and Bourbaki, “Algèbres tensorielles, algèbres extérieures, algèbres symétriques”, §10, for more general and complete definitions.

the module Ω^{n+1} has rank 1 and is freely generated by $\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$, denoted ω. The module Ω^n has rank $n+1$ and is freely generated by the elements ξ_i defined by

$$\xi_i \stackrel{\text{def}}{=} (-1)^i \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n.$$

Exterior derivative. — The differential map d, from A to Ω^1, extends to an endomorphism of Ω, called *exterior derivative*, such that for $\alpha \in \Omega^p$ and $\beta \in \Omega$,

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta.$$

In particular $\mathrm{d}(\Omega^p)$ is included in Ω^{p+1} and $\mathrm{d}^2 = 0$. For a n-form β, written as $\sum_i b_i \xi_i$, we check that $\mathrm{d}\beta$ equals $\left(\sum_i \partial_i b_i\right) \omega$. The exterior derivative gives rise to a complex

$$0 \longrightarrow A \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} \Omega^{n+1} \longrightarrow 0$$

which is exact.

Homogeneity. — The degree of a monomial $x^I \mathrm{d}x^J$ is defined to be $|I| + |J|$. A form is called *homogeneous of degree k* if it is a linear combination of monomials of degree k. If α and β are two homogeneous forms of degree k_α and k_β respectively, then $\mathrm{d}\alpha$ is a homogeneous form of degree k_α and $\alpha \wedge \beta$ is a homogeneous form of degree $k_\alpha + k_\beta$.

Koszul complex. — The exterior product with $\mathrm{d}f$ gives a map from Ω^p to Ω^{p+1}, and since $\mathrm{d}f \wedge \mathrm{d}f$ vanishes we can consider the chain complex

$$\mathcal{K}(\mathrm{d}f) : 0 \longrightarrow A \xrightarrow{\mathrm{d}f} \Omega^1 \xrightarrow{\mathrm{d}f} \cdots \xrightarrow{\mathrm{d}f} \Omega^n \xrightarrow{\mathrm{d}f} \Omega^{n+1} \longrightarrow 0,$$

known as the *Koszul complex* of A with respect to $\mathrm{d}f$, and its cohomology $H\mathcal{K}(\mathrm{d}f)$ defined by

$$H^p\mathcal{K}(\mathrm{d}f) = \frac{\Omega^p \cap \ker \mathrm{d}f}{\mathrm{d}f \wedge \Omega^{p-1}}.$$

For a n-form β, written as $\sum_i b_i \xi_i$, the exterior product $\mathrm{d}f \wedge \beta$ is $\left(\sum_i b_i \partial_i f\right) \omega$. Thus $H^{n+1}\mathcal{K}(\mathrm{d}f)$ is isomorphic to $A/\operatorname{Jac} f$, with a shift of $n+1$ in the natural grading, where $\operatorname{Jac} f$ is the Jacobian ideal $(\partial_0 f, \ldots, \partial_n f)$.

Let Syz be the kernel of the product by $\mathrm{d}f$ on Ω^n. It is the syzygy module of the sequence $\partial_0 f, \ldots, \partial_n f$. Let Syz' be $\mathrm{d}f \wedge \Omega^{n-1}$, the module of trivial syzygies, generated by the elements $\partial_i f \, \xi_j - \partial_j f \, \xi_i$. In particular $H^n\mathcal{K}(\mathrm{d}f)$ is $\mathrm{Syz}/\mathrm{Syz}'$.

2.2. Chain complex T^p. — For an integer q, let T^p_q be the subspace of Ω^p generated by the homogeneous elements of degree qN. Let T^p be the direct sum $\oplus_q T^p_q$ and let $F_q T^p$ be $\oplus_{q' \leqslant q} T^p_{q'}$. Note that $\mathrm{d}f \wedge$ maps T^n_q to T^{n+1}_{q+1} and that d maps T^n_q to T^{n+1}_q. Let $\mathcal{S}$ (resp. $\mathcal{S}'$) be the intersection of T^n and Syz (resp. Syz'). The component of degree qN of an element α of T is denoted α_q.

The space T^{n+1}_q is the equivalent of M_q, as defined in the introductory remarks: the elements of T^{n+1}_q represent numerators of rational functions whose denominator is f^q. We define the linear map h from T^{n+1} to A_f by

$$h : a\omega \in T^{n+1}_q \longmapsto (q-1)! \frac{a}{f^q} \in A_f.$$

$$
\begin{tikzcd}
T^n_{q+1} \arrow[r, "\mathrm{d}"] & T^{n+1}_{q+1} \arrow[r, "\mathrm{d}"] & 0 & \\
{} \arrow[r] & T^n_q \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & T^{n+1}_q \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & 0 \\
{} \arrow[r] & T^{n-1}_{q-1} \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & T^n_{q-1} \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & T^{n+1}_{q-1} \arrow[u, "\mathrm{d}f"] \\
& {} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}
$$

Figure 1. Rham–Koszul double complex

Of course h is not injective since $h(f\alpha) = qh(\alpha)$, for $\alpha \in T^{n+1}_q$. Finally let D_f, the *twisted differential*, from T^p to T^{p+1} be the map defined by $D_f\alpha = \mathrm{d}\alpha - \mathrm{d}f \wedge \alpha$. Note that D_f maps F_qT^n to $F_{q+1}T^{n+1}$. The anticommutation $\mathrm{d}(\mathrm{d}f \wedge \beta) = -\mathrm{d}f \wedge \mathrm{d}\beta$ ensures that $D_f \circ D_f = 0$, so that T^p forms a chain complex.

Remark. — Consider the spaces T^{p+q}_q arranged within a grid. This forms a double complex, known as *Rham-Koszul double complex*,[31] with the *horizontal* differential being d and the *vertical* one being $\mathrm{d}f\wedge$, see Figure 1. This arrangement may help visualize some of the proofs in this article.

The following theorem has been proved by Dimca[32] and, independently, by Malgrange[33] and Deligne.[34]

Theorem 5. — *(Dimca) For all $p \geqslant 1$, $H^{p+1}T \simeq H^p_{Rham}(\mathbb{P}^n_{\mathbb{K}} \setminus V(f))$, where $H^{p+1}T$ is defined as $(T^{p+1} \cap \ker D_f)/D_f(T^p)$ and where $H^p_{Rham}(\mathbb{P}^n_{\mathbb{K}} \setminus V(f))$ is the p-th de Rham cohomology group of the variety $\mathbb{P}^n_{\mathbb{K}}$.*

When p is n, the cohomology group $H^{n+1}T$ is $T^{n+1}/D_f(T^n)$ and $H^n_{\mathrm{Rham}}(\mathbb{P}^n_{\mathbb{K}} \setminus V(f))$ is isomorphic to the subspace of $A_f / \sum_i \partial_i A_f$ of homogeneous elements of degree $-n-1$, and the isomorphism is given by the map induced by $h : T^{n+1} \to A_f$.

Proposition 6. — *$h(D_f(T^n)) \subset \sum_{i=0}^n \partial_i A_f$. In other words, the map h induces a map from $T^{n+1}/D_f(T^n)$ to $A_f / \sum_i \partial_i A_f$.*

Proof. — Let $\beta = \sum_{i=0}^n b_i\xi_i$ be an element of T^n_q, then

$$h\left(D_f\left(\sum_{i=0}^n b_i\xi_i\right)\right) = \sum_{i=0}^n h(\partial_i b\omega) - h(b_i\partial_i f\omega) = \sum_{i=0}^n (q-1)!\frac{\partial_i b_i}{f^q} - q!\frac{b_i\partial_i f}{f^{q+1}}$$

[31] Dimca, *Singularities and topology of hypersurfaces*.
[32] Dimca, "On the Milnor fibrations of weighted homogeneous polynomials", Theorem 1.8.
[33] Malgrange, *Lettre à Pierre Deligne*.
[34] Deligne, *Lettre à Bernard Malgrange*.

$$= (q-1)! \sum_{i=0}^{n} \partial_i \left(\frac{b_i}{f^q} \right). \quad \square$$

As a special case of Theorem 5, we have

Corollary 7. — *The induced map h from $T^{n+1}/D_f(T^n)$ to $A_f / \sum_i \partial_i A_f$ is injective.*

This way, the goal of computing normal forms modulo derivatives of rational functions can be reformulated as computing normal forms of elements of T^{n+1} modulo $D_f(T^n)$.

Example 8. — With $f = x^2y - z^3$, Equation (4) rewrites

$$x^3 \mathrm{d}x\mathrm{d}y\mathrm{d}z = D_f(\tfrac{2}{7}x^4\mathrm{d}y\mathrm{d}z + \tfrac{1}{7}x^3\mathrm{d}x\mathrm{d}z).$$

The rewriting is not always direct but Corollary 7 asserts that it is always possible.

3. Griffiths-Dwork reduction

We reword the Griffiths-Dwork reduction, presented in Section 1, in the above setting. Let us choose a monomial ordering on A, denoted $\prec$. For a linear subspace V of A and an element a of A, let $\operatorname{rem}_V a$, or $\operatorname{rem}(V, a)$, be the unique b in A such that $a - b$ is in V and no monomials of b is divided by the leading monomial of some element of V. If V is an ideal of A, this can be computed using a Gröbner basis of V, and if it is a finite-dimensional subspace, then Gaussian elimination following the monomial ordering computes $\operatorname{rem}_V a$.

The elementary step of the Griffiths-Dwork reduction is the following. Let α be an element of T_q^{n+1}. Write α as $\rho + \mathrm{d}f \wedge \beta$, with β in T^n and ρ minimal, that is equal to $\operatorname{rem}(\mathrm{d}f \wedge T^n, \alpha)$. The *elementary reduction of $a\omega$ in degree q* is defined to be

$$\operatorname{red}_q^{\mathrm{GD}}(\alpha) \stackrel{\text{def}}{=} \operatorname{rem}(\mathrm{d}f \wedge T^n, \alpha) + \mathrm{d}\beta.$$

We make the reasonable assumptions that β depends linearly on α and that β is zero if α equals ρ. For α in T_k^{n+1}, for some k different from q, we define $\operatorname{red}_q^{\mathrm{GD}}(\alpha) = \alpha$.

This reduction step is very easy to compute using a Gröbner basis of the Jacobian ideal $\operatorname{Jac} f = (\partial_0 f, \dots, \partial_n f)$ and its cofactors. Indeed, the multivariate division algorithm gives a decomposition of a polynomial a as $\operatorname{rem}(\operatorname{Jac} f, a) + \sum_{i=0}^n b_i \partial_i f$. If α is $a\omega$, then $\operatorname{rem}(\mathrm{d}f \wedge T^n, \alpha)$ is $\operatorname{rem}(\operatorname{Jac} f, a)\omega$ and β may be chosen equal to $\sum_i b_i \xi_i$. In this way, the assumptions on β are naturally satisfied. See Section 7 for more details about the implementation.

Proposition 9. — *For all α in T_q^{n+1}:*

(1) $\operatorname{red}_q^{\mathrm{GD}}(\operatorname{red}_q^{\mathrm{GD}} \alpha) = \operatorname{red}_q^{\mathrm{GD}} \alpha$
(2) $\operatorname{red}_q^{\mathrm{GD}} \alpha$ *is in* $T_q^{n+1} + T_{q-1}^{n+1}$
(3) $\operatorname{red}_q^{\mathrm{GD}} \alpha \equiv \alpha \mod D_f(F_{q-1}T^{n+1})$

(4) *if* α *is in* $D_f(F_{q-1}T^{n+1}) + F_{q-1}T^{n+1}$*, then* $\operatorname{red}_q^{\mathrm{GD}} \alpha \equiv 0 \mod F_{q-1}T^{n+1}$.

Proof. — Point (1) is a consequence of the fact that the map $\operatorname{rem}_{\mathrm{d}f\wedge T^n}$ is idempotent and the assumption that, in the definition of $\operatorname{red}_q^{\mathrm{GD}}(\alpha)$, β is zero if $\operatorname{rem}_{\mathrm{d}f\wedge T^n} \alpha$ equals α.

Since $\mathrm{d}f$ is homogeneous of degree N, we can assume that β, in the definition of $\operatorname{red}_q^{\mathrm{GD}}(\alpha)$, is homogeneous of degree $(q-1)N$, that is β is in T^n_{q-1}, and so $\mathrm{d}\beta$ is in T^{n+1}_{q-1}. Point (2) follows.

Point (3) is given by the equalities

$$\alpha = \operatorname{rem}(\mathrm{d}f \wedge T^n, \alpha) + \mathrm{d}f \wedge \beta = \operatorname{rem}(\mathrm{d}f \wedge T^n, \alpha) + \mathrm{d}\beta - D_f(\beta).$$

Concerning point (4), let β be an element of $F_{q-1}T^{n+1}$. The component $(\mathrm{d}\beta)_q$ is $\mathrm{d}f \wedge \beta_{q-1}$, which is in $(\operatorname{Jac} f)\omega$, so $\operatorname{red}_q^{\mathrm{GD}} \beta$ is in $F_{q-1}T^{n+1}$. □

When translated into a relation between fractions, point (3) reflects integration by parts:

$$\oint b_i \partial_i (1/f^{q-1})\mathrm{d}\mathbf{x} = -\oint \partial_i b_i / f^{q-1} \mathrm{d}\mathbf{x}.$$

This reduction step can be iterated by induction on the degree by defining $[\alpha]_{\mathrm{GD}}$ as $\operatorname{red}_1^{\mathrm{GD}}(\cdots \operatorname{red}_q^{\mathrm{GD}}(\alpha)\cdots)$, for $\alpha \in F_qT^{n+1}$. Or, equivalently, as $\rho + [\mathrm{d}\beta]_{\mathrm{GD}}$, using the notations in the definitions of $\operatorname{red}_q^{\mathrm{GD}}$. Proposition 9, and an induction on the degree, gives the equivalence

$$a\omega \equiv [a\omega]_{\mathrm{GD}} \mod D_f(T^n).$$

The reduction $[\,]_{\mathrm{GD}}$ is known as the *Griffiths-Dwork reduction.* It is a multivariate and homogeneous analogue of Hermite reduction. In general, it does not have all the nice properties of Hermite reduction. Namely, it may happen that for some α in $D_f(T^n)$ the reduction $[\alpha]_{\mathrm{GD}}$ is not zero. Nevertheless, Griffiths has proven the following:

***Theorem 10* (Griffiths[35]).** — *If* $V(f)$ *is smooth in* $\mathbb{P}^n_{\mathbb{K}}$ *then*

(i) $\ker[\,]_{\mathrm{GD}} = D_f(T^n)$,

(ii) *for all* α *in* T^{n+1} *the reduction* $[\alpha]_{\mathrm{GD}}$ *is in* F_nT^{n+1}

The hypothesis "$V(f)$ is smooth" is equivalent to the fact that $\operatorname{Jac} f$ is a zero-dimensional ideal, that is $A/\operatorname{Jac} f$ is finite-dimensional over $\mathbb{K}$. It is also equivalent to the equality of $\mathcal{S}$ and $\mathcal{S}'$, respectively the syzygies and the trivial syzygies in T^n. The main step of the proof of Theorem 10 is

***Theorem 11* (Griffiths[36]).** — *Assume that* $V(f)$ *is smooth in* $\mathbb{P}^n_{\mathbb{K}}$*. For all* α *in* F_qT^{n+1}*, if there exists* β *in* T^n *such that* $\alpha + D_f\beta$ *is in* $F_{q-1}T^{n+1}$*, then there exists* β' *in* $F_{q-1}T^n$ *such that* $\alpha + D_f\beta'$ *is in* $F_{q-1}T^{n+1}$.

35 Griffiths, "On the periods of certain rational integrals. I, II", §4
36 Ibid., Theorem 4.3

In the singular case, it is never true that $\ker[\,]_{\mathrm{GD}} = D_f(T^n)$. Worse still, the quotient $T^{n+1}/\ker[\,]_{\mathrm{GD}}$ is never finite dimensional. Indeed, we have

$$\frac{F_q T^{n+1}}{F_q T^{n+1} \cap \ker[\,]_{\mathrm{GD}} + F_{q-1}T^{n+1}} \simeq (A/\operatorname{Jac} f)_{qN-n-1},$$

so that the quotient is finite dimensional if and only if $\operatorname{Jac} f$ is a zero-dimensional ideal.

4. Computation of higher order relations

Let W_q^1 be the subspace of $T_q^{n+1} + T_{q-1}^{n+1}$ defined by

$$W_q^1 \overset{\text{def}}{=} D_f(T_{q-1}^{n+1}) = \left\{ -\mathrm{d}f \wedge \beta + \mathrm{d}\beta \;\middle|\; \beta \in T_{q-1}^n \right\}, \tag{6}$$

so that $\ker(\mathrm{red}_q^{\mathrm{GD}}) + T_{q-1} = W_q^1 + T_{q-1}$. Following the idea developed in Section 1, we define, for $r \geqslant 1$ and $q \geqslant 0$

$$W_q^{r+1} \overset{\text{def}}{=} W_q^1 + W_{q+1}^r \cap T_q^{n+1}.$$

Compared to Section 1, the space M_q has been replaced by T_q^{n+1} and the product $M_q \times M_{q-1}$ by the direct sum $T_q^{n+1} + T_{q-1}^{n+1}$.

Proposition 12. — *For all $r \geqslant 1$ and $q \geqslant 0$,*

$$W_q^r = D_f\big(\sum_{k=1}^{r} T_{q+k-2}^n\big) \bigcap F_q T^{n+1}.$$

Proof. — By induction on r. For $r = 1$, the claim reduces to $W_q^1 = D_f(T_{q-1}^{n+1})$, which is the definition. Then, let us prove that the right-hand side satisfies the recurrence relation defining W_q^r, that is:

$$D_f(T_{q-1}^{n+1}) + D_f\big(\sum_{k=1}^{r} T_{q+k-1}^n\big) \bigcap T_q^{n+1} = D_f\big(\sum_{k=1}^{r+1} T_{q+k-2}^n\big) \bigcap F_q T^{n+1}.$$

The right-to-left inclusion is clear. Conversely, let β in $\sum_{k=1}^{r+1} T_{q+k-2}^n$ be such that $D_f\beta$ is in $F_q T^{n+1}$. Let β' be $\beta - \beta_{q-1}$, where β_{q-1} is the component in T_{q-1}^n of β. The form $D_f\beta'$ is in T_q^{n+1}. Indeed, for $k > q$, $(D_f\beta')_k$ equals $(D_f\beta)_k$, which is zero, by hypothesis. And for $k < q$, $(D_f\beta')_k$ equals $\mathrm{d}\beta'_k - \mathrm{d}f \wedge \beta_{k-1}$ which is zero too. Thus $D_f\beta'$ is in $D_f\big(\sum_{k=1}^{r} T_{q+k-1}^n\big) \bigcap T_q^{n+1}$. Moreover $D_f\beta_{q-1}$ is in $D_f(T_{q-1}^{n+1})$. □

Example 13. — With $f = xy^2 - z^3$, we find that $W_1^1 = 0$ and

$$W_2^1 = \left\langle x^2y, xy^2, y^3, xyz, y^2z, xz^2, yz^2, z^3, 1 \right\rangle \omega.$$

Thus $W_2^1 \cap T_1^3$ is $\langle\omega\rangle$ and W_1^2 is $\langle\omega\rangle$.

Together with Corollary 7, the following claim proves Theorem 4.

Corollary 14. — *For all $\alpha \in T_q^{n+1}$. The following assertions are equivalent:*

q	0	1	2	3	4	$q>4$
$\dim E_q^0$	0	10	165	680	1771	$\binom{6q-1}{3} \sim 36q^3$
$\dim E_q^1$	0	10	86	102	120	$18q+48$
$\dim E_q^2$	0	10	7	6	6	6
$\dim E_q^3$	0	9	1	0	0	0

Table 1. Some dimensions related to the polynomial $2x_1x_2x_3(x_0-x_1)(x_0-x_2)(x_0-x_3) - x_0^3(x_0^3 - x_0^2x_3 + x_1x_2x_3)$.

(i) there exists a form β in $F_{q+r-2}T^n$ such that $\alpha = D_f\beta$;
(ii) there exists an α' in $D_f(F_{q+r-2}T^n) \cap F_{q-1}T^{n+1}$ such that $\alpha \equiv \alpha' \mod W_q^r$.

Proof. — Point (ii) implies point (i) because W_r^q is included in $D_f(F_{q+r-2}T^n)$, by Proposition 12. Conversely, let β in $F_{q+r-2}T^n$ such that $\alpha = D_f\beta$. The form β decomposes as $\beta' + \varepsilon$ with β' in $\sum_{k=1}^r T_{q+k-2}^n$ and ε in $F_{q-2}T^n$. The form $D_f\varepsilon$ is in $F_{q-1}T^{n+1}$, so that $D_f\beta'$ is in F_qT^{n+1} because $D_f\beta$ is. Thus the form $D_f\beta'$ is in W_q^r, and $\alpha \equiv D_f\varepsilon \mod W_q^r$. □

In view of Proposition 12, it is useful to introduce the spaces

$$E_q^r \stackrel{\text{def}}{=} \frac{F_qT^{n+1}}{W_r^q + F_{q-1}T^{n+1}} = \frac{F_qT^{n+1}}{D_f(F_{q+r-2}T^n) \cap F_qT^{n+1} + F_{q-1}T^{n+1}}.$$

In other words, E_q^r is F_qT^{n+1} modulo elements which are reducible to $F_{q-1}T^{n+1}$ by elements of $D_f(F_{q+r-2}T^n)$. It is clear that E_q^0 is $F_qT^{n+1}/F_{q-1}T^{n+1}$, which is isomorphic to F_q^{n+1}. Moreover, as a reformulation of Proposition 9, the space E_q^1 is

$$E_q^1 = \frac{F_qT^{n+1}}{\{\alpha \in F_qT^{n+1} \mid [\alpha]_{\mathrm{GD}} \in F_{q-1}T^{n+1}\}}.$$

Example 15. — Let us consider the polynomial f

$$f \stackrel{\text{def}}{=} 2x_1x_2x_3(x_0-x_1)(x_0-x_2)(x_0-x_3) - x_0^3(x_0^3 - x_0^2x_3 + x_1x_2x_3)$$

coming from an integral for the Apéry numbers, see Example 1. In this case $n=3$ and $N=6$. The dimension of the singular locus of $V(f)$ in $\mathbb{P}^3_{\mathbb{K}}$ is 1.

The dimensions of the first few E_q^r are shown in Table 1. This illustrates the successive dimension falls. At $r=3$, the equality $\dim E_1^3 = 9$ means that $\dim W_1^3/F_{q-1}T^{n+1}$ is one. It is generated by $(2x_1^2 - 2x_2^2 - x_0(x_1-x_2))\omega$, which equals $D_f\beta$ for some β in F_2T^4 but no such β is small enough to be reproduced here.

A basis of W_q^r can be computed by elementary linear algebra, either by induction or by Proposition 12. We can do much better. The next three sections propose successive refinements in the computation of W_q^r. Section 4.4 puts them all together.

4.1. Using Griffiths-Dwork reduction. — As we already remarked, the Griffiths-Dwork reduction is efficient and easy to implement. In the singular case, it does not catch all relations modulo $D_f(T^n)$, but still catches a lot of them. More precisely, the dimension of E_q^0 is $\binom{Nq-1}{n}$, which is equivalent to $N^n q^n/n!$ when $q \to \infty$. By contrast, the dimension of E_q^1 is $\mathcal{O}(q^\nu)$, where ν is the dimension of the singular locus of $V(f)$ in $\mathbb{P}^n_{\mathbb{K}}$. A natural improvement to the computation of W_q^r is to reuse the Griffiths-Dwork reduction as much as possible.

Let $\dot W_q^1$ be $\mathrm{d}(\mathcal{S}_{q-1})$ and

$$\dot W_q^{r+1} \stackrel{\text{def}}{=} \dot W_q^1 + \mathrm{red}_q^{\mathrm{GD}}\left(\dot W_{q+1}^r \cap T_q^{n+1}\right). \tag{7}$$

The dimension of $\dot W_q^1$ is smaller than the dimension of W_q^1, so the time spent in linear algebra operations is reduced, at the cost of computing $\mathrm{red}^{\mathrm{GD}}$.

Lemma 16. — *$\mathrm{red}_q^{\mathrm{GD}}\,\alpha$ equals α for all $\alpha \in \dot W_q^r$.*

Proof. — For $r = 1$, this is true since $\dot W_q^1$ is included in T_{q-1}^{n+1}. The claim follows by induction on r from the fact that $\mathrm{red}_q^{\mathrm{GD}}$ is idempotent (Proposition 9). □

Let G_q be the kernel of $\mathrm{red}_q^{\mathrm{GD}}$, which is a subset of $T_q^{n+1} + T_{q-1}^{n+1}$. Note that $G_q \cap F_{q-1}T^{n+1}$ is null.

Proposition 17. — *$\dot W_q^r \oplus G_q = W_q^r$, for all $r \geqslant 1$ and $q \geqslant 0$.*

Proof. — For $r = 1$. The inclusion of G_q in W_q^1 is a consequence of Proposition 9, point (3). Let us prove that $(G_{q+1} + \dot W_{q+1}^r) \cap T_q^{n+1}$ equals $\dot W_{q+1}^r \cap T_q^{n+1}$. Let α in G_{q+1} and β in $\dot W_{q+1}^r$ be such that $\alpha + \beta$ is in T_q^{n+1}. Then $\mathrm{red}_{q+1}^{\mathrm{GD}}(\alpha+\beta)$ equals $\alpha + \beta$. But it also equals $\mathrm{red}_{q+1}^{\mathrm{GD}}(\beta)$ since α is in $\ker \mathrm{red}_{q+1}^{\mathrm{GD}}$. Since $\mathrm{red}_{q+1}^{\mathrm{GD}}(\beta) = \beta$, we obtain $\alpha = 0$, which gives the claim.

By induction on r,

$$\begin{aligned}
W_q^{r+1} &= W_q^1 + W_{q+1}^r \cap T_q^{n+1} && \text{by definition}\\
&= G_q + \dot W_q^1 + (G_{q+1} + \dot W_{q+1}^r) \cap T_q^{n+1} && \text{by induction hypothesis}\\
&= G_q + \dot W_q^1 + \dot W_{q+1}^r \cap T_q^{n+1} && \text{by the claim above}\\
&= G_q + \dot W_q^{r+1}.
\end{aligned}$$

Lemma 16 implies that $\dot W_q^r \cap G_q$ is null. □

4.2. Removing trivial syzygies. — As shown in Section 1, the trivial syzygies does not bring new relations when passing from W_q^1 to W_q^2. We can simply discard them.

Let A_q be a complementary subspace of $\mathcal{S}'_q$ in $\mathcal{S}_q$, that is $\mathcal{S}_q$ equals $\mathcal{S}'_q \oplus A_q$. We define $\ddot W_q^1 \stackrel{\text{def}}{=} \mathrm{d}A_{q-1}$, and

$$\ddot W_q^{r+1} \stackrel{\text{def}}{=} \ddot W_q^1 + \mathrm{red}_q^{\mathrm{GD}}\left(\ddot W_q^1 \cap T_q^{n+1}\right). \tag{8}$$

q	0	1	2	3	4	$q>4$
$\dim \mathcal{S}_q$	0	21	522	2429	6604	$\sim 144q^3$
$\dim \mathcal{S}_q/\mathcal{S}'_q$	0	1	92	132	168	$36q+24$
$\dim E^1_q - \dim E^2_q$	0	0	79	96	114	$18q+42$

Table 2. Gain of dimension by discarding trivial syzygies and number of new relations generated by the syzygies

The dimension of $\ddot{W}^1_q$ is much smaller than the dimension of $\dot{W}^1_q$, so the time spent in linear algebra operations is again reduced, at the expense of computing A_{q-1}.

Proposition 18. — *$\ddot{W}^r_q + \mathrm{d}\mathcal{S}'_{q-1} = \dot{W}^r_q$, for all $r \geqslant 1$ and $q \geqslant 0$.*

Proof. — Since $\mathcal{S}_{q-1} = A_{q-1} + \mathcal{S}'_{q-1}$, the claim when $r=1$ follows form the definition of $\ddot{W}^1_q$ and $\dot{W}^1_q$. Then, by induction on r,

$$\begin{aligned}
\dot{W}^{r+1}_q &= \dot{W}^1_q + \operatorname{red}^{\mathrm{GD}}_q\left(\dot{W}^r_{q+1} \cap T^{n+1}_q\right) && \text{by definition}\\
&= \mathrm{d}\mathcal{S}'_{q-1} + \ddot{W}^1_q + \operatorname{red}^{\mathrm{GD}}_q\left((\mathrm{d}\mathcal{S}'_q + \ddot{W}^r_{q+1}) \cap T^{n+1}_q\right) && \text{by induction hypothesis}\\
&= \mathrm{d}\mathcal{S}'_{q-1} + \ddot{W}^1_q + \operatorname{red}^{\mathrm{GD}}_q(\mathrm{d}\mathcal{S}'_q) + \operatorname{red}^{\mathrm{GD}}_q\left(\ddot{W}^r_{q+1} \cap T^{n+1}_q\right) && \text{because } \mathrm{d}\mathcal{S}'_q \subset T^{n+1}_q\\
&= \mathrm{d}\mathcal{S}'_{q-1} + \ddot{W}^{r+1}_q + \operatorname{red}^{\mathrm{GD}}_q(\mathrm{d}\mathcal{S}'_q).
\end{aligned}$$

We conclude with diagram chasing in Figure 1. Let $\mathrm{d}f \wedge \beta$ be an element of $\mathcal{S}'_q$, with β in T^{n-1}_{q-1}. Then $\mathrm{d}(\mathrm{d}f \wedge \beta)$ equals $-\mathrm{d}f \wedge \mathrm{d}\beta$, so that $\operatorname{red}^{\mathrm{GD}}_q(\mathrm{d}(\mathrm{d}f \wedge \beta))$ is $\mathrm{d}\varepsilon$, for some ε in T^n_{q-1} such that $\mathrm{d}f \wedge \varepsilon = -\mathrm{d}f \wedge \mathrm{d}\beta$. In particular $\varepsilon + \mathrm{d}\beta$ is in $\mathcal{S}_{q-1}$, so that it decomposes in $\varepsilon' + \eta$, with ε' in A_{q-1} and η in $\mathcal{S}'_{q-1}$. In the end $\operatorname{red}^{\mathrm{GD}}_q(\mathrm{d}(\mathrm{d}f \wedge \beta))$ equals $\mathrm{d}\varepsilon' + \mathrm{d}\eta$, because $\mathrm{d}(\mathrm{d}\beta) = 0$, which belongs to $\ddot{W}^1_q + \mathrm{d}\mathcal{S}'_{q-1}$, so that $\dot{W}^{r+1}_q$ equals $\mathrm{d}\mathcal{S}'_{q-1} + \ddot{W}^{r+1}_q$. □

Example 19. — Continuing Example 15, we illustrate the computational gain obtained by discarding trivial syzygies, see Table 2. Computing $\dot{W}^2_q$ involves the computation of $\operatorname{red}^{\mathrm{GD}}_q$ and linear algebra on $\dim \mathcal{S}_q$ different vectors, while the computation of $\ddot{W}^2_q$ involves only $\dim \mathcal{S}_q/\mathcal{S}'_q$ vectors. The cost of computing these vectors is negligible in practice, compared to the cost of $\operatorname{red}^{\mathrm{GD}}_q$ and the linear algebra. Finally, this number of vectors is compared to the difference $\dim E^1_q - \dim E^2_q$ which indicates how many new relations are generated by the syzygies. The asymptotics, when $q \to \infty$,

$$\dim \mathcal{S}_q \sim (n+1)N^n q^n/n! \quad \text{and} \quad \dim \mathcal{S}_q/\mathcal{S}'_q = \mathcal{O}(q^\nu)$$

are as expected.

4.3. Removing more redundant computations. — Let $\dddot{W}_q^1$ be $\ddot{W}_q^1$, that is $\mathrm{d}A_{q-1}$, where A_{q-1} is a complementary subspace of $\mathcal{S}'_{q-1}$ in $\mathcal{S}_q$, as above. Assuming that $\dddot{W}_q^r$ and $\dddot{W}_{q+1}^r$ are defined, let U_q^r be a complementary subspace of T_{q-1}^{n+1} in $\dddot{W}_q^r$ and let

$$\dddot{W}_q^{r+1} \overset{\text{def}}{=} U_q^r + \mathrm{red}_q^{\mathrm{GD}}\left(\dddot{W}_q^r \cap T_q^{n+1}\right). \tag{9}$$

Proposition 20. — *For all $r \geqslant 1$ and $q \geqslant 0$*

(i) $\dddot{W}_q^r + \ddot{W}_q^{r-1} \cap T_{q-1}^{n+1} = \ddot{W}_q^r$,

(ii) $U_q^r \oplus (\ddot{W}_q^r \cap T_{q-1}^{n+1}) = \ddot{W}_q^r$.

Proof. — Point (i) implies that

$$\dddot{W}_q^r \cap T_{q-1}^{n+1} + \ddot{W}_q^{r-1} \cap T_{q-1}^{n+1} = \ddot{W}_q^r \cap T_{q-1}^{n+1}. \tag{10}$$

Thus point (i) implies (ii) because

$$\begin{aligned} U_q^r + (\ddot{W}_q^r \cap T_{q-1}^{n+1}) &= U_q^r + \dddot{W}_q^r \cap T_{q-1}^{n+1} + \ddot{W}_q^{r-1} \cap T_{q-1}^{n+1} && \text{by Eq. (10)} \\ &= \dddot{W}_q^r + \ddot{W}_q^{r-1} \cap T_{q-1}^{n+1} && \text{by def. of } U_r^q \\ &= \ddot{W}_q^r && \text{by point (i).} \end{aligned}$$

The fact that the sum is direct follows from $U_q^r \cap T_{q-1}^{n+1} = 0$, by definition.

Concerning point (i), the base case $r = 1$ is clear, because $\ddot{W}_q^1 = \dddot{W}_q^1$. Then, by induction on r,

$$\begin{aligned} \ddot{W}_q^{r+1} &= \ddot{W}_q^r + \mathrm{red}_q^{\mathrm{GD}}\left(\ddot{W}_q^r \cap T_q^{n+1}\right) && \text{by definition} \\ &= \ddot{W}_q^r + \mathrm{red}_q^{\mathrm{GD}}\left(\dddot{W}_{q+1}^r \cap T_q^{n+1}\right) + \mathrm{red}_q^{\mathrm{GD}}\left(\ddot{W}_{q+1}^{r-1} \cap T_q^{n+1}\right) && \text{by Eq. (10)} \\ &= \ddot{W}_q^r + \mathrm{red}_q^{\mathrm{GD}}\left(\dddot{W}_{q+1}^r \cap T_q^{n+1}\right) \end{aligned}$$

because $\mathrm{red}_q^{\mathrm{GD}}\left(\ddot{W}_{q+1}^{r-1} \cap T_q^{n+1}\right)$ is a subspace of $\ddot{W}_q^r$

$$\begin{aligned} &= U_q^r + \ddot{W}_q^r \cap T_{q-1}^{n+1} + \mathrm{red}_q^{\mathrm{GD}}\left(\dddot{W}_{q+1}^r \cap T_q^{n+1}\right) && \text{by induction hypothesis} \\ &= \dddot{W}_q^{r+1} + \ddot{W}_q^r \cap T_{q-1}^{n+1}, \end{aligned}$$

which concludes the proof. □

4.4. Reduction of order r. — Putting all together, we obtain:

Theorem 21. — *For all $\alpha \in T_q^{n+1}$. The following assertions are equivalent:*

(i) there exists a form β in $F_{q+r-2}T^n$ such that $\alpha = D_f\beta$;

(ii) there exists an α' in $D_f(F_{q+r-2}T^n) \cap F_{q-1}T^{n+1}$ such that $\mathrm{red}_q^{\mathrm{GD}}\alpha$ equals α' modulo $\dddot{W}_q^r$.

Proof. — Point (ii) implies point (ii) of Corollary 12 because $\ker(\operatorname{red}_q^{\mathrm{GD}})$ and $\dddot{W}_q^r$ are subspaces of W_q^r.

Conversely, assume point (ii) of Corollary 12, that is there exists an α' in $F_{q-1}T^{n+1}$ and a β in $F_{q+r-2}T^n$ such that $\alpha = \alpha' + D_f\beta$.

Propositions 17, 18 and 20 prove that

$$W_q^r + F_{q-1}T^{n+1} = \dddot{W}_q^r + G_q + F_{q-1}T^{n+1},$$

thus there exist α'', γ and δ in $F_{q-1}T^{n+1}$, $\dddot{W}_q^r$ and G_q respectively such that α equals $\alpha'' + \gamma + \delta$. By definition of G_q, $\operatorname{red}_q^{\mathrm{GD}} \delta = 0$. By Lemma 16 and the inclusion $\dddot{W}_q^r \subset \dot{W}_q^r$, we obtain

$$\operatorname{red}_q^{\mathrm{GD}} \alpha = \alpha'' \mod \dddot{W}_q^r.$$

And α'' is in $D_f(F_{q+r-2}T^n)$ because α is. □

The higher order analogue of $\operatorname{red}_q^{\mathrm{GD}}$ is

$$\operatorname{red}_q^r \alpha \stackrel{\text{def}}{=} \operatorname{rem}(U_q^r, \operatorname{red}_q^{\mathrm{GD}} \alpha),$$

for α in T^{n+1}. As for the Griffiths-Dwork reduction, we define $[\alpha]_r$ as $\operatorname{red}_1^r \circ \cdots \circ \operatorname{red}_q^r(\alpha)$, for α in F_qT^{n+1}. By construction, if α is in $\dddot{W}_q^r + F_{q-1}T^{n+1}$, then $\operatorname{red}_q^r \alpha$ is in $F_{q-1}T^{n+1}$.

Corollary 22. — *Let $r \geqslant 1$, and $q \geqslant 0$. For all $\alpha \in F_qT^{n+1}$*

(i) $\alpha \equiv [\alpha]_r \mod D_f(T^n)$,

(ii) $[\alpha]_r \equiv 0 \mod F_{q-1}T^{n+1}$ *if and only if α is in $D_f(F_{q+r-2}T^{n+1}) + F_{q-1}T^{n+1}$.*

Moreover $[\,]_1$ coincides with the Griffiths-Dwork reduction $[\,]_{\mathrm{GD}}$.

Corollary 23. — *If β is in F_qT^n, then $[D_f\beta]_{q+1} = 0$. In particular*

$$D_f(T^n) = \bigcup_{r \geqslant 1} \ker[\,]_r.$$

This gives a proof of Theorem 11: if $\mathcal{S} = \mathcal{S}'$, then $\ddot{W}_q^1$ is null and so is $\ddot{W}_q^r$. Thus $[\,]_r$ equals $[\,]_{\mathrm{GD}}$ for all $r \geqslant 0$. By Corollary 23, this implies that $D_f(T^n)$ equals $\ker[\,]_{\mathrm{GD}}$.

5. Extensions of Griffiths' theorems

Given α in $D_f(T^n)$, how can we compute a r such that if α is in $D_f(T^n)$ then $[\alpha]_r$ equals zero? Corollary 23 lacks effective bounds and does not answer this question. The question is more commonly addressed in terms rational functions, but results are easily translated from one setting to the other with a more precise version of Corollary 7.

Let E_q^r be the quotient space, as introduced above,

$$E_q^r \stackrel{\text{def}}{=} \frac{F_qT^{n+1}}{D_f(F_{q+r-2}T^n) \cap F_qT^{n+1} + F_{q-1}T^{n+1}} = \frac{F_qT^{n+1}}{\{\alpha \in F_qT^{n+1} \mid [\alpha]_r \in F_{q-1}T^{n+1}\}},$$

where the second equality comes from Corollary 22. The space E_q^{r+1} is a quotient of E_q^r, so we can consider E_q^∞, the inductive limit $\varinjlim E_q^r$, that is

$$E_q^\infty \stackrel{\text{def}}{=} \frac{F_q T^{n+1}}{D_f(T^n) \cap F_q T^{n+1} + F_{q-1} T^{n+1}} = \frac{F_q T^{n+1}}{\{\alpha \in F_q T^{n+1} \mid \exists r, [\alpha]_r \in F_{q-1} T^{n+1}\}}.$$

The analogous spaces in terms of fractions are

$$\bar{E}_q^r \stackrel{\text{def}}{=} \frac{F_q^{n+1} A_f}{(\sum_i \partial_i F_{q+r-2}^n A_f) \cap F_q^{n+1} A_f + F_{q-1}^{n+1} A_f},$$

and

$$\bar{E}_q^\infty \stackrel{\text{def}}{=} \frac{F_q^{n+1} A_f}{(\sum_i \partial_i A_f) \cap F_q^{n+1} A_f + F_{q-1}^{n+1} A_f},$$

where $F_s^t A_f$ is the space of all homogeneous fractions of degree $-t$ whose denominator divides f^s, that is

$$F_s^t A_f \stackrel{\text{def}}{=} \{a/f^s \mid a \in \mathbb{K}[\mathbf{x}]_{sN+t}\}.$$

It is only a matter of decoding notations to check that the map $h : T^{n+1} \to A_f$, defined in Section 2.2, induces a map $E_r^q \to \bar{E}_r^q$. Dimca has proven the following refinement of Corollary 7:

Theorem 24 **(Dimca[37])**. — *For $r \geqslant 1$ and $q \geqslant 0$, the induced map $h : E_q^r \to \bar{E}_q^r$ is an isomorphism.*

It is clear that E_q^{r+1} is a quotient of E_q^r. Since E_q^0 is finite dimensional, there exists an r such that E_q^r equals E_q^∞. Let $r(q)$ be the least such r, and let $s(q)$ be $\max_{k \leqslant q} r(q)$. The value of $s(q)$ is particularly interesting because for all α in $F_q T^{n+1}$, if α is in $D_f(T^n)$, then $[\alpha]_{s(q)}$ equals zero.

The value $r(q)$ is in fact bounded independently of q:

Theorem 25 **(Dimca[38])**. — *There exists an integer C, depending only on f, such that for any rational function a/f^q, homogeneous of degree $-n-1$, of the form $\sum_i \partial_i(b_i/f^s)$ for some polynomials b_i and some integer s, the function a/f^q equals $\sum_i \partial_i(b_i'/f^{q+C-2})$ for some polynomials b_i'.*

An immediate reformulation of Theorem 25, together with Theorem 24 is

Corollary 26. — *There exists a C such that $E_q^C = E_q^\infty$ for all q. In particular* $\ker[\,]_C$ *equals $D_f(T^n)$.*

Let C_f be the least such C. Unfortunately, while explicit, this integer C_f is not easy to compute: in Dimca's proof it is expressed in terms of a resolution of the singularities of the projective variety $V(f)$. By contrast, the point (ii) of Theorem 10 fully generalizes to singular cases:

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[37] Dimca, "On the Milnor fibrations of weighted homogeneous polynomials", Theorem 1.8

[38] Dimca, "On the de Rham cohomology of a hypersurface complement", Theorem B and Corollary 2

Theorem 27 **(Dimca[39])**. — *For any rational function a/f^q homogeneous of degree $-n-1$, there exists another rational function a'/f^n, homogeneous of degree $-n-1$ such that*

$$a/f^q = a'/f^n + \sum_{i=0}^{n} \partial_i (b_i/f^s)$$

for some polynomials b_i and some integer s.

Together with Theorems 24 and 25 we obtain

Corollary 28. — *For all α in T^{n+1} the reduction $[\alpha]_{C_f}$ is in $F_n T^{n+1}$.*

Dimca[40] conjectured that

Conjecture 29. — $C_f \leqslant n+1$.

Example 30. — Needless to say, the polynomial f of Example 15 confirms these results. It seems that C_f equals 3.

As far as I know, computations on explicit examples confirm this conjecture. Moreover the bound is tight when n is 2. A proof of this conjecture would have very interesting algorithmic consequences: the reduction algorithm is extensible to the computation of the whole cohomology of T, not just the top cohomology. Only the bound $C_f \leqslant n+1$ lacks for obtaining an efficient algorithm for computing the de Rham cohomology of the complement of a projective hypersurface.

For some applications, such that the computation of annihilating operators of periods with a parameter, Corollary 28 gives an efficient workaround. Consider an algorithm which computes reductions $[\alpha]_r$, for some forms α and some fixed integer r, and does it as long as the reductions it computes are linearly independent. Then either all the $[\alpha]_r$ are in the finite dimensional space $F_n T^{n+1}$, and then the algorithm terminates; or some $[\alpha]_r$ is not in $F_n T^{n+1}$, and then $r < C_f$, by Corollary 28. When the second case is encountered, we abort the algorithm, increment r and start over. This may happen only if $r < C_f$, so the first case happens eventually and the algorithm terminates.

[39] Dimca, "On the Milnor fibrations of weighted homogeneous polynomials", Theorem 2.7
[40] Dimca, "On the de Rham cohomology of a hypersurface complement".

PART II
PERIODS WITH A PARAMETER

We apply the reduction algorithm to the computation of Picard-Fuchs equations.

6. Algorithms

6.1. Setting. — Let $\mathbb{K}$ be a field of characteristic zero with a derivation δ. Typically $\mathbb{K}$ is $\mathbb{Q}(t)$ and δ is the usual derivation with respect to t. Let $\mathbb{K}\langle\delta\rangle$ be the algebra of differential operators in δ: it is the associative algebra with unity generated over $\mathbb{K}$ by δ and subject to the relations

$$\delta x = x\delta + \delta(x)$$

for all x in $\mathbb{K}$, where $\delta(x)$ denotes the application of δ to x whereas δx is the operator that multiplies by x and then applies δ. On $\mathbb{K}(x_0, \dots, x_n)$, let ∂_i denote the derivation with respect to x_i. The derivation δ extends to $\mathbb{K}(x_0, \dots, x_n)$ uniquely by setting $\delta(x_i) = 0$. In particular $\delta \circ \partial_i = \partial_i \circ \delta$.

This section describes an algorithm which takes as input a rational function R in $\mathbb{K}(x_1, \dots, x_n)$ and outputs an operator $\mathcal{L}$ in $\mathbb{K}\langle\delta\rangle$ such that there exist other rational functions $C_1, \dots, C_n$ with

$$\mathcal{L}(R) = \sum_{i=1}^{n} \partial_i C_i.$$

Moreover, the irreducible factors of the denominators of the C_i divide the denominator of R. Such an operator will be called an *annihilating operator of the periods of* R, or *a differential equation for* $\oint R$. The minimal annihilating operator of $\oint R$ is called the *Picard-Fuchs equation* (of $\oint R$). The output operator $\mathcal{L}$ is not necessarily the Picard-Fuchs equation but it is of course a left multiple of it.

Being based on the reduction algorithm of Section I, the algorithm does not compute the C_i. It is worth a word because while only $\mathcal{L}$ matters, the size of the C_i, say the size of a binary dense representation, is usually much larger than the size of $\mathcal{L}$. To be able to compute $\mathcal{L}$ without computing the C_i is certainly a good point toward practical efficiency. The fractions C_i are called *certificate*: they allow to check *a posteriori* that $\mathcal{L}$ is indeed an annihilating operator of $\oint R$.

6.2. Homogenization. — The reduction algorithm works in an homogeneous setting. If we are interested in computing the Picard-Fuchs equation of the integral of an inhomogeneous function, the problem can be homogenized as follows. Let R_{hom} be the homogenization of R in degree $-n-1$ defined by

$$R_{\text{hom}} = x_0^{-n-1} R\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{K}(\mathbf{x}),$$

where $\mathbf{x}$ denotes $x_0, \ldots, x_n$ hereafter. A fraction $F(\mathbf{x})$ in $\mathbb{K}(\mathbf{x})$ is *homogeneous of degree* d if

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n),$$

or, equivalently, if $F = u/v$ where u and v are homogeneous polynomials such that $\deg u - \deg v$ equals d.

Let us write R_{hom} as a/f^q, with a and f two homogeneous polynomials and q an integer. Usually f will be chosen square-free but it is not necessary. Let N be the degree of f. Since R_{hom} is homogeneous of degree $-n-1$, the degree of a is $qN-n-1$. This is the main reason for considering homogeneous fractions: the degree of the denominator determines the degree of the numerator, there is no *invisible* pole at infinity. The degree $-n-1$ is crucial to ensure that:

Lemma 31. — *If $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ is a annihilating operator of $\oint R_{hom}$ then $\mathcal{L}$ is also a annihilating operator of $\oint R$.*

Proof. — Assume that $\mathcal{L}(R_{\mathrm{hom}})$ equals $\sum_{i=0}^n \partial_i(b_i/f^m)$, for some polynomials b_i and some integer m. Substituting x_0 by 1 gives

$$\mathcal{L}(R) = \partial_0(b_0/f^m)|_{x_0=1} + \sum_{i=1}^n \partial_i(b_i/f^m|_{x_0=1}).$$

Since R_{hom} is homogeneous of degree $-n-1$, we may assume that each b_i/f^m is homogeneous of degree $-n$. Euler's relation gives

$$-nb_0/f^m = \sum_{i=0}^n x_i\partial_i(b_0/f^m) = \sum_{i=0}^n \left(\partial_i(x_i b_0/f^m) - b_0/f^m\right).$$

This proves that

$$0 = \partial_0(b_0/f^m)|_{x_0=1} + \sum_{i=1}^n \partial_i(x_i b_0/f^m|_{x_0=1}),$$

and the claim follows. □

The Picard-Fuchs equation of $\oint R_{\mathrm{hom}}$ may not be the Picard-Fuchs equation of $\oint R$. However, it is the case if x_0 divides f, which is possible to assume, up to replacing f by $x_0 f$ and a by $x_0^q a$. From now on I focus exclusively on the homogeneous case.

6.3. Computation of Picard-Fuchs equations. — The derivation δ is extended to T coefficient-wise, and we define the twisted derivative $\tilde{\delta}$ by

$$\tilde{\delta} : \alpha \in T \mapsto \delta(\alpha) - \delta(f)\alpha \in T.$$

It commutes with the map h, and the differential D_f, as a consequence of δ commuting with ∂_i.

To highlight the difference between the smooth and the singular cases, I recall first how the Griffiths-Dwork reduction applies to the computation of Picard-Fuchs

Algorithm 1. Computation of annihilating operators of the periods of a rational function, smooth case

Input — a/f^q a homogeneous rational function in $\mathbb{K}(\mathbf{x})$ of degree $-n-1$, with $V(f)$ smooth in $\mathbb{P}^n_{\mathbb{K}}$
Output — $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ the Picard-Fuchs equation of $\oint R$
procedure PicardFuchs(a/f^q)
  $\rho_0 \leftarrow [a\omega]_{\mathrm{GD}}$
  for m from 0 to ∞ **do**
    if $\operatorname{rank}_{\mathbb{K}}(\rho_0, \ldots, \rho_m) = m+1$ **then**
      $\rho_{m+1} \leftarrow [\delta(\rho_m)]_{\mathrm{GD}}$
    else
      solve $\sum_{k=0}^{m-1} a_k \rho_k = \rho_m$ w.r.t. $a_0, \ldots, a_{m-1}$ in $\mathbb{K}$
      return $\delta^m - \sum_{k=0}^{m-1} a_k \delta^k$

equations. Let a/f^q be a homogeneous fraction of degree $-n-1$. We define ρ_0 as $[a\omega]_{\mathrm{GD}}$ and

$$\rho_{k+1} \stackrel{\text{def}}{=} [\tilde{\delta}(\rho_k)]_{\mathrm{GD}}.$$

Since $\tilde{\delta}$ commutes with D_f, it is clear that ρ_k equals $\tilde{\delta}^k(a\omega)$ modulo $D_f(T^n)$. Hence Theorem 10 implies that ρ_k equals $[\tilde{\delta}^k(a\omega)]_{\mathrm{GD}}$. Thus, by Theorem 10 and Corollary 7, for $u_0, \ldots, u_m$ in $\mathbb{K}$,

$$\sum_{k=0}^{m} u_k \delta^k(a/f^q) \in \sum_{k=0}^{n} \partial_k A_f \text{ if and only if } \sum_{k=0}^{m} u_k \rho_k = 0.$$

This leads to Algorithm 1.

Proposition 32. — *Algorithm 1 applied to a fraction R satisfying the regularity assumption terminates and outputs the Picard-Fuchs equation of $\oint R$.*

Proof. — Correction has just been proven. Termination follows from Theorem 10, point (ii), which implies that the ρ_i lie in a finite-dimensional space, so they are linearly dependent. □

If Conjecture 29 were proven, it would be enough to replace $[\,]_{\mathrm{GD}}$ by $[\,]_{n+1}$ in Algorithm 1 to obtain an algorithm which provably outputs the Picard-Fuchs equation of a rational integral in the singular case. While this gives good results in practice, the absence of a proof is embarrassing.

It is worth mentioning the treatment of singular cases by a generic deformation: to compute a differential for $\oint R$, for some $R = a/f$, we may change R into

$$R_\lambda = \frac{a}{f + \lambda \sum_{i=0}^{n} x_i^{\deg f}},$$

where λ is a free variable. The denominator of R_λ always satisfy the smoothness hypothesis, so Algorithm 1 applies, over $\mathbb{K}(\lambda)$, and gives the Picard-Fuchs equation of $\oint R_\lambda$, say $\mathcal{L}$ in $\mathbb{K}(\lambda)\langle\delta\rangle$. Then $(\lambda^a \mathcal{L})_{|\lambda=0}$, where a is the unique integer which makes this evaluation neither zero nor singular, is a differential equation for $\oint R$. This method achieves a good computational complexity[41] but its practical efficiency is terrible.

Another approach, using the reductions $[\,]_r$, is to loop over r. We begin by fixing r to an initial value, for example 1, and we introduce another variable M, a positive integer. Then we compute ρ_0, ρ_1, etc. as in Algorithm 1 but replacing $[\,]_{\mathrm{GD}}$ by $[\,]_r$, up to ρ_M. If there is no linear dependency relation between the ρ_k then we increase both r and M and repeat the procedure. At some point, the parameter r will exceed C_f and M will exceed the order of the Picard-Fuchs equation of $\oint R$. There, a relation will be found between the ρ_k and it will give the Picard-Fuchs equation. It is possible that a relation is found before the condition $r \geqslant C_f$ is met: it gives of course a differential equation, but it need not be the minimal one.

Theorem 27 and its corollary allow for an interesting variant of this approach. As above, we loop over r. For a given value of r, the forms ρ_0, ρ_1, etc. are computed as in Algorithm 1 but replacing $[\,]_{\mathrm{GD}}$ by $[\,]_r$. Contrary to the previous approach, the number of ρ_i we compute before moving to the next value of r is not bounded *a priori*. Instead, we compute ρ_0, ρ_1, etc. as long as ρ_k stays in $F_n T^{n+1}$. Since $F_n T^{n+1}$ is finite dimensional, we have the following alternative: either there exists a relation between the ρ_k, or there exists a k such that ρ_k is not in $F_n T^{n+1}$. In the first case, the relation gives a differential equation for $\oint R$. In the second case, we increase r. Corollary 28 assures that as soon as $r \geqslant C_f$, the second condition is never met, so a relation will eventually be found. Algorithm 2 details the procedure.

Theorem 33. — *Algorithm 2 terminates and outputs an annihilating operator of $\oint R$.*

7. Implementation

Algorithm 2 has been implemented in the computer algebra system Magma,[42] with $\mathbb{Q}(t)$ as base field $\mathbb{K}$, with the usual derivation.[43] To be able to treat large examples—like the ones in Section 8—the coefficient swell makes it necessary to implement a randomized evaluation-interpolation scheme which splits a computation over $\mathbb{Q}(t)$ into several analogous computations over different finite fields. However it comes at a price: since we lack tight *a priori* bounds on the size of the output—order, degree, size of the coefficients—the reconstruction step is not certified to be correct, even though the probability of failure can be made arbitrarily small. There are also

[41] Bostan, Lairez, and Salvy, "Creative telescoping for rational functions using the Griffiths–Dwork method".
[42] Bosma, Cannon, and Playoust, "The Magma algebra system. I. The user language".
[43] The implementation is available at `http://github.com/lairez/periods`.

Algorithm 2. Computation of annihilating operators of the periods of a rational function

Input — a/f^q a homogeneous rational function in $\mathbb{K}(\mathbf{x})$ of degree $-n-1$
Output — $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ a differential equation for $\oint R$
procedure PICARDFUCHS(a/f^q)
  for r from 1 to ∞ **do**
    $\rho_0 \leftarrow [a\,\omega]_r$ ▷ Compute the subspaces U_r^q as they are needed.
    for m from 0 to ∞ while $\deg \rho_m \leqslant n \deg f$ **do**
      if $\operatorname{rank}_{\mathbb{K}}(\rho_0, \dots, \rho_m) = m+1$ **then**
        $\rho_{m+1} \leftarrow [\delta(\rho_m) - \delta(f)\rho_m]_r$
      else
        solve $\sum_{k=0}^{m-1} a_k \rho_k = \rho_m$ w.r.t. $a_0, \dots, a_{m-1}$ in $\mathbb{K}$
        return $\delta^m - \sum_{k=0}^{m-1} a_k \delta$

several ways to cross-check the result independently. The variant is described in Section 7.2.

7.1. Implementation of $[\,]_r$ with Gröbner bases. — Let M be the module $\Omega^{n+1} + \Omega^n$, that is the free module generated by ω and the ξ_i. A convenient way to implement the reduction $[\,]_r$ is to compute a reduced Gröbner basis,[44] say G, of the submodule P of M generated by the $\partial_i f \omega - \xi_i$, that is $\mathrm{d}f \wedge \xi_i - \xi_i$. We choose on M a monomial ordering, denoted $\succ$, such that for all multi-indices I and J, and all integer j

$$|I| + 1 \geqslant |J| + N \Longrightarrow x^I \omega \succ x^J \xi_j. \tag{11}$$

For example, any position-over-term (POT) ordering with $\omega \succ \xi_0 \succ \xi_1 \succ \cdots$ is fine. But a term-over-position (TOP), with $\omega \succ \xi_0 \succ \xi_1 \succ \cdots$, extending a graded ordering on A works as well. This gives some flexibility in the implementation. Let rem_G denote the remainder on division by G. The condition (11) on the order is enough to ensure that $\succ$ behaves like an order eliminating ω.

The reason is the following. If we give to ω the degree 1 and to each ξ_i the degree N, then P is a homogeneous submodule of M. Thus any reduced Gröbner basis G of P, whatever the monomial order, contains only homogeneous elements and the remainder on division by G of a homogeneous element of degree d is homogeneous of degree d. In particular we have the

Lemma 34. — *Let α be an element of Ω^{n+1}. Then the coefficient of ω in $\operatorname{rem}_G \alpha$ is zero if and only if α is in the ideal $\mathrm{d}f \wedge \Omega^n$. In this case $\alpha = \mathrm{d}f \wedge \operatorname{rem}_G \alpha$.*

[44] See Cox, Little, and O'Shea, *Using algebraic geometry*, chap. 5, for details about Gröbner bases for modules, the division algorithm, etc.

Proof. — By definition of G there exist polynomials c_i such that

$$\alpha = \operatorname{rem}_G(a\omega) + \sum_{i=0}^{n} c_i(\mathrm{d}f \wedge \xi_i - \xi_i).$$

If the coefficient of ω in $\operatorname{rem}_G(\alpha)$ is zero then $\operatorname{rem}_G(\alpha)$ is in Ω^n. Identifying the components gives

$$\alpha = \mathrm{d}f \wedge \sum_{i=0}^{n} c_i\xi_i = \left(\sum_{i=0}^{n} c_i\partial_i f\right)\omega \quad \text{and} \quad \operatorname{rem}_G(\alpha) = \sum_i c_i\xi_i.$$

Conversely, assume that α is in $\mathrm{d}f \wedge \beta$, for some β in Ω^n. We may assume that α is homogeneous of degree d and that β is homogeneous of degree $d-N$. In particular $\alpha-\beta$ is in P and $\operatorname{rem}_G(a\omega - \beta) = 0$, since G is a Gröbner basis of P. By linearity $\operatorname{rem}_G(a\omega)$ equals $\operatorname{rem}_G(\beta)$.

For the grading introduced above, the element β is homogeneous of degree $d-n$, thus so is $\operatorname{rem}_G(\beta)$. Furthermore, the leading monomial of $\operatorname{rem}_G(\beta)$, with respect to $\succ$, is at most the leading monomial of β, which has the form $x^I\xi_i$ with $|I| = d - N - n$. The claim follows since no monomial of the form $x^J\omega$ has degree $d-n$ (with the alternative grading) and is less than $x^I\xi_i$, thanks to hypothesis (11). $\square$

In the same way we prove that

Lemma 35. — *The intersection $G \cap \Omega^n$ is a Gröbner basis of* Syz.

Together with a Gröbner basis of Syz$'$, this Gröbner basis can be used to compute a basis of $\mathcal{S}_q/\mathcal{S}'_q$ in the following way. Using the Gröbner bases, we compute the set

$$S \stackrel{\text{def}}{=} \{\operatorname{lm}(\alpha) \mid \alpha \in \mathcal{S}_q\} \setminus \left\{\operatorname{lm}(\alpha) \;\middle|\; \alpha \in \mathcal{S}'_q\right\}.$$

Then, for each element α of S we pick an element of $\mathcal{S}_q$ whose leading monomial is α. Those elements form a basis of $\mathcal{S}_q/\mathcal{S}'_q$.

Gröbner bases in the module M can be *emulated* by Gröbner bases in the polynomial ring A with two extra variables, say u and v. Let A' be $A[u,v]$, let ω' be u^{n+1} and ξ'_i be $u^{n-i}v^{i+1}$. Let M' be the A-submodule of $A[u,v]$ generated by ω' and ξ'_i. Let P' be ideal of A generated by $\partial_i f\omega' - \xi'_i$ and all the monomials u^pv^q, with $p+q = n+2$. Let φ be the A-linear map from M' to M sending ω' to ω and ξ'_i to ξ_i. Finally, let G' be a Gröbner basis with respect to any graded monomial ordering $\succ'$, say the graded reverse lexicographic ordering, with $u \succ v \succ x_0 \succ \cdots \succ x_n$.

If $\succ$, the monomial ordering for M, is the TOP ordering proposed above, then the following holds:

$$\varphi(\operatorname{rem}_{G'}\alpha) = \operatorname{rem}_G \varphi(\alpha),$$

and the proof is left to the reader.

The computation of U_q^r and $[\,]_r$ is detailed in Algorithm 3. The function Echelon takes as input a finite subset S of T^{n+1} and outputs a basis in echelon form of Vect(S), with respect to the monomial order $\succ$: that is, a basis B of Vect(S) such that for all

Algorithm 3. Computation of $[\,]_r$

Input — α an element of T^{n+1} and q an integer
Output — $\mathrm{red}_q^{\mathrm{GD}}(\alpha)$ as defined in §3
procedure RedStep(α, q)
  $\alpha' \leftarrow \alpha - \alpha_q$
  $\rho + \beta \leftarrow \mathrm{rem}(G, \alpha_q)$, with $\rho \in \Omega^{n+1}$ and $\beta \in \Omega^n$
  return $\alpha' + \rho + \mathrm{d}\beta$

Input — $r \geqslant 1$ and $q \geqslant 0$ integers
Output — a basis of U_q^r, as defined in §4.3
procedure BasisU(r, q)
  return $\{\alpha \in \mathrm{BasisW}(r, q) \mid \deg \alpha = qN\}$

Input — $r \geqslant 1$ and $q \geqslant 0$ integers
Output — a basis of $\ddot{W}_q^r$, as defined in §4.3
procedure BasisW(r, q)
  if $r = 1$ **then**
    return $\left\{\mathrm{d}\beta \mid \beta \in (\text{a basis of } \mathcal{S}_{q-1}/\mathcal{S}'_{q-1})\right\}$
  else
    $U \leftarrow \mathrm{BasisU}(r-1, q)$
    $W \leftarrow \mathrm{BasisW}(r-1, q+1)$
    return $\mathrm{Echelon}(U \cup \{\mathrm{RedStep}(\alpha, q) \in W \mid \deg \alpha = qN\})$

Input — α an element of T^{n+1}, r a positive integer
Output — $[\alpha]_r$ as defined in §4.4
procedure Reduction(α, r)
  $q \leftarrow \deg \alpha / N$ and $\alpha' \leftarrow \alpha - \alpha_q$
  $\rho \leftarrow \mathrm{rem}(\mathrm{BasisU}(r, q), \mathrm{RedStep}(\alpha_q, q))$
  return $\rho_q + \mathrm{Reduction}(\alpha' + \rho_{q-1}, r)$

element b of B, the leading monomial of b does not appear with a non-zero coefficient in the other elements of B.

7.2. Evaluation and interpolation scheme. — Let $h(t) = p/q$ be an element of $\mathbb{Q}(t)$ such that q is a unitary polynomial. Let d be the maximum of $\deg p$ and $\deg q$, and M be the maximum of the absolute values of numerators and denominators of the coefficients of p and q. Given distinct primes $p_1, \dots, p_n$, distinct rational numbers $u_1, \dots, u_m$ and the evaluations

$$a_{i,j} \equiv h(u_j) \mod p_i,$$

the fraction h can be reconstructed given that no p_i divides the denominator of some coefficient of q, no u_j annihilates q, $\prod_{i=1}^m p_i > 2M$ and $m > 2d$. To do so, we first compute a_i in $\mathbb{F}_{p_i}(t)$ such that $a_i \equiv h \pmod{p_i}$, using Cauchy interpolation.[45] Then, by the Chinese remainder theorem, we compute A such that $A \equiv h \pmod{\prod_i p_i}$. And then, using rational reconstruction[46] to each coefficient of A, we recover h. Without *a priori* bounds on h, it is still possible to try to reconstruct it with the method above. Assume that we obtain a result h', and let M' and d' be the analogues of M and d for h'. Under randomness assumptions, the bigger $\prod_{i=1}^m p_i - 2M'$ and $m - 2d'$ are, the higher is the probability that $h' = h$.

Any algorithm which inputs and outputs elements of $\mathbb{Q}(t)$ and which performs only ring operations—addition, multiplication, negation, constant one, zero test, inversion—in $\mathbb{Q}(t)$ can be turned into a randomized evaluation-interpolation algorithm, simply by evaluating the input at $t = u$ and reducing it in $\mathbb{F}_p$, for several p and u, and proceeding to the computation over $\mathbb{F}_p$. Indeed, the execution of the algorithm requires a finite number of operations, either field operations, which commute with ν, or zero test. For generic values of p and u, these tests yield the same result on evaluated or unevaluated data. For specific values of p and u, the computation over $\mathbb{F}_p$ may fail or return a result which is not the evaluation of the result of the computation over $\mathbb{Q}(t)$. It is important to be able to test that in order to exclude bad evaluations.

The number of evaluation points (p, u) is chosen, *a priori* or on-the-fly, so that the reconstruction of the outputs is possible with high probability of success. If *a priori* bounds on the output are known it may be possible to certify the result. If no bounds are known, then the evaluation-interpolation algorithm may return a false result, but the probability of this event can be made arbitrarily small. This evaluation-interpolation approach is classical in computer algebra for avoiding the problem of coefficient swell.

Algorithm 2 depends on the derivation δ, which is not a field operation, so the conversion to an evaluation-interpolation algorithm is not completely straightforward.

7.2.1. Principle. — Let u be in $\mathbb{Q}$ and p be a prime number. Let ν be the partial function $\mathbb{Q}(t) \to \mathbb{F}_p$, which consists in evaluating t in u and reducing modulo p. The function ν is extended coefficient-wise to $\mathbb{Q}(t)[\mathbf{x}]$, Ω, matrices, etc.

Let f be a polynomial in $\mathbb{Z}[t][\mathbf{x}]$, and $\nu(f)$ be its evaluation in $\mathbb{F}_p[\mathbf{x}]$. We can consider the reductions $[\,]_r$ associated to f, but also the *evaluated* reduction, denoted $[\,]_r^\nu$, associated to $\nu(f)$, over $\mathbb{F}_p$. For given $\alpha \in T^{n+1}$, and for generic values of p and u, the evaluations $\nu(\alpha)$ and $\nu([\alpha]_r)$ are defined and

$$\nu([\alpha]_r) = [\nu(\alpha)]_r^\nu.$$

However, the value of $\nu(\delta(a))$ for some form a cannot be deduced from $\nu(a)$, so that Algorithm 2 requires an adaptation to fit into an evaluation-interpolation scheme.

[45] Gathen and Gerhard, *Modern computer algebra*, §5.8.
[46] Ibid., §5.10.

As in Section 6, let $R = a/f^q$ be a rational function in $\mathbb{Q}(t)$, homogeneous of degree $-n-1$ with respect to the variables $\mathbf{x}$. Let α be $a\omega$. Once the value of r is fixed, Algorithm 2 computes the terms of the sequence $(\rho_i)_{i\in\mathbb{N}}$, defined by $\rho_0 = [\alpha]_r$ and $\rho_{i+1} = [\tilde{\delta}(\rho_i)]_r$, until it finds a linear dependency relation between the ρ_i. For a prime p and an evaluation point u, can we compute $\nu(\rho_i)$ using only operations in $\mathbb{F}_p$? The answer seems to be negative, but there are two ways to circumvent this issue.

The first one is to define ρ_i to be $[\tilde{\delta}^i(\alpha)]_r$. With this definition, the principle and the halting condition $\deg \rho_i \leqslant nN$ of Algorithm 2 remain valid. And given $\nu(\tilde{\delta}^i(\alpha))$, which is certainly easy to compute, it is possible in this case to compute $\nu(\rho_i)$ using only operations in $\mathbb{F}_p$. This approach is feasible but it becomes terrible if i reaches high values: indeed, the degree of $\tilde{\delta}^i(\alpha)$ is $\deg \alpha + iN$.

Another approach is to compute the matrix of the linear map, say m, such that

$$\rho_{i+1} = \delta(\rho_i) + m(\rho_i),$$

where $\delta(\rho_i)$ denotes the coefficient-wise differentiation of ρ_i, as opposed to $\tilde{\delta}(\rho_i)$ which is $\delta(\rho_i) - \delta(f)\rho_i$. Such a linear map exists and its matrix in a certain basis can be computed by evaluation-interpolation.

7.2.2. The matrix of δ. — Let J_r be the image $[T^{n+1}]_r$ of the reduction map $[\,]_r$. By construction, the reduction $[\,]_r$ is idempotent, that is $[\alpha]_r = \alpha$ for all $\alpha \in J_r$. The evaluation-interpolation algorithm relies on the following property of the reduction map $[\,]_r$:

Proposition 36. — *The space J_r is stable under component-wise differentiation.*

Sketch of the proof. — This is a consequence of the fact that J_r is generated by monomials. More precisely, let B be the, finite of infinite, minimal sequence $(b_0, \dots)$ of monomials of T^{n+1} which generates $T^{n+1}/\ker[\,]_r$. Minimal with respect to the lexicographic order on sequences of monomials, where the monomials are compared with $\prec$. Then B is a basis of J_r containing only monomials. ☐

As a consequence $[\tilde{\delta}(\rho)]_r = \delta(\rho) - [\delta(f)\rho]_r$, for all $\rho \in J_r$.

Let $\mathcal{M}$ be the least set of monomials of T^{n+1} such that $\operatorname{Vect}\mathcal{M}$ contains ρ_0 and is stable under the map $m : \rho \mapsto [\delta(f)\rho]_r$. And let A be the matrix in $\mathbb{Q}(t)^{\mathcal{M}\times\mathcal{M}}$ of the map $m_{|\operatorname{Vect}\mathcal{M}}$ in the basis $\mathcal{M}$. For generic values of p and u, the basis $\mathcal{M}$, the matrix $\nu(A)$ and ρ_0 are all computable using only operations in $\mathbb{F}_p$, once given $\nu(f)$, $\nu(\delta(f))$ and $\nu(\alpha)$. Once $\mathcal{M}$, A and ρ_0 are reconstructed over $\mathbb{Q}(t)$, the ρ_i are easily computed with $\rho_{i+1} = \delta(\rho_i) - m(\rho_i)$, and the minimal operator $\mathcal{L} = \sum_i a_i(t)\delta^i$ such that $\sum_i a_i(t)\rho_i = 0$ can be deduced. It seems to be a good idea to reconstruct A and ρ_0 over $\mathbb{F}_p(t)$ and compute $\mathcal{L}$ modulo p, and only then to use several moduli to reconstruct $\mathcal{L}$ over $\mathbb{Q}(t)$. The full procedure is summarized by Algorithm 4.

Algorithm 4. Computation of annihilating operators of the periods of a rational function, randomized evaluation-interpolation method

Input — $R = a/f^q$ a rational function in $\mathbb{Q}(t)(\mathbf{x})$, homogeneous of degree $-n-1$ w.r.t. $\mathbf{x}$
Output — $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ an annihilating operator of $\oint R$, with high probability
procedure PICARDFUCHS(a/f^q)
  loop
    $p \leftarrow$ random prime number
    Compute $\mathcal{M}$, ρ_0 and $\operatorname{Mat}_{\mathcal{M}} m$, as defined in §7.2.2, over $\mathbb{F}_p(t)$ by repeated evaluation of t and rational interpolation.
    Compute $\rho_0, \rho_1, \ldots$ over $\mathbb{F}_p(t)$, with $\rho_{i+1} = \delta(\rho_i) - m(\rho_i)$, until finding a relation $\rho_n + \sum_{i=0}^{n-1} a_i\rho_i = 0$ over $\mathbb{F}_p(t)$.
    Using the Chinese remainder theorem and computations modulo previous values of p, try to lift the a_i in $\mathbb{Q}(t)$.
    if possible **then**
      return the lifting.

7.2.3. Estimation of the probability of success. — Let $\mathcal{M}$, ρ_0 and $A = \operatorname{Mat}_{\mathcal{M}} m$ as in Section 7.2.2, computed over $\mathbb{Q}(t)$. For some u in $\mathbb{Q}$ and some prime p, let $\mathcal{M}'$, ρ_0' and A' be the analogues computed over $\mathbb{F}_p$. It is not hard to check that $\nu(\ker[\,]_r)$ equals $\ker[\,]_r^\nu$, where $\nu(\ker[\,]_r)$ is the set of all α in $\ker[\,]_r$ such that $\nu(\alpha)$ is defined. Let α be an element of T^{n+1}, whose coefficients are polynomials in t with integer coefficients. Do we have $\nu([\alpha]_r) = [\nu(\alpha)]_r^\nu$? The fact that J_r is generated by monomials implies that $[\alpha]_r$ equals $\operatorname{rem}(\ker[\,]_r, \alpha)$, and that $[\nu(\alpha)]_r^\nu$ equals $\operatorname{rem}(\ker[\,]_r^\nu, \alpha)$. The equality is equivalent to

$$\nu(\operatorname{rem}_{\ker[\,]_r} \alpha) = \operatorname{rem}_{\nu(\ker[\,]_r)} \alpha.$$

A sufficient condition is that the set L of leading monomials of elements of $\ker[\,]_r$ equals the set L' of leading monomials of $\nu(\ker[\,]_r)$. Since $\mathcal{M}$ (resp. $\mathcal{M}'$) is the complement of L (resp. L') in the set of all monomials of T^{n+1}, we obtain

Lemma 37. — *If $\mathcal{M} = \mathcal{M}'$ then $A' = \nu(A)$ and $\rho_0' = \nu(\rho_0)$.*

Let P be the probability that $\mathcal{M}' = \mathcal{M}$. Assume for simplicity that $\deg\alpha \leqslant nN$ and that J_r is included in F_nT^{n+1}. Let V be the subspace $\ker[\,]_r \cap F_{n+1}T^{n+1}$ and let $\mathcal{B}$ be an echelonized basis of V, formed by elements of T^{n+1} whose coefficients are in $\mathbb{Z}[t]$. For the above equalities to hold, it is enough that for all b in $\mathcal{B}$, the evaluation $\nu(\operatorname{lc} b)$ of the leading coefficient of b is not zero.

Under the assumption, somewhat excessive, that for random p and u the $\nu(\operatorname{lc} b)$, with $b \in \mathcal{B}$ are independent and uniformly distributed in $\mathbb{F}_p$, the probability P

equals $(1-\frac{1}{p})^{\#\mathcal{B}}$. Of course $\#\mathcal{B} \leqslant \dim F_{n+1}T^{n+1}$ and

$$\dim F_{n+1}T^{n+1} = \sum_{q=0}^{n+1} \binom{qN-1}{n} \leqslant \frac{(n+3/2)^{n+1}N^n}{(n+1)!}.$$

So that

$$\log P \leqslant \frac{(n+3/2)^{n+1}N^n}{(n+1)!p} \leqslant -\frac{e^{n+3/2}N^n}{p}. \tag{12}$$

The set $\mathcal{M}$ is not computed, so it is not possible to compare it with $\mathcal{M}'$. However, we can compare the different $\mathcal{M}'$ obtained for different values of p and u. Typically, most of them will be mutually equal—and hopefully equal to $\mathcal{M}$—and a few will differ. We simply drop the pairs (p, u) giving odd $\mathcal{M}'$.

8. Application to periods arising from mirror symmetry

Batyrev and Kreuzer[47] have recently constructed a family of 210 smooth Calabi–Yau varieties of dimension three with Hodge number $h^{1,1}$ equal to one. Their method is based on toric varieties of reflexive polytopes. To each variety is associated a one-parameter mirror family of varieties and we look for the Picard-Fuchs equation of a distinguished principal period. This computation is the first step toward the computation of other important invariants, like, mirror maps, *instanton* numbers, etc.[48] The 210 varieties gather together into 68 different classes of diffeomorphic manifolds.[49] The principal periods associated to diffeomorphic varieties need not coincide but they are typically expected to differ only by a rational change of variable.

In concrete terms, we look for a differential equation satisfied by periods of rational integrals in the form

$$F(t) \stackrel{\text{def}}{=} \oint_\gamma \frac{1}{1-tg(x_1,\dots,x_4)} \frac{\mathrm{d}x_1}{x_1}\frac{\mathrm{d}x_2}{x_2}\frac{\mathrm{d}x_3}{x_3}\frac{\mathrm{d}x_4}{x_4}, \tag{13}$$

where g is a Laurent polynomial and the integral is taken over the cycle γ defined by $|x_i| = \varepsilon$, with ε a small positive real number. Here g is $\sum_v x^v$, where the sum ranges over the vertices of a reflexive lattice polytope. For the 210 polytopes under consideration, Batyrev and Kreuzer claim that $F(t)$ satisfies a linear differential equation of order 4, as a consequence of $h^{1,1}$ being 1. Moreover, this differential equation should have maximally unipotent monodromy at $t = 0$.

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47 Batyrev and Kreuzer, “Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions”.

48 For an introduction to the topic, see Batyrev and Straten, “Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties”; Cox and Katz, *Mirror symmetry and algebraic geometry*.

49 Batyrev and Kreuzer, “Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions”, table 3.

A power series expansion of the integrand with respect to t shows that

$$F(t) = \sum_n \mathrm{ct}(g^n) t^n, \tag{14}$$

where $\mathrm{ct}(g^n)$ stands for the constant term of f^n. Batyrev and Kreuzer have computed Picard-Fuchs operators for topologies #37, #40 and #43–68 of their list. They used the *guessing* method presented in the introduction: they computed the power series expansion of $F(t)$, using equation (14), until they reached a degree d such that they could find a non-zero solution to the equation

$$\left(\sum_{i=0}^{4} \sum_{j=0}^{d} a_{i,j} t^j \theta^i \right) \cdot F(t) = \mathcal{O}(t^{5(d+1)+1}).$$

The issue with this technique is not the reconstruction step which can be done efficiently—with respect to the size of the computed operator—but the computation of the power series expansion: the number of monomials in g^k is $\Theta(k^4)$, so the computation of N terms of $F(t)$ with this technique take $\Theta(N^5)$ operations in $\mathbb{Z}$, and we may add an order of magnitude to reflect the binary complexity.

Metelitsyn[50] computed four more equations for topologies #24, #38, #39 and #41. His method is also guessing, with modular evaluation techniques, but he managed to improve the space complexity, not the time complexity though, in the power expansion step and he provided an implementation optimized with GPU programming. Moreover, Almkvist[51] reports that Straten, Metelitsyn and Schömer have computed one operator for the topology #17. To the best of my knowledge, no other computation succeeded in the remaining topologies (#1–16, #18-23 , #25–36, #42).

With the implementation described in Section 7, I have been able to compute a differential equation for the 136 remaining integrals, associated to 35 different topologies.[52]

8.1. Minimal equation and crosschecking. — The equations obtained from the algorithm are not always minimal, for two reasons. Firstly they were obtained with $r = 2$ but a higher value might have caught a lower order equation. Secondly, the algorithm computes an annihilating operator of all the periods of a given rational function; a period associated to a given cycle may satisfy a lower order equation.

Nevertheless, once any differential equation $\mathcal{L}$ for $F(t)$ is obtained, it is easy to compute efficiently thousands of terms of its power series expansion: the relation $\mathcal{L}(F) = 0$ translates into a linear recurrence relation on the coefficients of the power series expansion and the initial conditions are given by Equation (14). Thus we may try to reconstruct the minimal equation $\mathcal{L}_0$. By contrast to the guessing method, the

[50] Metelitsyn, "How to compute the constant term of a power of a Laurent polynomial efficiently".
[51] Almkvist, "The art of finding Calabi-Yau differential equations".
[52] The results are available at `http://pierre.lairez.fr/supp/periods`.

reconstructed equation $\mathcal{L}_0$ can be proven correct: it is enough to check that it is a right divisor of $\mathcal{L}$, and that it annihilates the first few terms[53] of $F(t)$. If the power series expansion does not reveal a lower order differential equation, we may conjecture that $\mathcal{L}$ is minimal. Proving it may be done using methods by Hoeij,[54] see §8.2.2 for an example.

Since Algorithm 4 is randomized, it is desirable to have criteria to crosscheck the result. The Picard-Fuchs equations of periods of rational integrals are known to have strong arithmetic properties: regular singularities with rational exponents and nilpotent p-curvature for all prime p, with a finite number of exceptions.[55] Checking these properties is a good confirmation of the correctness of the output: these properties are so strong that a bad reconstruction would most probably break them. In addition, the computation of many terms of the power series expansion of $F(t)$ using a annihilating operator $\mathcal{L}$ can also be used as a crosschecking: if the coefficients computed are all integers, as expected in view of Equation (14), this is also strong indication that the operator is indeed correct.

8.2. Description of the results. — In depth treatment is a work in progress with Jean-Marie Maillard. This section presents two examples.[56]

8.2.1. Topology #42, polytope v25.59. — For the period (13) with

$$g = wxyz + wxy + \frac{1}{wxy} + wxz + \frac{1}{wxz} + \frac{wy}{z} + \frac{z}{wy} + wy + \frac{1}{wy} + \frac{1}{wz} + w + \frac{1}{w} + \frac{xz}{y} + \frac{y}{xz} + \frac{1}{xy} + xz + \frac{1}{xz} + x + \frac{1}{x} + \frac{z}{y} + \frac{y}{z} + y + \frac{1}{y} + z + \frac{1}{z},$$

where the first few terms of the power series expansion are

$$F(t) = 1 + 22t^2 + 204t^3 + 3474t^4 + 57000t^5 + 1031080t^6 + 19368720t^7 + \mathcal{O}(t^8).$$

I have computed the following Picard-Fuchs equation

$$\begin{aligned}
&t^3(7t+1)^2(25t-1)^2(2t+1)^3(101t+43)^3(3t+1)^3\partial^4\\
&\quad + 2t^2(7t+1)(25t-1)(2t+1)^2(101t+43)^2(3t+1)^2\\
&\quad (848400t^5 + 1012956t^4 + 413041t^3 + 62473t^2 + 1819t - 129)\partial^3\\
&\quad + t(7t+1)(25t-1)(2t+1)(101t+43)(3t+1)(4627173600t^8 + 10573386192t^7\\
&+10004988192t^6+5027593832t^5+1423146511t^4+219009622t^3+15394840t^2+182234t-12943)\partial^2
\end{aligned}$$

[53] Up to the maximal integral root of the indicial polynomial of $\mathcal{L}_0$ at zero.
[54] Hoeij, "Factorization of differential operators with rational functions coefficients".
[55] Katz, "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin".
[56] There are two numberings. The first one, used in Table 3 of Batyrev and Kreuzer, "Constructing new Calabi-Yau 3-folds and their mirrors via conifold transitions", numbers the 68 different topologies, ordered by increasing $h^{1,2}$ number, covering the 210 smooth Calabi-Yau threefolds with Picard number 1. The second one, used in the database `http://hep.itp.tuwien.ac.at/~kreuzer/math/0802`, numbers in the form v$x.y$ the 198849 reflexive 4D polytopes satisfying an extra property. The letter x indicates the number of vertices.

$$+ (7t+1)(25t-1)(2t+1)(101t+43)(3t+1)(6169564800t^8 + 13061530080t^7 + 11311205016t^6 + 5112706620t^5 + 1268815538t^4 + 164341135t^3 + 9051543t^2 + 74605t - 1849)\partial + 8t(7t+1)(25t-1)(2t+1)(101t+43)(3t+1)(192798900t^6 + 375787872t^5 + 294032949t^4 + 116697469t^3 + 24254991t^2 + 2406495t + 81356),$$

or, with $\theta = t\partial$, in a form which highlights the maximally unipotent monodromy,

$$\begin{aligned} &1849\theta^4 - 43t\theta(142\theta^3 + 890\theta^2 + 574\theta + 129)\\ &\quad - t^2(647269\theta^4 + 2441818\theta^3 + 3538503\theta^2 + 2423953\theta + 650848)\\ &\quad - t^3(7200000\theta^4 + 34423908\theta^3 + 65337898\theta^2 + 57379329\theta + 19251960)\\ &\quad - t^4(37610765\theta^4 + 220029964\theta^3 + 499781264\theta^2 + 511393545\theta + 194039928)\\ &\quad - 2t^5(\theta+1)(549781121\theta^3 + 3247373370\theta^2 + 6650662226\theta + 466789876)\\ &\quad - t^6(\theta+2)(\theta+1)(1851815547\theta^2 + 9159314425\theta + 1176131796)\\ &\quad - 1212t^7(1389790\theta + 413408)(\theta+3)(\theta+2)(\theta+1)\\ &\quad - 64266300t^8(\theta+4)(\theta+3)(\theta+2)(\theta+1). \end{aligned}$$

This equation satisfies the conditions given by Almkvist, Enckevort, Straten, and Zudilin[57] and it is not in their database.[58] The computation took 80 seconds and 30 megabytes of memory on a laptop.

Note that formula (13), and homogeneization, give a rational function a/f with f of degree 8 with respect to the integration variables. The change of variables which maps x to $1/x$ and w to w/y lowers this degree down to 5. This improves dramatically the computation time. This kind of monomial substitution can be found by random trials and errors. Among the substitutions that lead to degree 5, some are better than others in terms of computation time; but this seems hard to predict.

8.2.2. Topology #27, polytope v23.289. — For the period (13) with

$$f = \frac{1}{w} + w + \frac{1}{x} + \frac{w}{x} + x + \frac{x}{w} + \frac{1}{y} + \frac{w}{y} + \frac{1}{xy} + \frac{w}{xy} + y + \frac{y}{w} + \frac{xy}{w} + \frac{1}{z} + \frac{w}{z} + \frac{x}{z} + \frac{1}{yz} + \frac{w}{yz} + \frac{w}{xyz} + z + \frac{z}{w} + \frac{z}{x} + \frac{z}{wx},$$

where the first few terms of the power series expansion are

$$F(t) = 1 + 18t^2 + 138t^3 + 2070t^4 + 29040t^5 + 452610t^6 + 7308000t^7 + \mathcal{O}(t^8),$$

I have computed an annihilating operator of order 6 and degree 29, let us denote it $\mathcal{L}_6$, which is too large to be reproduced here. The operator is not of order 4 and has not maximally unipotent monodromy. Is it the minimal equation of $F(t)$? Van Hoeij has

[57] Almkvist, Enckevort, Straten, and Zudilin, *Tables of Calabi–Yau equations*.
[58] Straten, *Calabi-Yau Operators Database*.

proved[59] that if $\mathcal{L}_6$ admits a right factor of order 4 then the degree of the coefficients of this factor is at most 88. Thus, admitting that $\mathcal{L}_6$ is indeed an annihilating operator of $F(t)$, if the minimal annihilating operator of $F(t)$ has order 4, it would have degree at most 88. Zero being the only solution to the system of linear equations

$$\sum_{i=0}^{4}\sum_{j=0}^{88} a_{i,j} t^j f^{(i)}(t) = \mathcal{O}(t^{405}),$$

where the unknowns are the $a_{i,j}$, this shows that the minimal annihilating operator of $F(t)$ is not of order 4. The argument holds for orders 1, 2, 3 and 5 with respective degree bounds 10, 16, 45 and 125. This is rather surprising since it contradicts the claims of Batyrev and Kreuzer. The topology #17, polytope v18.16766, shows the same behavior with a minimal equation of order 6. This has been first reported by Almkvist,[60] referring to a computation by Straten, Metelitsyn and Schömer. As Almkvist wrote about topology #17, "this example leaves some doubts about the reflexive polytopes." I can only corroborate. The remaining operators have not been studied in depth yet, but it seems that only one of the 137 newly computed periods has a minimal equation of order 4.

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[59] Using methods introduced in (Hoeij, "Factorization of differential operators with rational functions coefficients")

[60] Almkvist, "The art of finding Calabi-Yau differential equations".

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