# Computing periods of rational integrals

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# COMPUTING PERIODS OF RATIONAL INTEGRALS

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Abstract. A period of a rational integral is the result of integrating, with respect to one or several variables, a rational function over a closed path. This work focuses particularly on periods depending on a parameter: in this case the period under consideration satisfies a linear differential equation, the Picard-Fuchs equation. I give a reduction algorithm that extends the Griffiths-Dwork reduction and apply it to the computation of Picard-Fuchs equations. The resulting algorithm is elementary and has been successfully applied to problems that were previously out of reach.

## Introduction

This work studies periods of rational integrals, that is, the result of the integration, with respect to one or several variables, of a rational function over a closed path. I focus especially on the case where the period depends on a parameter. The fact that periods depending on a parameter of rational or algebraic integrals satisfy linear differential equations with polynomial coefficients has emerged from the work of Euler [24, §7] and his computation of a differential equation[1] for the perimeter of an ellipse as a function of eccentricity. Since then, these differential equations, known as *Picard-Fuchs equations*, have proven to be useful in numerous domains such as combinatorics [11], number theory [6] or physics [39]. They play also a key role in mirror symmetry [38]. Research in computer algebra has devoted great efforts to provide algorithms for computing integrals and, in particular, Picard-Fuchs equations. Nevertheless the practical efficiency of current methods is not satisfactory in many cases. One reason might be the high level of generality of most algorithms, which apply to the integration of general holonomic functions. Rational functions are certainly very specific among holonomic functions, but the numerous applications of Picard-Fuchs equations as well as the fundamental nature of rational functions make it worth developing specific methods for them.

**The problem.** Let $R$ be a rational function in the variables $x_1, \dots, x_n$, denoted $\mathbf{x}$, and a parameter $t$, with coefficients in $\mathbb{C}$. Let $\gamma$ be a $n$-cycle in $\mathbb{C}^n$, e.g. an embedding of the sphere $\mathbb{S}^n$ in $\mathbb{C}^n$, on which $R$ is continuous when $t$ ranges over some connected open set $U$ of $\mathbb{C}$. We can form the following integral, depending on $t \in U$,

$$P(t) \stackrel{\text{def}}{=} \oint_\gamma R(t, \mathbf{x}) \mathrm{d}\mathbf{x}, \tag{1}$$

where $\mathrm{d}\mathbf{x}$ stands for $\mathrm{d}x_1 \cdots \mathrm{d}x_n$.

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[1] $(t - t^3)y'' + (1 - t^2)y' + ty = 0$

*Example* 1. For $t \in \mathbb{C}$, with $|t| < 17 - 12\sqrt{2}$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = \frac{1}{(2\pi i)^3} \oint_\gamma \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{1 - (1-xy)z - txzy(1-x)(1-y)(1-z)},$$

where the cycle of integration $\gamma$ is $\left\{(x,y,z) \in \mathbb{C}^3 \;\middle|\; |x| = |y| = |z| = 1/2\right\}$. This is the generating function of Apéry numbers [6].

These integrals, for different cycles $\gamma$, are called the *periods* of the integral $\oint R$. It is well-known that $P(t)$ satisfies a linear differential equation with polynomial coefficients. It is a consequence of the finiteness of the algebraic de Rham cohomology of $\mathbb{A}^n \setminus V(f)$ with $\mathbb{C}(t)$ as base field [28; 37]. Let $\mathcal{L}_{R,\gamma}$ denote the differential operator in $t$ and $\partial_t$ which corresponds to the minimal-order equation of $P(t)$. That is to say $\mathcal{L}_{R,\gamma}$ is the non zero operator $\sum_{k=0}^{r} a_k(t)\partial_t^k$ with coprime polynomial coefficients and minimal $r$, such that

$$\mathcal{L}_{R,\gamma}(P) \stackrel{\text{def}}{=} \sum_{k=0}^{r} a_k(t) P^{(k)}(t) = 0.$$

Every linear differential equation for $P(t)$ translates into an operator which is a left multiple of $\mathcal{L}_{R,\gamma}$.

It often happens that the description of the cycle $\gamma$ is analytic or topological, sometimes not even explicit, and, to say the least, unsuitable to a formal algorithmic treatment. In fact there is no harm in simply discarding $\gamma$: there exists a differential equation satisfied by all the periods of $\oint R$. In other words, there exists an operator in $t$ and $\partial_t$ which is a left multiple of all $\mathcal{L}_{R,\gamma}$. Let $\mathcal{L}_R$ denote the least common left multiple of the $\mathcal{L}_{R,\gamma}$. The classical result which allows the algorithmic computation of $\mathcal{L}_R$ is that it is the minimal operator $\mathcal{L}$ such that

$$\mathcal{L}(R) = \sum_{i=1}^{n} \partial_i(B_i) \tag{2}$$

for some rational functions $B_i$ in $\mathbb{C}(t, \mathbf{x})$ whose denominators divide a power of the denominator of $R$, and where $\partial_i$ denotes $\partial/\partial x_i$. This article presents an algorithm that compute the operator $\mathcal{L}_R$, or at least a left multiple of it.

*Example* 2. In the case of Example 1, the operators $\mathcal{L}_R$ and $\mathcal{L}_{R,\gamma}$ both equal

$$\mathcal{L}_R = t^2(t^2 - 34t + 1)\partial_t^3 + 3t(2t^2 - 51t + 1)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + (t - 5).$$

Note that integrals of algebraic functions are easily translated into integrals of rational functions with one variable more: if $W(t, \mathbf{x})$ is a function such that $P(t, \mathbf{x}, W) = 0$ for some polynomial $P$ in $\mathbb{C}[t, \mathbf{x}, y]$, elementary residue calculus shows that

$$W(t, \mathbf{x}) = \frac{1}{2\pi i} \oint_\tau \frac{y\partial_y P}{P} \mathrm{d}y$$

over some adequate contour $\tau$ and where $\partial_y$ denotes the derivation $\partial/\partial y$, so that

$$\oint_\gamma W(t, \mathbf{x})\mathrm{d}\mathbf{x} = \frac{1}{2\pi i} \oint_{\gamma \times \tau} \frac{y\partial_y P}{P} \mathrm{d}\mathbf{x}\mathrm{d}y.$$

**Contributions.** Following the principle of the reduction of the pole order, I define a family of finer and finer reductions $[\,]_r$, for $r \geqslant 1$, that given a rational function $R$ in several variables produces another rational function $[R]_r$ that differs from $R$ only by a sum of partial derivatives of other rational functions (Section 4). The first reduction $[\,]_1$ is the Griffiths-Dwork reduction (Section 3).

When applied to the case of periods depending on a parameter, these reductions can solve Equation (2), and hence compute Picard-Fuchs equations of rational integrals (Section 6). A major difficulty is to fix an $r$ such that the $r^{\text{th}}$ reduction map $[\,]_r$ will be fine enough to ensure the termination of the algorithm. It is solved by applying a theorem of Dimca (Section 5).

The new algorithm has been implemented and shows excellent performance (Section 7). For example, I applied it to compute 137 periods coming from mathematical physics that were previously out of reach [4] (Section 8).

**Reduction of pole order.** The principle of the method originates from Hermite reduction [29]. It is a procedure for computing a normal form of a univariate function modulo derivatives. Hermite introduced his method as a way to compute the algebraic part of the primitive of a univariate rational function without computing the roots of its denominator, as opposed to the classical partial fraction decomposition method. Let $[R]$ denote the reduction of a fraction $R$. It is defined as follows. Let $a/f^q$ be a rational function in $\mathbb{C}(x)$, with $f$ a square-free polynomial and $q$ a positive integer. Every fraction can be written in this way since $a$ and $f$ are not assumed to be relatively prime. If $q > 1$, then we first write $a$ as $uf + vf'$, using the assumption that $f$ is square-free, and we observe that

$$\frac{a}{f^q} = \frac{u + \frac{1}{q-1}v'}{f^{q-1}} - \left(\frac{1}{q-1}\frac{v}{f^{q-1}}\right)'.$$

This leads to the following recursive definition of $[a/f^q]$:

$$\left[\frac{a}{f^q}\right] = \left[\frac{u + \frac{1}{q-1}v'}{f^{q-1}}\right].$$

When $q = 1$, the reduction $[a/f]$ is defined to be $r/f$, where $r$ is the remainder in the Euclidean division of $a$ by $f$. Hermite reduction enjoys the following properties: it is linear; the fractions $[R]$ and $R$ differ only by a derivative of a rational function; and $[R]$ is zero if and only if $R$ is the derivative of a rational function.

The principle of Hermite reduction gives an efficient way to compute the Picard-Fuchs equation of univariate integrals [8]. Let $R$ be a rational function in $\mathbb{C}(t, x)$. Hermite reduction can be performed without modification over the field with one parameter $\mathbb{C}(t)$. To compute $\mathcal{L}_R$, it is sufficient to compute the reductions $[\partial_t^k R]$, for $k \geqslant 0$, until finding a linear dependency relation over $\mathbb{C}(t)$

$$\sum_{k=0}^{r} a_k(t)[\partial_t^k R] = 0.$$

Then the properties of the Hermite reduction assure that $\mathcal{L}_R$ is $\sum_{k=0}^r a_k(t)\partial_t^k$. The computations of all the reductions $[\partial_t^k R]$ is improved significantly when noting the inductive formula $\left[\partial_t^{k+1} R\right] = \left[\partial_t[\partial_t^k R]\right]$.

With several variables, the construction of a normal form modulo derivatives is considerably harder than with a single variable. Nonetheless, as soon as we obtain such a normal form, it is possible to compute Picard-Fuchs equations as above, by finding linear relations between the $[\partial_t^k R]$.

Part 1 deals with the construction of the maps $[\,]_r$ whereas Part 2 deals with the computation of Picard-Fuchs equations.

**Related works.** Several existing algorithms are applicable to the computation of $\mathcal{L}_R$. The reader may refer to [14] for an extensive survey of "creative telescoping" approaches. A first family, originating in the work of Fasenmyer [25] and Verbaeten [45], gave rise to an algorithm by Wilf and Zeilberger [46], refined by Apagodu and Zeilberger [3], applicable to proper hyperexponential terms, which includes rational functions. The idea is to transform Equation (2) into a linear system over $\mathbb{C}(t)$ by bounding *a priori* the order of a left multiple of $\mathcal{L}_R$ and the degree of the polynomials appearing in the fractions $B_i$. While being an interesting method, especially because it gives *a priori* bounds, the order of the linear system to be solved is large even for moderate sizes of the input.

Zeilberger's "fast algorithm" [47] for hypergeometric summation is the origin of a different family of algorithms, whose key idea is to reduce the resolution of Equation (2) to the computation of rational solutions of systems of ordinary linear differential equations. Interestingly, Picard used this idea much earlier in a method for computing double rational integrals [41]. Chyzak's algorithm [13] and Koutschan's semi-algorithm [33]—termination is not proven—belong to this line and apply to $D$-finite ideals in Ore algebras. Rational functions are a very specific case.

A last family of algorithms coming from $\mathcal{D}$-module theory has given algorithms for numerous operations on $\mathcal{D}$-modules and, in particular, an algorithm by Oaku and Takayama [40] to compute the de Rham cohomology of the complement of an affine hypersurface, which would allow, in theory, to compute Picard-Fuchs equations. It is worth noting that an algorithm to compute the integration of a holonomic $\mathcal{D}$-module does not give as such an algorithm applicable to our problem: computing the annihilator of a rational function in the Weyl algebra is far from being an easy task [40].

The domain of application of each of these three families is much larger than just rational integrals: any comparison with the present algorithm must be done with this point in mind.

The *guessing* method, or *equation reconstruction*, a totally different method, applies to the computation of $\mathcal{L}_{R,\gamma}$. It often happens that beside the integral formula for $P(t)$ one has a way to compute a power series expansion. After computing sufficiently many terms, it is possible to recover $\mathcal{L}_{R,\gamma}$ *via* Hermite-Padé approximants. It may be difficult to prove that the operator computed is indeed correct, but not too hard to get convinced. The simplicity of this method counterbalances a certain lack of delicacy and justifies its ample use. When the power series expansion of $P(t)$ is, for some reason, easy to compute, it can find Picard-Fuchs equations which are far out of reach of any existing algorithms [*e.g.* 32]. Most of the time, though, the power series expansion of $P(t)$ is expensive to compute. For example, I am aware of no general method allowing to compute directly the first $p$ terms of a diagonal of a rational function in $n$ variables in less than $p^n$ arithmetic operations. However, space complexity can be improved [36].

Picard and Simart have studied the case of simple and double integrals of algebraic functions and gave methods to compute normal forms modulo derivatives extensively [42]. Chen, Kauers, and Singer [12] gave an algorithm in this direction, for double rational integrals. This algorithm is an echo, independently discovered, of one of the methods of Picard [41]. Interestingly, it has two steps: a first one based on a reduction *à la* Hermite and another one based on creative telescoping.

Well later after Picard, Griffiths resumed the search for a normal form in the setting of de Rham cohomology of smooth projective hypersurfaces, defining what is now known as the Griffiths-Dwork reduction [22, §3; 23, §8; 27, §4]. This reduction is in many respects similar to the Hermite reduction. It can be applied to the computation of Picard-Fuchs equations in the same way as Hermite reduction applies to univariate integrals. The smoothness hypothesis can be worked around with a generic deformation. This leads to an interesting complexity result about the computation of Picard-Fuchs equations [9] but to disappointing practical efficiency in singular cases. The direction of Griffiths and Dwork was extended, in particular, by Dimca [19; 18] and Saito [21], and some results are known in the case of a singular hypersurface.

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## Part 1. Reduction of periods

Let $\mathbb{K}$ be a field of characteristic zero, and let $A$ be the polynomial ring $\mathbb{K}[x_0, \dots, x_n]$, for some integer $n$. Let $f$ be an homogeneous element of $A$ and let $A_f$ be the localized ring $A[1/f]$. The degree of $f$ is denoted $N$. We focus here on integrals $\oint R\mathrm{d}\mathbf{x}$ which are homogeneous of degree zero, this means that

$$R(\lambda x_0, \dots, \lambda x_n)\mathrm{d}(\lambda x_0)\cdots \mathrm{d}(\lambda x_n) = R(x_0, \dots, x_n)\mathrm{d}x_0 \cdots \mathrm{d}x_n,$$

or equivalently that $R$ is a homogeneous rational function of degree $-n-1$. Every integral can be homogenized with a new variable, see §6.2.

This part addresses the problem of finding an algorithm *à la* Hermite that computes an idempotent linear map $R \mapsto [R]$, from $A_f$ to itself such that $[R]$ equals zero if and only if $R$ is in the linear subspace $\sum_{i=0}^n \partial_i A_f$. This problem is solved by the Hermite reduction when $n$ is 1 and by the Griffiths-Dwork reduction when $f$ satisfies an additional regularity hypothesis (see Theorems 3 and 10). To this purpose, a family of maps, denoted $[\,]_r$, is constructed such that $[\,]_1$ is the Griffiths-Dwork reduction and such that $[\,]_{r+1}$ factors through $[\,]_r$. I give an efficient algorithm to compute these maps. Conjecturally, $[\,]_{n+1}$ satisfies the desired properties. Fortunately, other results allow to avoid relying on this conjecture when dealing with periods depending on a parameter.

## 1. Overview

**1.1. Griffiths-Dwork reduction.** To achieve a normal form modulo derivatives, the guiding principle is the *reduction of pole order*. Let us first consider the decision problem: given a rational function $a/f^q$, decide whether it lies in $\sum_{i=0}^n \partial_i A_f$. A major actor of the study is $\operatorname{Jac} f$, the Jacobian ideal of $f$. It is the ideal of $A$ generated by the partial derivatives $\partial_0 f, \dots, \partial_n f$. The basic observation is that the differentiation formula

$$\sum_{i=0}^n \partial_i \left( \frac{b_i}{f^{q-1}} \right) = \frac{\sum_{i=0}^n \partial_i b_i}{f^{q-1}} - (q-1) \frac{\sum_{i=0}^n b_i \partial_i f}{f^q} \tag{3}$$

implies, by reading it right-to-left, that if $a \in \operatorname{Jac} f$ and $q > 1$ then $a/f^q$ equals $a'/f^{q-1}$ modulo derivatives, for some polynomial $a'$. Namely, if $a = \sum_i b_i \partial_i f$ then

$$\frac{a}{f^q} \equiv \frac{\frac{1}{q-1}\sum_{i=0}^n \partial_i b_i}{f^{q-1}} \mod \sum_{i=0}^n \partial_i A_f.$$

Griffiths [27] proved the converse property in the case when $\operatorname{Jac} f$ is zero-dimensional or, equivalently, when the projective variety defined by $f$ is smooth. Under this hypothesis, if $q > 1$ and if $a/f^q \equiv a'/f^{q-1}$, modulo derivatives, for some polynomial $a'$, then $a \in \operatorname{Jac} f$. This gives an algorithm to solve the decision problem, by induction on the pole order $q$.

1.2. **Singular cases.** In presence of singularities, Griffiths' theorem always fails. For example, with $f$ equal to $xy^2 - z^3$,

$$\frac{x^3}{f^2} = \partial_x \left( \frac{\frac{2}{7}x^4}{f^2} \right) - \partial_y \left( \frac{\frac{1}{7}x^3 y}{f^2} \right), \tag{4}$$

but $x^3$ is not in $\operatorname{Jac} f$, which is here the ideal $(xy, y^2, z^2)$. This identity is a consequence of the following particular case of Equation (3):

$$\sum_{i=0}^n b_i \partial_i f = 0 \Rightarrow \sum_{i=0}^n \partial_i \left( \frac{b_i}{f^q} \right) = \frac{\sum_{i=0}^n \partial_i b_i}{f^q}. \tag{5}$$

Tuples of polynomials $(b_0, \dots, b_n)$ such that $\sum_{i=0}^n b_i \partial_i f$ are called *syzygies* (of the sequence $\partial_0 f, \dots, \partial_n f$). Therefore, in order to complete the reduction of pole order strategy, we should not only consider elements of the Jacobian ideal, but also elements of the form $\sum_i \partial_i b_i$, where $(b_b, \dots, b_n)$ is a syzygy. Such elements are called *differentials of syzygies.*

Considering differential of syzygies is not always enough. For example, with $f$ equal to $x_0^4 x_1 - x_0^2 x_1 x_2^2 + x_0 x_2^4$:

$$\frac{x_1^7}{f^2} = \frac{1062347}{276480} \frac{89x_0^2 + 96x_0x_1 + 712x_2^2}{f} + \sum_{i=0}^2 \partial_i \left( \frac{b_i}{f^3} \right),$$

for some lengthy polynomials $b_i$, whereas $x_1^7$ is not a sum of a differential of a syzygy and of an element of $\operatorname{Jac} f$. Note the exponent 3 appearing in $\partial_i(b_i/f^3)$, it is the least possible.

1.3. **Higher order relations.** Let $M_q$ be the set of rational functions of the form $a/f^q$. Let $W_q^1$ be the subset of $M_q \times M_{q-1}$ defined by

$$W_q^1 = \left\{ \left( (q-1)\frac{\sum_{i=0}^n b_i \partial_i f}{f^q}, \frac{\sum_{i=0}^n \partial_i b_i}{f^{q-1}} \right) \;\middle|\; b_i \in A \right\}.$$

An element $(R, R')$ of $W_q^1$ relates a rational function $R$ with a pole order at most $q$ with another rational function $R'$, with pole order at most $q-1$, which is equivalent to $R$ modulo derivatives. The following statement is a rewording of Griffiths' result:

**Theorem 3** (Griffiths)**.** *Assume that $V(f)$ is smooth. For all $R$ in $M_q$, homogeneous of degree $-n-1$, the following assertions are equivalent:*

*(1) $R$ is in $\sum_i \partial_i A_f$;*
*(2) there exists $R'$ in $M_{q-1} \cap \sum_i \partial_i A_f$ such that $(R, R')$ is in $W_q^1$.*

The starting point of the method in the general case is to observe that $W_q^1$ contains ordered pairs in the form $(0, R')$. Namely, if $b_0, \dots, b_n$ is a syzygy, then $(0, \sum_i \partial_i b_i / f^{q-1})$ is in $W_q^1$. They seem to be useless relation in view of Theorem 3. However, for all such pairs $(0, R')$, the rational function $R'$ is in $\sum_i \partial_i A_f$, since it is equivalent to 0 modulo derivatives.

But it is possible, as remarked above, that $R'$ is not part of a pair $(R', R'')$ in $W_{q-1}^1$. This motivates the definition of $W_q^2$ as

$$W_q^2 \stackrel{\text{def}}{=} W_q^1 + \left\{ (R, 0) \;\middle|\; (0, R) \in W_{q+1}^1 \right\}.$$

Of course, this can be iterated:

$$W_q^{r+1} \stackrel{\text{def}}{=} W_q^r + \left\{ (R, 0) \;\middle|\; (0, R) \in W_{q+1}^r \right\}.$$

The basic property that is preserved through this induction is that for all $(R, R')$ in $W_q^r$, the first element $R$ has a pole of order at most $q$ and is equivalent, modulo derivatives, to the second element $R'$, which has a pole of order at most $q-1$. When $V(f)$ is smooth, then $W_q^r = W_q^1$, for all $q$, but when $V(f)$ is singular, the spaces $W_q^r$, with $r > 1$, bring new relations. This construction is somehow exhaustive. The first result is the following, with no assumption on $V(f)$:

**Theorem 4.** *There exists an integer $r \geqslant 1$, depending only on $f$, such that for all $q$ and all $R$ in $M_q$, homogeneous of degree $-n-1$, the following assertions are equivalent:*

1. *$R$ is in $\sum_i \partial_i A_f$;*
2. *there exists $R'$ in $M_{q-1} \cap \sum_i \partial_i A_f$ such that $(R, R')$ is in $W_q^r$.*

The algorithm presented in this article is based on this theorem. The definition of the $W_q^r$ gives readily an algorithm to compute these spaces: it is only a matter of linear algebra. The second result is a method to achieve efficiency. The two main ingredients are the use of Gröbner bases, and the computation of a basis of *non-trivial* syzygies to catch most elements of $W_q^r$ at reasonnable cost.

1.4. **Trivial syzygies.** The space $W_q^2$ is made from $W_q^1$ and elements in the form $(\sum_i \partial_i b_i / f^q, 0)$, where $b_0, \dots, b_n$ is a syzygy, that is $\sum_i b_i \partial_i f$ vanishes.

Among syzygies, the *trivial syzygies* do not bring new relations to the relations already in $W_r^1$. A syzygy $b_0, \dots, b_n$ is called *trivial* if there exist polynomials $c_{i,j}$, with $c_{i,j} = -c_{j,i}$, such that

$$b_i = \sum_{j=0}^{n} c_{i,j} \partial_j f.$$

The antisymmetry property implies that this defines a syzygy, and we check that

$$\sum_{i=0}^{n} \partial_i b_i = \sum_{j=0}^{n} \left( \sum_{i=0}^{n} \partial_i c_{ij} \right) \partial_j f + \underbrace{\sum_{i,j=0}^{n} c_{i,j} \partial_i \partial_j f}_{=0},$$

so that $\sum_{i=0}^{n} \partial_i b_i$ is in the Jacobian ideal. Moreover

$$\sum_j \partial_j \left( \sum_i \partial_i c_{ij} \right) = 0.$$

It follows that the ordered pair $(\sum_i \partial_i b_i/f^q, 0)$ is already in $W_q^1$. Thus, in order to compute $W_q^2$, one may discard trivial syzygies. Quantitatively, the trivial syzygies are numerous among the syzygies—see, for example, Table 2—so that discarding them is a tremendous improvement. A basis of *non-trivial* syzygies can be computed efficiently by means of Gröbner bases.

1.5. **Reduction procedure.** Let $R = a/f^q$ be a fraction in $M_q$. The reduced form $[R]_r$ is defined by induction on $q$ in the following way. We decompose $R$ as $R' + S$ where $R'$ is minimal in some sense and where $S$ is the first element of a pair $(S, T)$ in $W_q^r$. Then $[R]_r$ is defined to be $R' + [T]_r$. By construction $[R]_r \equiv R$ modulo derivatives. The constraint on the homogeneity degree of $R$ will ensure that $T$ is zero at some point of the induction.

## 2. Exponential isomorphism

The *exponential isomorphism*, Theorem 6, allows to manipulate polynomials rather than rational functions. It is folklore, for an account see [18]. We work in a homogeneous setting and we deal only with homogeneous fractions $R$ of degree $-n-1$. (So that $R\mathrm{d}x_0 \cdots \mathrm{d}x_n$ is homogeneous of degree 0.) A fraction $a/f^q$ is therefore represented solely by its numerator $a$: if $a/f^q$ is homogeneous of degree $-n-1$, the numerator $a$ is homogeneous of degree $q \deg f - n - 1$, so that $q$ may be recovered from $a$. To the usual partial derivative $\partial_i$ on the rational side corresponds the *twisted* derivative on the polynomial side

$$\partial'_i a \overset{\text{def}}{=} \partial_i a - (\partial_i f) a = e^f \partial_i (a e^{-f}).$$

The exponential isomorphism relates, on the one hand, homogeneous fractions $a/f^q$ of degree $-n-1$ modulo derivatives and, on the other hand, homogeneous polynomials, with degree in $(\deg f)\mathbb{Z} - n - 1$, modulo twisted derivatives.

2.1. **Differential forms.** This section is a short reminder about differential forms, or simply *forms*.[2] Let $\Omega^1$ denote the polynomial differential 1-forms: it is the free $A$-module of rank $n+1$, and the basis is denoted by the symbols $\mathrm{d}x_0, \dots, \mathrm{d}x_n$. The differential map d from $A$ to $\Omega^1$ is defined by

$$\mathrm{d}a = \sum_{i=0}^{n} \partial_i a \, \mathrm{d}x_i.$$

The $A$-algebra of differential forms, denoted $\Omega$, is the exterior algebra over $\Omega^1$. Its multiplication is denoted $\wedge$, it is generated by the $\mathrm{d}x_i$ and is subject to the relations $\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i$. The $A$-module of $p$-forms, denoted $\Omega^p$, is the submodule of $\Omega$ generated by the $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_p}$. With the multi-index notation, this is denoted $\mathrm{d}x^I$, with $I = (i_1, \dots, i_p)$. $\Omega^p$ is a free module of rank $\binom{n}{p}$. The module of 0-forms $\Omega^0$ is identified with $A$. As a module, $\Omega$ decomposes as $\oplus_{p=0}^n \Omega^p$. Specifically, the module $\Omega^{n+1}$ has rank 1 and is freely generated by $\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$, denoted $\omega$. The module $\Omega^n$ has rank $n+1$ and is freely generated by the elements $\xi_i$ defined by

$$\xi_i \overset{\text{def}}{=} (-1)^i \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n.$$

[2]See, for example [35, chap. 10] and [10, §10], for more general and complete definitions.

2.1.1. *Exterior derivative.* The differential map d, from $A$ to $\Omega^1$, extends to an endomorphism of $\Omega$, called *exterior derivative*, such that for $\alpha \in \Omega^p$ and $\beta \in \Omega$,

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta.$$

In particular $\mathrm{d}(\Omega^p)$ is included in $\Omega^{p+1}$ and $\mathrm{d}^2 = 0$. For a $n$-form $\beta$, written as $\sum_i b_i \xi_i$, we check that $\mathrm{d}\beta$ equals $(\sum_i \partial_i b_i)\,\omega$. The exterior derivative gives rise to a complex

$$0 \longrightarrow A \xrightarrow{\mathrm{d}} \Omega^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n \xrightarrow{\mathrm{d}} \Omega^{n+1} \longrightarrow 0$$

which is exact.

2.1.2. *Homogeneity.* The degree of a monomial $x^I \mathrm{d}x^J$ is defined to be $|I| + |J|$. A form is called *homogeneous of degree* $k$ if it is a linear combination of monomials of degree $k$. If $\alpha$ and $\beta$ are two homogeneous forms of degree $k_\alpha$ and $k_\beta$ respectively, then $\mathrm{d}\alpha$ is a homogeneous form of degree $k_\alpha$ and $\alpha \wedge \beta$ is a homogeneous form of degree $k_\alpha + k_\beta$.

2.1.3. *Koszul complex.* The exterior product with $\mathrm{d}f$ gives a map from $\Omega^p$ to $\Omega^{p+1}$, and since $\mathrm{d}f \wedge \mathrm{d}f$ vanishes we can consider the chain complex

$$\mathcal{K}(\mathrm{d}f) : 0 \longrightarrow A \xrightarrow{\mathrm{d}f} \Omega^1 \xrightarrow{\mathrm{d}f} \cdots \xrightarrow{\mathrm{d}f} \Omega^n \xrightarrow{\mathrm{d}f} \Omega^{n+1} \longrightarrow 0,$$

known as the *Koszul complex* of $A$ with respect to $\mathrm{d}f$, and its cohomology $H\mathcal{K}(\mathrm{d}f)$ defined by

$$H^p\mathcal{K}(\mathrm{d}f) = \frac{\Omega^p \cap \ker \mathrm{d}f}{\mathrm{d}f \wedge \Omega^{p-1}}.$$

For a $n$-form $\beta$, written as $\sum_i b_i \xi_i$, the exterior product $\mathrm{d}f \wedge \beta$ is $(\sum_i b_i \partial_i f)\,\omega$. Thus $H^{n+1}\mathcal{K}(\mathrm{d}f)$ is isomorphic to $A/\operatorname{Jac} f$, with a shift of $n+1$ in the natural grading, where $\operatorname{Jac} f$ is the Jacobian ideal $(\partial_0 f, \dots, \partial_n f)$.

Let Syz be the kernel of the product by $\mathrm{d}f$ on $\Omega^n$. It is the syzygy module of the sequence $\partial_0 f, \dots, \partial_n f$. Let $\mathrm{Syz}'$ be $\mathrm{d}f \wedge \Omega^{n-1}$, the module of trivial syzygies, generated by the elements $\partial_i f\, \xi_j - \partial_j f\, \xi_i$. In particular $H^n\mathcal{K}(\mathrm{d}f)$ is $\mathrm{Syz}/\mathrm{Syz}'$.

2.2. **Chain complex $T^p$.** For an integer $q$, let $T^p_q$ be the subspace of $\Omega^p$ generated by the homogeneous elements of degree $qN$. Let $T^p$ be the direct sum $\oplus_q T^p_q$ and let $F_q T^p$ be $\oplus_{q' \leqslant q} T^p_{q'}$. Note that $\mathrm{d}f \wedge$ maps $T^n_q$ to $T^{n+1}_{q+1}$ and that d maps $T^n_q$ to $T^{n+1}_q$. Let $\mathcal{S}$ (resp. $\mathcal{S}'$) be the intersection of $T^n$ and Syz (resp. $\mathrm{Syz}'$). The component of degree $qN$ of an element $\alpha$ of $T$ is denoted $\alpha_q$.

The space $T^{n+1}_q$ is the equivalent of $M_q$, as defined in the introductory remarks: the elements of $T^{n+1}_q$ represent numerators of rational functions whose denominator is $f^q$. We define the linear map $h$ from $T^{n+1}$ to $A_f$ by

$$h : a\omega \in T^{n+1}_q \longmapsto (q-1)! \frac{a}{f^q} \in A_f.$$

Of course $h$ is not injective since $h(f\alpha) = qh(\alpha)$, for $\alpha \in T^{n+1}_q$. Finally let $D_f$, the *twisted differential*, from $T^p$ to $T^{p+1}$ be the map defined by $D_f \alpha = \mathrm{d}\alpha - \mathrm{d}f \wedge \alpha$. Note that $D_f$ maps $F_q T^n$ to $F_{q+1} T^{n+1}$. The anticommutation $\mathrm{d}(\mathrm{d}f \wedge \beta) = -\mathrm{d}f \wedge \mathrm{d}\beta$ ensures that $D_f \circ D_f = 0$, so that $T^p$ forms a chain complex.

$$\begin{tikzcd}
T^{n}_{q+1} \arrow[r, "\mathrm{d}"] & T^{n+1}_{q+1} \arrow[r, "\mathrm{d}"] & 0 & \\
{} \arrow[r] & T^{n}_{q} \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & T^{n+1}_{q} \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & 0 \\
{} \arrow[r] & T^{n-1}_{q-1} \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & T^{n}_{q-1} \arrow[u, "\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & T^{n+1}_{q-1} \arrow[u, "\mathrm{d}f"] \\
& {} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

*Figure 1.* Rham–Koszul double complex

*Remark* 5. The spaces $T_q^{p+q}$ arranged within a grid form a double complex, known as *Rham-Koszul double complex* [20], with the *horizontal* differential being d and the *vertical* one being $\mathrm{d}f\wedge$, see Figure 1. This arrangement may help visualize some of the proofs in this article.

For $p \geqslant 0$, let $H^p_{\mathrm{Rham}}(\mathbb{P}^n_{\mathbb{K}} \setminus V(f))$ be the $p^{\mathrm{th}}$ de Rham cohomology group of the variety $\mathbb{P}^n_{\mathbb{K}} \setminus V(f)$,[3] and let $H^{p+1}T$ be the $p^{\mathrm{th}}$ cohomology group of the complex $T$, that is $(T^p \cap \ker D_f)/D_f(T^{p-1})$. The following theorem has been proved in numerous occasions under several appearances, it goes back at least to Dwork. In this exact form, I am aware of proofs by Dimca [19, Theorem 1.8], Malgrange [34] and Deligne [17].

**Theorem 6.** *$H^{p+1}T \simeq H^p_{Rham}(\mathbb{P}^n_{\mathbb{K}} \setminus V(f))$, for all $p \geqslant 1$.*

We will only make use of Theorem 6 in the case where $p = n$. The cohomology group $H^{n+1}T$ is $T^{n+1}/D_f(T^n)$ and $H^n_{\mathrm{Rham}}(\mathbb{P}^n_{\mathbb{K}} \setminus V(f))$ is isomorphic to the subspace of $A_f/\sum_i \partial_i A_f$ generated by the homogeneous elements of degree $-n-1$, and the isomorphism is the map induced by $h : T^{n+1} \to A_f$:

**Proposition 7.** *$h(D_f(T^n)) \subset \sum_{i=0}^n \partial_i A_f$. In other words, the map $h$ induces a map from $T^{n+1}/D_f(T^n)$ to $A_f/\sum_i \partial_i A_f$.*

*Proof.* Let $\beta = \sum_{i=0}^n b_i \xi_i$ be an element of $T^n_q$, then

$$h\left(D_f\left(\sum_{i=0}^n b_i\xi_i\right)\right) = \sum_{i=0}^n h(\partial_i b\omega) - h(b_i \partial_i f\omega) = \sum_{i=0}^n (q-1)!\frac{\partial_i b_i}{f^q} - q!\frac{b_i \partial_i f}{f^{q+1}}$$
$$= (q-1)!\sum_{i=0}^n \partial_i\left(\frac{b_i}{f^q}\right). \quad \square$$

This way, the goal of computing normal forms modulo derivatives of rational functions can be reformulated as computing normal forms of elements of $T^{n+1}$ modulo $D_f(T^n)$.

*Example* 8. With $f = x^2y - z^3$, Equation (4) rewrites

$$x^3\mathrm{d}x\mathrm{d}y\mathrm{d}z = D_f\left(\tfrac{2}{7}x^4\mathrm{d}y\mathrm{d}z + \tfrac{1}{7}x^3\mathrm{d}x\mathrm{d}z\right).$$

[3] See [28] for a general definition and [27] for a definition in the specific case of a complement of a projective hypersurface

The rewriting is not always as simple as in this example but Theorem 6 asserts that it is always possible.

2.3. **Filtered maps.** The space $T^{n+1}$ admits a filtration given by the subspaces $F_q T^{n+1}$ with $q \in \mathbb{Z}$. In the next sections, we will define reduction maps which will be *filtered* endomorphisms of $T^{n+1}$, that is to say linear maps $u : T^{n+1} \to T^{n+1}$ such that $u(F_q T^{n+1}) \subset F_q T^{n+1}$ for all $q \in \mathbb{Z}$. Two filtered endomorphisms of $T^{n+1}$, say $u$ and $v$, are *equivalent* if for all $q \in \mathbb{Z}$ and all $\alpha \in F_q T^{n+1}$ we have $u(\alpha) \equiv v(\alpha)$ modulo $F_{q-1}T^{n+1}$.

For all filtered map $u$, we can define the *associated graded map* as

$$\operatorname{Gr} u : \alpha \in T^{n+1} \mapsto \sum_{q \geqslant 0} u(\alpha_q)_q \in T^{n+1}.$$

Two filtered maps are equivalent if and only if their associated graded maps are equal.

## 3. Griffiths-Dwork reduction

We reword the Griffiths-Dwork reduction, presented in Section 1, in the above setting. Let us choose a monomial ordering on $A$, denoted $\prec$. For a linear subspace $V$ of $A$ and an element $a$ of $A$, let $\operatorname{rem}(a, V)$, be the unique $b$ in $A$ such that $a - b$ is in $V$ and no monomials of $b$ is divided by the leading monomial of some element of $V$. If $V$ is an ideal of $A$, this can be computed using a Gröbner basis of $V$, and if it is a finite-dimensional subspace, then Gaussian elimination following the monomial ordering computes $\operatorname{rem}(a, V)$.

The elementary step of the Griffiths-Dwork reduction is the following. Let $\alpha$ be an element of $T_q^{n+1}$. By definition there is a $\beta$ in $T^n$ such that $\alpha = \operatorname{rem}(\alpha, \mathrm{d}f \wedge T^n) + \mathrm{d}f \wedge \beta$. We choose $\beta$ in such a way that: it depends linearly on $\alpha$ ; $\beta = 0$ if $\alpha$ is in $D_f T^n$ ; and $\beta$ is in $T_{q-1}^n$. The *elementary reduction of $\alpha$ in degree $q$* is then defined to be

$$\operatorname{red}_q^{\mathrm{GD}}(\alpha) \stackrel{\text{def}}{=} \operatorname{rem}(\alpha, \mathrm{d}f \wedge T^n) + \mathrm{d}\beta. \tag{6}$$

For $\alpha$ in $T_k^{n+1}$, for some $k$ different from $q$, we define $\operatorname{red}_q^{\mathrm{GD}}(\alpha) = \alpha$. The definition of $\operatorname{red}_q^{\mathrm{GD}}$ depends on the choice of $\beta$; however, the equivalence class of $\operatorname{red}_q^{\mathrm{GD}}$ as a filtered map does not.

This elementary reduction is very easy to compute using a Gröbner basis of the Jacobian ideal $\operatorname{Jac} f = (\partial_0 f, \dots, \partial_n f)$ and its cofactors. Indeed, the multivariate division algorithm gives a decomposition of a polynomial $a$ as $\operatorname{rem}(a, \operatorname{Jac} f) + \sum_{i=0}^n b_i \partial_i f$. If $\alpha$ is $a\omega$, then $\operatorname{rem}(\alpha, \mathrm{d}f \wedge T^n)$ is $\operatorname{rem}(a, \operatorname{Jac} f)\omega$ and $\beta$ may be chosen equal to $\sum_i b_i \xi_i$. In this way, the assumptions on $\beta$ are naturally satisfied. See Section 7 for more details about the implementation.

By construction, $\alpha - \operatorname{red}_q^{\mathrm{GD}}(\alpha) = -D_f \beta$, so that $\operatorname{red}_q^{\mathrm{GD}}$ is an idempotent map whose kernel is included in $D_f(T^n)$. When translated into a relation between fractions, this reflects integration by parts:

$$\oint b_i \partial_i (1/f^{q-1}) \mathrm{d}\mathbf{x} = -\oint \partial_i b_i / f^{q-1} \mathrm{d}\mathbf{x}.$$

This reduction step can be iterated and for $\alpha \in F_q T^{n+1}$, the *Griffiths-Dwork reduction* of $\alpha$, denoted $[\alpha]_{\mathrm{GD}}$, is defined as

$$[\alpha]_{\mathrm{GD}} \stackrel{\text{def}}{=} \operatorname{red}_1^{\mathrm{GD}} \circ \cdots \circ \operatorname{red}_q^{\mathrm{GD}}(\alpha).$$

We check the following recursive relation: $[\alpha]_{\mathrm{GD}} = \operatorname{rem}(\alpha, \mathrm{d}f \wedge T^n) + [\mathrm{d}\beta]_{\mathrm{GD}}$, where $\beta$ is the one in the equation 6. Again, the map $[\,]_{\mathrm{GD}}$ depend on the choice of $\beta$ but its equivalence class, as a filtered map, does not.

**Proposition 9.** *The map $[\,]_{\mathrm{GD}}$ is filtered, idempotent and $\ker[\,]_{\mathrm{GD}} \subset D_f(T^n)$. In particular $\alpha \equiv [\alpha]_{\mathrm{GD}}$ modulo $D_f(T^n)$ for all $\alpha \in T^{n+1}$. Moreover, for all $q \geqslant 0$ and $\alpha \in F_q T^{n+1}$, $[\alpha]_{\mathrm{GD}} \in F_{q-1}T^{n+1}$ if and only if $\alpha \in D_f(F_{q-1}T^n) + F_{q-1}T^{n+1}$.*

*Proof.* It is straightforward that the map $[\,]_{\mathrm{GD}}$ is filtered, idempotent and that $\ker[\,]_{\mathrm{GD}}$ is included in $D_f(T^n)$. Concerning the second point, let $\alpha \in F_q T^{n+1}$. By contruction $[\alpha]_{\mathrm{GD}} = \alpha + D_f\beta$ for some $\beta \in F_{q-1}T^n$. So if $[\alpha]_{\mathrm{GD}}$ is in $F_{q-1}T^{n+1}$ then $\alpha$ is in $D_f(F_{q-1}T^n) + F_{q-1}T^{n+1}$.

Conversely, let $\alpha = D_f\beta + \varepsilon$, with $\beta \in F_{q-1}T^n$ and $\varepsilon \in F_{q-1}T^{n+1}$. Then $\alpha_q = \mathrm{d}f \wedge \beta_{q-1}$, and so $\operatorname{rem}(\alpha_q, \mathrm{d}f \wedge T^n) = 0$. By Equation (6), $\operatorname{red}_q^{\mathrm{GD}}(\alpha) \in F_{q-1}T^{n+1}$, and so $[\alpha]_{\mathrm{GD}}$ as well. □

The Griffiths-Dwork reduction $[\,]_{\mathrm{GD}}$ is a multivariate and homogeneous analogue of Hermite reduction. In general, it does not have all the nice properties of Hermite reduction: it may happen that for some $\alpha$ in $D_f(T^n)$ the reduction $[\alpha]_{\mathrm{GD}}$ is not zero and it may fail at reducing the degree. Nevertheless, Dwork [22, §3; 23, §8] and Griffiths [27, §4] have proven the following:

**Theorem 10** (Dwork, Griffiths)**.** *If $V(f)$ is smooth in $\mathbb{P}^n_{\mathbb{K}}$ then*

*(i)* $\ker[\,]_{\mathrm{GD}} = D_f(T^n)$,

*(ii) for all $\alpha$ in $T^{n+1}$ the reduction $[\alpha]_{\mathrm{GD}}$ is in $F_n T^{n+1}$*

*Remark* 11. This theorem still holds if we replace $[\,]_{\mathrm{GD}}$ by any equivalent filtered and idempotent map $u$ whose kernel is included in $D_f(T^n)$. Indeed, in this case $\ker \operatorname{Gr} u = \ker \operatorname{Gr}[\,]_{\mathrm{GD}} = \operatorname{Gr}(D_f T^n)$ Since $\ker \operatorname{Gr} u$ is $\operatorname{Gr}(\ker u)$, this implies that $\ker u = D_f(T^n)$. Moreover, the point (ii) implies that $[F_q T^{n+1}]_{\mathrm{GD}} \subset F_{q-1}T^{n+1}$ for all $q > n$. Since $[\,]_{\mathrm{GD}}$ and $u$ are equivalent, the same holds for $u$. And since $u$ is assumed to be idempotent, this implies that $u(T^{n+1}) \subset F_n T^{n+1}$.

The hypothesis "$V(f)$ is smooth" is equivalent to the fact that $\operatorname{Jac} f$ is a zero-dimensional ideal, that is $A/\operatorname{Jac} f$ is finite-dimensional over $\mathbb{K}$. It is also equivalent to the equality of $\mathcal{S}$ and $\mathcal{S}'$, respectively the syzygies and the trivial syzygies in $T^n$. The main step of the proof of Theorem 10 is [27, Theorem 4.3]:

**Theorem 12** (Dwork, Griffiths)**.** *If $V(f)$ is smooth in $\mathbb{P}^n_{\mathbb{K}}$ then $D_f(T^n) \cap F_q T^{n+1}$ is contained in $D_f(F_{q-1}T^n)$ for all $q \geqslant 0$.*

In the singular case, it is never true that $\ker[\,]_{\mathrm{GD}} = D_f(T^n)$. Worse still, the cokernel $T^{n+1}/\ker[\,]_{\mathrm{GD}}$ is never finite dimensional. Indeed, we have

$$\frac{F_q T^{n+1}}{F_q T^{n+1} \cap \ker[\,]_{\mathrm{GD}} + F_{q-1}T^{n+1}} \simeq (A/\operatorname{Jac} f)_{qN-n-1},$$

so the quotient is finite dimensional if and only if $\operatorname{Jac} f$ is a zero-dimensional ideal.

## 4. Computation of higher order relations

4.1. **Construction.** Let $W_q^1$ be the subspace of $T_q^{n+1} + T_{q-1}^{n+1}$ defined by

$$W_q^1 \stackrel{\text{def}}{=} D_f(T_{q-1}^n) = \left\{ -\mathrm{d}f \wedge \beta + \mathrm{d}\beta \;\middle|\; \beta \in T_{q-1}^n \right\}. \tag{7}$$

$$
\begin{tikzcd}
0\ (= -\mathrm{d}f \wedge \beta_{q+2}) & & & & \\
\beta_{q+2} \arrow[u, "-\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & 0\ (= \mathrm{d}\beta_{q+2} - \mathrm{d}f \wedge \beta_{q+1}) & & & \\
& \beta_{q+1} \arrow[u, "-\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & 0 & & \\
& & \beta_q \arrow[u, "-\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & \alpha_q & \\
& & & \beta_{q-1} \arrow[u, "-\mathrm{d}f"] \arrow[r, "\mathrm{d}"] & \alpha_{q-1}
\end{tikzcd}
$$

*Figure 2.* A $n$-form $\beta \in T^n_{q-1} + \cdots + T^n_{q+2}$ such that $D_f\beta \in F_qT^{n+1}$, thus giving an element $\alpha$ of $W^4_q$.

Following the idea developed in Section 1, we define, for $r \geqslant 1$ and $q \geqslant 0$

$$W_q^{r+1} \overset{\text{def}}{=} W_q^1 + W_{q+1}^r \cap F_qT^{n+1}.$$

Compared to Section 1, the space $M_q$ has been replaced by $T_q^{n+1}$ and the product $M_q \times M_{q-1}$ by the direct sum $T_q^{n+1} \oplus T_{q-1}^{n+1}$.

**Proposition 13.** *For all $r \geqslant 1$ and $q \geqslant 0$,*

$$W_q^r = D_f\Big(\sum_{k=1}^{r} T^n_{q+k-2}\Big) \bigcap F_qT^{n+1}.$$

*Proof.* By induction on $r$. For $r = 1$, the claim reduces to $W_q^1 = D_f(T^n_{q-1})$, which is the definition. Then, let us prove that the right-hand side satisfies the recurrence relation defining $W_q^r$, that is:

$$D_f\Big(\sum_{k=1}^{r+1} T^n_{q+k-2}\Big) \bigcap F_qT^{n+1} = D_f(T^n_{q-1}) + D_f\Big(\sum_{k=1}^{r} T^n_{q+k-1}\Big) \bigcap F_qT^{n+1},$$

which follows simply from $D_f(T^n_{q-1}) \subset F_qT^{n+1}$. □

Figure 2 depicts what are elements of $W_q^r$.

*Example* 14. With $f = xy^2 - z^3$, we find that $W_1^1 = 0$ and

$$W_2^1 = \big\langle x^2y, xy^2, y^3, xyz, y^2z, xz^2, yz^2, z^3, 1\big\rangle\,\omega.$$

Thus $W_2^1 \cap T_1^3 = \langle\omega\rangle$ and $W_1^2 = \langle\omega\rangle$.

4.2. **Reductions of order $r$.** The higher order analogue of $\mathrm{red}_q^{\mathrm{GD}}$, denoted $\mathrm{red}_q^r$ is the linear map $T^{n+1} \to T^{n+1}$ defined by

$$\mathrm{red}_q^r\,\alpha \overset{\text{def}}{=} \mathrm{rem}(\alpha, W_q^r),$$

for $\alpha$ in $T_q^{n+1}$, and $\operatorname{red}_q^r \alpha = \alpha$ for $\alpha \in T_k^r$ with $k \neq q$. As for the Griffiths-Dwork reduction, we define for $\alpha$ in $F_q T^{n+1}$

$$[\alpha]_r \stackrel{\text{def}}{=} \operatorname{red}_1^r \circ \cdots \circ \operatorname{red}_q^r(\alpha).$$

This reduction map enjoys the following properties, to be compared with Proposition 9 relative to $[\,]_{\mathrm{GD}}$.

**Proposition 15.** *Let $r \geqslant 1$. The map $[\,]_r$ is filtered and idempotent, its kernel is included in $D_f(T^n)$ and $[\,]_{r+1} \circ [\,]_r = [\,]_{r+1}$. Moreover, for all $q \geqslant 0$ and $\alpha \in F_q T^{n+1}$, $[\alpha]_r \in F_{q-1} T^{n+1}$ if and only if $\alpha \in D_f(F_{q+r-2} T^{n+1}) + F_{q-1} T^{n+1}$.*

*Proof.* It is straightforward that the map $[\,]_r$ is filtered and idempotent. Since $W_q^r \subset D_f(T^n)$, for all $q$, we have $\ker[\,]_r \subset D_f(T^n)$. And since $W_q^r \subset W_q^{r+1}$ we have $[\,]_{r+1} \circ [\,]_r = [\,]_{r+1}$.

Let $\alpha \in F_q T^{n+1}$ such that $[\alpha]_r \in F_{q-1} T^{n+1}$. From the definition, $[\alpha]_r \equiv \operatorname{red}_q^r \alpha \pmod{F_{q-1} T^{n+1}}$ and $\operatorname{red}_q^r \alpha \equiv \alpha \pmod{W_q^r}$. So $\alpha \equiv 0 \pmod{W_q^r + F_{q-1} T^{n+1}}$ and $\alpha \in D_f(F_{q+r-2} T^{n+1}) + F_{q-1} T^{n+1}$.

Conversely, let us assume that $\alpha = D_f \beta + \alpha'$, with $\beta$ in $F_{q+r-2} T^{n+1}$ and $\alpha'$ in $F_{q-1} T^{n+1}$. The form $\beta$ splits as $\beta' + \varepsilon$, with $\beta' \in \sum_{k=1}^r T_{q+k-2}^n$ and $\varepsilon \in F_{q-2} T^n$. We check that $D_f \beta' \in F_q T^{n+1}$, so $D_f \beta' \in W_q^r$, by Proposition 13. And $\operatorname{red}_q^r(D_f \beta') \in F_{q-1} T^{n+1}$, by definition of $\operatorname{red}_q^r$. Thus

$$\operatorname{red}_q^r(\alpha) = \operatorname{red}_q^r(D_f \beta') + \operatorname{red}_q^r(D_f \varepsilon + \alpha') \in F_{q-1} T^{n+1},$$

and $[\alpha]_r$, which equals $[\operatorname{red}_q^r(\alpha)]_r$, is in $F_{q-1} T^{n+1}$ as well. □

**Corollary 16.** $D_f(T^n) = \bigcup_{r \geqslant 1} \ker[\,]_r$.

*Proof.* Let $\beta \in T^n$ such that $D_f \beta \neq 0$. Let $q \geqslant 0$ be the least integer such that $D_f \beta \in F_q T^{n+1}$. Let $r \geqslant 1$ such that $\beta \in F_{q+r-2} T^{n+1}$. By Proposition 15, $[D_f \beta]_r$ is in $F_{q-1} T^n$, and it is also in $D_f(T^n)$ because $D_f \beta \equiv [D_f \beta]_r$ modulo $D_f(T^n)$. By induction on $q$, there exists an $s \geqslant r$ such that $[[D_f \beta]_r]_s = 0$. Since $[\,]_s \circ [\,]_r = [\,]_s$, the result follows. □

*Remark* 17. The reductions $[\,]_{\mathrm{GD}}$ and $[\,]_1$ do not necessarily coincide, but they are equivalent filtered maps.

Thus, we have a family of finer and finer reductions which generalize the Griffiths-Dwork reduction and which are exhaustive in the sense that they reduce to zero every $D_f \beta$ if $r$ is large enough. However, two problems remains. The first on is practical: as defined, the computation of $[\,]_r$, for a given $r$, involves the resolution of huge linear systems, both when computing the spaces $W_q^r$ and when computing $\operatorname{red}_q^r$. This is in contrast with $[\,]_{\mathrm{GD}}$ which only involve the computation of a Gröbner basis and reductions modulo it for computing $\operatorname{red}_q^{\mathrm{GD}}$. The §4.3 describe a faster way to compute $[\,]_r$. The second problem is theoretical: how to set the parameter $r$? This is addressed in Section 5.

4.3. **Faster computation.** There are two ingredient for computing $[\,]_r$ faster than with plain linear algebra. The first is the use of $\operatorname{red}^{\mathrm{GD}}$, whose implementation is efficient and which readily perform a great deal of reductions. Secondly, we discard trivial syzygies, as explained in §1.4.

Let $A_q$ be a complementary subspace of $\mathcal{S}'_q$ in $\mathcal{S}_q$, that is $\mathcal{S}_q$ equals $\mathcal{S}'_q \oplus A_q$. Let $X_q^1 \stackrel{\text{def}}{=} \mathrm{d}A_{q-1}$ and, for all $q \geqslant 0$ and $r \geqslant 1$,

$$X_q^{r+1} \stackrel{\text{def}}{=} \mathrm{d}A_{q-1} + \operatorname{red}_q^{\mathrm{GD}}\left(X_{q+1}^r \cap F_q T^{n+1}\right).$$

Since $\mathrm{d}A_{q-1} = D_f(A_{q-1})$, it is clear that $X_q^1 \subset D_f(F_{q-1}T^n)$, and by induction on $r$, we obtain that $X_q^r \subset D_f(F_{q+r-2}T^n)$. Moreover, we have $\operatorname{red}_q^{\mathrm{GD}} \alpha = \alpha$ for all $q$ and all $\alpha \in X_q^r$. Finally, let $\rho_q^r : T^{n+1} \to T^{n+1}$ the linear map defined by

$$\rho_q^r(\alpha) \stackrel{\text{def}}{=} \operatorname{rem}(\operatorname{red}_q^{\mathrm{GD}}(\alpha), X_q^r),$$

for $\alpha$ in $T_q^{n+1}$, and $\rho_q^r(\alpha) = \alpha$ for $\alpha \in T_k^r$ with $k \neq q$. For $\alpha \in F_q T^{n+1}$ we define

$$[\alpha]'_r \stackrel{\text{def}}{=} \rho_1^r \circ \cdots \circ \rho_q^r(\alpha).$$

This paragraph aims at proving the following:

**Theorem 18.** *For all $r \geqslant 1$, the map $[\,]'_r$ is filtered and idempotent, its kernel is included in $D_f(T^n)$ and $[\,]'_{r+1} \circ [\,]'_r = [\,]'_{r+1}$. Moreover, it is equivalent to $[\,]_r$, in particular, for all $q \geqslant 0$ and $\alpha \in F_q T^{n+1}$, $[\alpha]'_r \in F_{q-1}T^{n+1}$ if and only if $\alpha \in D_f(F_{q+r-2}T^{n+1}) + F_{q-1}T^{n+1}$.*

**Corollary 19.** $D_f(T^n) = \bigcup_{r\geqslant 1} \ker[\,]'_r$.

*Proof.* The proof is the same as Corollary 16. □

The map $[\,]'_r$ is easier to compute than $[\,]_r$ because the linear algebra involved in the computation of $X_q^r$ arises in much lower dimension than the one for $W_q^r$. It comes at the cost of using $\operatorname{red}^{\mathrm{GD}}$ and of computing the space $A_q$ of non trivial syzygies, which can be done efficiently through Gröbner basescomputations, see Section 7.

The main fact which allows to discard trivial syzygies is the following:

**Lemma 20.** $\operatorname{red}_q^{\mathrm{GD}}(\mathrm{d}\mathcal{S}'_q) \subset \mathrm{d}\mathcal{S}_{q-1}$, *for all $q \geqslant 0$.*

*Proof.* Recall that $\mathcal{S}'_q = \mathrm{d}f \wedge T_{q-1}^n$, so let $\beta \in T_{q-1}^n$. The differential anti-commutes with $\mathrm{d}f\wedge$ so that $\mathrm{d}(\mathrm{d}f \wedge \beta) = -\mathrm{d}f \wedge \mathrm{d}\beta$. By definition $\operatorname{red}_q^{\mathrm{GD}}(\mathrm{d}(\mathrm{d}f \wedge \beta))$ is thus $\mathrm{d}\gamma$ for some $\gamma \in T_{q-1}^n$ such that $\mathrm{d}f \wedge \gamma = -\mathrm{d}f \wedge \mathrm{d}\beta$. Thus $\gamma = -\mathrm{d}\beta + \varepsilon$, for some $\varepsilon \in \mathcal{S}_{q-1}$. Since $\mathrm{d}(\mathrm{d}\beta) = 0$, we obtain that $\operatorname{red}_q^{\mathrm{GD}}(\mathrm{d}(\mathrm{d}f \wedge \beta)) = \mathrm{d}\varepsilon$. □

Let $G_q \subset T^{n+1}$ be the kernel of $\operatorname{red}_q^{\mathrm{GD}}$. It is a subspace of $T_q^{n+1} \oplus T_{q-1}^{n+1}$.

**Proposition 21.** $W_q^r = X_q^r + G_q + \mathrm{d}\mathcal{S}'_{q-1}$, *for all $q \geqslant 0$ and $r \geqslant 1$.*

*Proof.* We proceed by induction on $r$. When $r = 1$, it boils down to proving that $D_f(T_{q-1}^n) = \mathrm{d}A_{q-1} + G_q + \mathrm{d}\mathcal{S}'_{q-1}$, that is $D_f(T_{q-1}^n) = G_q + \mathrm{d}\mathcal{S}_{q-1}$, using the fact that $\mathrm{d}A_{q-1} + \mathrm{d}\mathcal{S}'_{q-1} = \mathrm{d}\mathcal{S}_{q-1}$. Let $\beta \in T_{q-1}^n$. By definition of $\operatorname{red}_q^{\mathrm{GD}}$,

$$\operatorname{red}_q^{\mathrm{GD}}(D_f\beta) = -\operatorname{red}_q^{\mathrm{GD}}(\mathrm{d}f \wedge \beta) + \mathrm{d}\beta = \mathrm{d}(\beta - \beta'),$$

for some $\beta' \in T_{q-1}^n$ such that $\mathrm{d}f\wedge\beta' = \mathrm{d}f\wedge\beta$. Thus $\beta-\beta'$ lies in $\mathcal{S}_{q-1}$ and $\operatorname{red}_q^{\mathrm{GD}}(D_f\beta)$ is in $\mathrm{d}\mathcal{S}_{q-1}$. Moreover, since $\operatorname{red}_q^{\mathrm{GD}}$ is idempotent, $D_f\beta - \operatorname{red}_q^{\mathrm{GD}}(D_f\beta)$ is in $G_q$, and in the end $D_f\beta \in G_q + \mathrm{d}\mathcal{S}_{q-1}$. Conversely, $\mathcal{S}_{q-1} \subset T_{q-1}^n$, so it remains to prove that $G_q \subset D_f(T_{q-1}^n)$, which is easy from the definitions.

Now let $r \geqslant 1$. By definition, and by the induction hypothesis

$$W_q^{r+1} = W_q^1 + W_{q+1}^r \cap F_q T^{n+1}$$

$$= G_q + \mathrm{d}A_{q-1} + \mathrm{d}\mathcal{S}'_{q-1} + (X^r_{q+1} + \mathrm{d}\mathcal{S}'_q + G_{q+1}) \cap F_q T^{n+1}.$$

And we have

$$(X^r_{q+1} + \mathrm{d}\mathcal{S}'_q + G_{q+1}) \cap F_q T^{n+1} = X^r_{q+1} \cap F_q T^{n+1} + \mathrm{d}\mathcal{S}'_q.$$

Indeed $\mathrm{d}\mathcal{S}'_q \subset F_q T^{n+1}$, and if $\alpha \in X^r_{q+1}$ and $\alpha' \in G_{q+1}$ are such that $\alpha+\alpha' \in F_q T^{n+1}$, then $\alpha' = 0$ because

$$\alpha + \alpha' = \mathrm{red}^{\mathrm{GD}}_{q+1}(\alpha + \alpha') = \mathrm{red}^{\mathrm{GD}}_{q+1}(\alpha) + \mathrm{red}^{\mathrm{GD}}_{q+1}(\alpha') = \alpha + 0.$$

Thus $W^{r+1}_q = G_q + \mathrm{d}A_{q-1} + \mathrm{d}\mathcal{S}'_{q-1} + \mathrm{d}\mathcal{S}'_q + X^r_{q+1} \cap F_q T^{n+1}$. For any linear subspace $A \subset T^{n+1}$, the decomposition $\alpha \in A$ as $\mathrm{red}^{\mathrm{GD}}_q \alpha + (\alpha - \mathrm{red}^{\mathrm{GD}}_q \alpha)$ shows that $G_q + \mathrm{red}^{\mathrm{GD}}_q(A) = G_q + A$. Thus

$$W^{r+1}_q = G_q + \mathrm{d}A_{q-1} + \mathrm{d}\mathcal{S}'_{q-1} + \mathrm{red}^{\mathrm{GD}}_q(\mathrm{d}\mathcal{S}'_q) + \mathrm{red}^{\mathrm{GD}}_q(X^r_{q+1} \cap F_q T^{n+1}),$$

and the statement follows, by Lemma 20 and the definition of $X^{r+1}_q$. □

We may now prove Theorem 18.

*Proof of Theorem 18.* It is straightforward that $[\,]'_r$ is filtered and idempotent, that $\ker[\,]'_r \subset D_f(T^n)$ and that $[\,]'_{r+1} \circ [\,]'_r = [\,]'_{r+1}$.

To prove that $[\,]_r$ and $[\,]'_r$ are equivalent, it is enough to prove that $\mathrm{red}^r_q$ and $\rho^r_q$ are equivalent. And indeed, if $\alpha \in F_q T^{n+1}$ then

$$\rho^r_q(\alpha) \equiv \mathrm{rem}(\alpha, G_q + X^r_q) \mod F_{q-1}T^{n+1}$$
$$\text{and} \quad \mathrm{red}^r_q(\alpha) \equiv \mathrm{rem}(\alpha, \mathrm{d}\mathcal{S}'_{q-1} + G_q + X^r_q) \mod F_{q-1}T^{n+1},$$

using Proposition 21. Since $\mathrm{d}\mathcal{S}'_{q-1} \subset F_{q-1}T^{n+1}$ the claim follows. □

In what follows, $[\,]_r$ will stand for $[\,]'_r$. Except in terms of computational complexity, they have the same properties.

4.4. **Quantitative facts.** It is useful to introduce the spaces

$$E^r_q \stackrel{\text{def}}{=} \frac{F_q T^{n+1}}{D_f(F_{q+r-2}T^n) \cap F_q T^{n+1} + F_{q-1}T^{n+1}}.$$

It is clear that $E^0_q$ is $F_q T^{n+1}/F_{q-1}T^{n+1}$, which is isomorphic to $T^{n+1}_q$. Moreover, as a reformulation of Proposition 9, the space $E^1_q$ is

$$E^1_q = \mathrm{coker}(\mathrm{Gr}[\,]_{\mathrm{GD}})_q \stackrel{\text{def}}{=} \frac{F_q T^{n+1}}{\{\alpha \in F_q T^{n+1} \mid [\alpha]_{\mathrm{GD}} \in F_{q-1}T^{n+1}\}} \simeq \frac{T^{n+1}_q}{\mathrm{d}f \wedge T^n_{q-1}}.$$

And by Proposition 15, this generalizes to the isomorphism $E^r_q \simeq \mathrm{coker}(\mathrm{Gr}[\,]_r)_q$. In other words, $E^r_q$ is $F_q T^{n+1}$ modulo elements which are reducible to $F_{q-1}T^{n+1}$ by $[\,]_r$ The space $E^{r+1}_q$ is a quotient of $E^r_q$, and the dimension fall represents how many new relations in degree $qN$ are computed by $[\,]_{r+1}$ compared to $[\,]_r$. For $r = 2$, we check that

$$E^2_q \simeq \frac{T^{n+1}_q}{\mathrm{d}f \wedge T^n_{q-1} + \mathrm{d}\mathcal{S}_q} = \frac{T^{n+1}_q}{\mathrm{d}f \wedge T^n_{q-1} + \mathrm{d}A_q}.$$

The dimension of $E^0_q$ is $\binom{Nq-1}{n}$, which is equivalent to $N^n q^n/n!$ when $q \to \infty$. The dimension of $E^1_q$ is $\mathcal{O}(q^\nu)$, where $\nu$ is the dimension of the singular locus of $V(f)$ in $\mathbb{P}^n_{\mathbb{K}}$. There is no easy estimate of the dimension of $E^2_q$, but $\dim A_{q-1}$ is also $\mathcal{O}(q^\nu)$. By contrast, $\dim \mathcal{S}_q \sim (n+1)N^n q^n/n!$. For the computation of $[\,]_2$ (or rather $[\,]'_2$),

| $q$ | 0 | 1 | 2 | 3 | 4 | $q > 4$ |
|---|---|---|---|---|---|---|
| $\dim E_q^0$ | 0 | 10 | 165 | 680 | 1771 | $\binom{6q-1}{3} \sim 36q^3$ |
| $\dim E_q^1$ | 0 | 10 | 86 | 102 | 120 | $18q+48$ |
| $\dim E_q^2$ | 0 | 10 | 7 | 6 | 6 | 6 |
| $\dim E_q^3$ | 0 | 9 | 1 | 0 | 0 | 0 |
| $\dim E_q^r$, $r \geqslant 4$ | 0 | 9 | 1 | 0 | 0 | 0 |

*Table 1.* Some dimensions related to Example 22

it is thus a substantial improvement to consider the non-trivial syzygies $A_q$ rather than all the syzgies $\mathcal{S}_q$ .

*Example* 22. To illustrate precisely what does bring the maps $[\,]_r$ in comparaison with $[\,]_{\mathrm{GD}}$, let us consider the polynomial $f$

$$f \stackrel{\text{def}}{=} 2x_1x_2x_3(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) - x_0^3(x_0^3 - x_0^2x_3 + x_1x_2x_3)$$

coming from an integral for the Apéry numbers, see Example 1. In this case $n = 3$ and $N = 6$. The dimension of the singular locus of $V(f)$ in $\mathbb{P}^3_{\mathbb{K}}$ is 1.

The dimensions of the first few $E_q^r$ are shown in Table 1. This illustrates the successive dimension falls. Noticeably, at $r = 3$ a new relation appears in $F_1T^{n+1}$. It is $(2x_1^2 - 2x_2^2 - x_0(x_1 - x_2))\omega$, which equals $D_f\beta$ for some $\beta$ in $F_2T^n$ but no such $\beta$ is small enough to be reproduced here.

Illustrating the same polynomial $f$, Table 2 shows the numbers of syzygies and non-trivial syzygies at a given degree. It also displays the difference $\dim E_q^1 - \dim E_q^2$, that is how many new relations are really generated from the syzygies.

## 5. Extensions of Griffiths' theorems

Given $\alpha$ in $T^{n+1}$, how can we compute a $r$ such that if $\alpha$ is in $D_f(T^n)$ then $[\alpha]_r$ equals zero? Corollaries 16 and 19 are lacking effective bounds and do not answer this question. Dimca proved two theorems [18, Th. B and Cor. 2; 19, Th. 2.7] which generalize Theorem 10. While they do not give a full answer, they allow to give enough guarantees on $[\,]_r$ to design algorithms that terminates.

**Theorem 23** (Dimca)**.** *There exists an integer $C$, depending only on $f$, such that $D_f(T^n) \cap F_qT^{n+1} \subset D_f(F_{q+C-2}T^n)$ for all $q \geqslant 0$.*

This statement is to be compared with Theorem 12. Given $f$ and $q$, it is easy to prove that there exists a $C$ such that $D_f(T^n) \cap F_qT^{n+1} \subset D_f(F_{q+C-2}T^n)$, because the left-hand side is a finite dimensional space and it is included in $\cup_{C\geqslant 0} D_f(F_{q+C-2}T^n)$. It is remarkable that one can choose a $C$ which does not depend on $q$. Let $r_f$ be the least such $C$.

**Corollary 24.** $\ker[\,]_{r_f} = D_f(T^n)$.

*Proof.* Let $\beta \in T^n$ and $q \geqslant 0$ the least integer such that $D_f\beta \in F_qT^{n+1}$. By Theorem 23, there exists $\beta' \in F_{q+r_f-2}T^n$ such that $D_f\beta' = D_f\beta$. Thus, by Theorem 18, $[D_f\beta]_{r_f}$ is in $F_{q-1}T^{n+1}$, and besides, it is also in $D_f(T^n)$. By induction on $q$, $[[D_f\beta]_{r_f}]_{r_f} = 0$. Since $[\,]_{r_f}$ is idempotent, the claim follows. ☐

| $q$ | 0 | 1 | 2 | 3 | 4 | $q > 4$ |
|---|---|---|---|---|---|---|
| $\dim \mathcal{S}_q$ | 0 | 21 | 522 | 2429 | 6604 | $\sim 144q^3$ |
| $\dim A_q$ | 0 | 1 | 92 | 132 | 168 | $36q + 24$ |
| $\dim E_q^1 - \dim E_q^2$ | 0 | 0 | 79 | 96 | 114 | $18q + 42$ |

*Table 2.* Gain of dimension by discarding trivial syzygies and number of new relations generated by the syzygies in the Example 22

Unfortunately, this integer $r_f$, while explicit, is not easy to compute: in Dimca's proof it is expressed in terms of a resolution of the singularities of the projective variety $V(f)$. By contrast, the point (ii) of Theorem 10 fully generalizes to singular cases:

**Theorem 25** (Dimca)**.** $D_f(T^n) + F_n T^{n+1} = T^{n+1}$.

**Corollary 26.** *For all $\alpha \in T^{n+1}$, the reduction $[\alpha]_{r_f}$ lie in $F_n T^{n+1}$.*

*Proof.* By Theorem 25, there exists $\beta \in T^n$ such that $\alpha + D_f\beta$ is in $F_n T^{n+1}$. Since $[\alpha]_{r_f} = [\alpha + D_f\beta]_{r_f} - [D_f\beta]_{r_f}$, the claim follows from Corollary 24. □

For some applications, such that the computation of annihilating operators of periods with a parameter, Theorem 25 gives an efficient workaround to the lack of *a priori* bounds for $r_f$. Consider an algorithm which computes reductions $[\alpha]_r$, for some forms $\alpha$ and some fixed integer $r$, and does it as long as the reductions it computes are linearly independent. Then either all the $[\alpha]_r$ are in the finite dimensional space $F_n T^{n+1}$, and then the algorithm terminates; or some $[\alpha]'_r$ is not in $F_n T^{n+1}$, and then $r < r_f$, by Theorem 25. When the second case is encountered, we abort the algorithm, increment $r$ and start over. This may happen only if $r < r_f$, and when it happens $r$ increases. So it may happen only finitely many times and the algorithm terminates.

Concerning the integer $r_f$ Dimca [18] conjectured that

**Conjecture 27.** $r_f \leqslant n + 1$.

As far as I know, computations on explicit examples confirm this conjecture. Moreover the bound is tight when $n = 2$. A proof of this conjecture would have very interesting algorithmic consequences: the reduction algorithm is extendable to the computation of the whole cohomology of $T$, not just the top cohomology. Only the bound $r_f \leqslant n + 1$ is lacking for obtaining an efficient algorithm for computing the de Rham cohomology of the complement of a projective hypersurface.

# Part 2. Periods with a parameter

We apply the reduction algorithm to the computation of Picard-Fuchs equations.

## 6. Algorithms

6.1. **Setting.** Let $\mathbb{K}$ be a field of characteristic zero with a derivation $\delta$. Typically $\mathbb{K}$ is $\mathbb{Q}(t)$ and $\delta$ is the usual derivation with respect to $t$. Let $\mathbb{K}\langle\delta\rangle$ be the algebra of differential operators in $\delta$: it is the associative algebra with unity generated over $\mathbb{K}$ by $\delta$ and subject to the relations $\delta x = x\delta + \delta(x)$ for all $x$ in $\mathbb{K}$, where $\delta(x)$

denotes the application of $\delta$ to $x$ whereas $\delta x$ is the operator that multiplies by $x$ and then applies $\delta$. On $\mathbb{K}(x_0,\dots,x_n)$, let $\partial_i$ denote the derivation with respect to $x_i$. The derivation $\delta$ extends to $\mathbb{K}(x_0,\dots,x_n)$ uniquely by setting $\delta(x_i)=0$. In particular $\delta\circ\partial_i=\partial_i\circ\delta$.

This section describes an algorithm which takes as input a rational function $R$ in $\mathbb{K}(x_1,\dots,x_n)$ and outputs an operator $\mathcal{L}$ in $\mathbb{K}\langle\delta\rangle$ such that there exist other rational functions $C_1,\dots,C_n$ with

$$\mathcal{L}(R)=\sum_{i=1}^n \partial_i C_i.$$

Moreover, the irreducible factors of the denominators of the $C_i$ divide the denominator of $R$. Such an operator will be called an *annihilating operator of the periods of* $R$, or *a differential equation for* $\oint R$. The minimal annihilating operator of $\oint R$ is called the *Picard-Fuchs equation* (of $\oint R$). The output operator $\mathcal{L}$ is not necessarily the Picard-Fuchs equation but it is of course a left multiple of it.

Being based on the reduction algorithm of Part 1, the algorithm does not compute the $C_i$. It is worth a word because while only $\mathcal{L}$ matters, the size of the $C_i$, say the size of a binary dense representation, is usually much larger than the size of $\mathcal{L}$ [9, Rem. 11]. To be able to compute $\mathcal{L}$ without computing the $C_i$ is certainly a good point toward practical efficiency. The fractions $C_i$ are called a *certificate*: they allow to check *a posteriori* that $\mathcal{L}$ is indeed an annihilating operator of $\oint R$.

6.2. **Homogenization.** The reduction algorithm works in an homogeneous setting. If we are interested in computing the Picard-Fuchs equation of the integral of an inhomogeneous function, the problem can be homogenized as follows. Let $R_{\mathrm{hom}}$ be the homogenization of $R$ in degree $-n-1$ defined by

$$R_{\mathrm{hom}}=x_0^{-n-1}R\left(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}\right)\in\mathbb{K}(\mathbf{x}),$$

where $\mathbf{x}$ denotes $x_0,\dots,x_n$ hereafter. The rational function $R_{\mathrm{hom}}(\mathbf{x})$ is *homogeneous of degree* $-n-1$, that is $R_{\mathrm{hom}}(\lambda x_0,\dots,\lambda x_n)=\lambda^{-n-1}R_{\mathrm{hom}}(x_0,\dots,x_n)$, or, equivalently, $R_{\mathrm{hom}}=b/g$ where $b$ and $g$ are homogeneous polynomials such that $\deg b+n+1=\deg g$.

Let us write $R_{\mathrm{hom}}$ as $a/f^q$, with $a$ and $f$ two homogeneous polynomials and $q$ an integer. Usually $f$ will be chosen square-free but we don't have to. Let $N$ be the degree of $f$. Since $R_{\mathrm{hom}}$ is homogeneous of degree $-n-1$, the degree of $a$ is $qN-n-1$. This is the main reason for considering homogeneous fractions: the degree of the denominator determines the degree of the numerator, there is no *hidden* pole at infinity. The degree $-n-1$ is crucial to ensure that:

**Lemma 28.** *If $\mathcal{L}\in\mathbb{K}\langle\delta\rangle$ is a annihilating operator of $\oint R_{hom}$ then $\mathcal{L}$ is also a annihilating operator of $\oint R$.*

*Proof.* Assume that $\mathcal{L}(R_{\mathrm{hom}})$ equals $\sum_{i=0}^n\partial_i(b_i/f^m)$, for some polynomials $b_i$ and some integer $m$. Substituting $x_0$ by 1 gives

$$\mathcal{L}(R)=\partial_0(b_0/f^m)|_{x_0=1}+\sum_{i=1}^n\partial_i(b_i/f^m|_{x_0=1}).$$

*Algorithm 1.* Computation of annihilating operators of the periods of a rational function, smooth case

*Input* — $a/f^q$ a homogeneous rational function in $\mathbb{K}(\mathbf{x})$ of degree $-n-1$, with $V(f)$ smooth in $\mathbb{P}^n_{\mathbb{K}}$
*Output* — $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ the Picard-Fuchs equation of $\oint R$
**procedure** $\textsc{PicardFuchs}(a/f^q)$
 $\rho_0 \leftarrow [a\omega]_{\mathrm{GD}}$
 **for** $m$ from 0 to $\infty$ **do**
  **if** $\operatorname{rank}_{\mathbb{K}}(\rho_0, \dots, \rho_m) = m+1$ **then**
   $\rho_{m+1} \leftarrow [\delta(\rho_m)]_{\mathrm{GD}}$
  **else**
   compute $a_0, \dots, a_{m-1} \in \mathbb{K}$ such that $\sum_{k=0}^{m-1} a_k \rho_k = \rho_m$
   **return** $\delta^m - \sum_{k=0}^{m-1} a_k \delta^k$

Since $R_{\mathrm{hom}}$ is homogeneous of degree $-n-1$, we may assume that each $b_i/f^m$ is homogeneous of degree $-n$. Euler's relation gives

$$-nb_0/f^m = \sum_{i=0}^{n} x_i \partial_i (b_0/f^m) = \sum_{i=0}^{n} \left(\partial_i(x_i b_0/f^m) - b_0/f^m\right).$$

This proves that $0 = \partial_0(b_0/f^m)|_{x_0=1} + \sum_{i=1}^n \partial_i(x_i b_0/f^m|_{x_0=1})$, and the claim follows. □

The Picard-Fuchs equation of $\oint R_{\mathrm{hom}}$ may not be the Picard-Fuchs equation of $\oint R$, but only a left multiple. However, it is the case if $x_0$ divides $f$, which is possible to assume, up to replacing $f$ by $x_0 f$ and $a$ by $x_0^q a$. From now on I focus exclusively on the homogeneous case.

6.3. **Computation of Picard-Fuchs equations.** The derivation $\delta$ is extended to the spaces $T^p$ of differential forms[4] by

$$\delta : \alpha \in T^p \mapsto \alpha^\delta - f^\delta \alpha \in T^p,$$

where $\bullet^\delta$ denotes component-wise differentiation. It commutes with the map $h$, and the differential $D_f$, as a consequence of $\delta$ commuting with $\partial_i$.

To highlight the difference between the smooth and the singular cases, I recall first how the Griffiths-Dwork reduction applies to the computation of Picard-Fuchs equations. Let $a/f^q$ be a homogeneous fraction of degree $-n-1$. We define $\rho_0 \stackrel{\text{def}}{=} [a\omega]_{\mathrm{GD}}$ and $\rho_{k+1} \stackrel{\text{def}}{=} [\delta(\rho_k)]_{\mathrm{GD}}$. Since $\delta$ commutes with $D_f$, it is clear that $\rho_k \equiv \delta^k(a\omega)$ modulo $D_f(T^n)$. Hence Theorem 10 implies that $\rho_k = [\delta^k(a\omega)]_{\mathrm{GD}}$. Thus, by Theorems 6 and 10 and, for $u_0, \dots, u_m$ in $\mathbb{K}$,

$$\sum_{k=0}^{m} u_k \delta^k(a/f^q) \in \sum_{k=0}^{n} \partial_k A_f \text{ if and only if } \sum_{k=0}^{m} u_k \rho_k = 0.$$

This leads to Algorithm 1.

**Proposition 29.** *Algorithm 1 applied to a fraction $R$ satisfying the regularity assumption terminates and outputs the Picard-Fuchs equation of $\oint R$.*

[4]See definition in §2.2.

*Proof.* Correctness has just been proven. Termination follows from Theorem 10, point (ii), which implies that the $\rho_i$ lie in a finite-dimensional space, so they are linearly dependent. □

If Conjecture 27 were proven, it would be enough to replace $[\,]_{\mathrm{GD}}$ by $[\,]_{n+1}$, or its efficient variant $[\,]'_{n+1}$, in Algorithm 1 to obtain an algorithm which provably outputs the Picard-Fuchs equation of a rational integral in the singular case. While assuming this conjecture gives good results in practice, the absence of a proof is embarrassing.

It is worth mentioning the treatment of singular cases by a generic deformation: to compute a differential for $\oint R$, for some $R = a/f$, we may change $R$ into

$$R_\lambda = \frac{a}{f + \lambda \sum_{i=0}^n x_i^{\deg f}},$$

where $\lambda$ is a free variable. The denominator of $R_\lambda$ always satisfy the smoothness hypothesis, so Algorithm 1 applies, over $\mathbb{K}(\lambda)$, and gives the Picard-Fuchs equation of $\oint R_\lambda$, say $\mathcal{L}$ in $\mathbb{K}(\lambda)\langle\delta\rangle$. Then $(\lambda^a \mathcal{L})_{|\lambda=0}$, where $a$ is the unique integer which makes this evaluation neither zero nor singular, is a differential equation for $\oint R$. This method achieves a good computational complexity, that is polynomial complexity with respect to the *generic size* of the output [9], but its practical efficiency is terrible because most Picard-Fuchs that are interesting to compute are much smaller than the generic Picard-Fuchs equation.

Another approach, using the reductions $[\,]_r$, is to loop over $r$. We begin by fixing $r$ to an initial value, for example 1, and we introduce another variable $M$, a positive integer. Then we compute $\rho_0$, $\rho_1$, etc. as in Algorithm 1 but replacing $[\,]_{\mathrm{GD}}$ by $[\,]_r$, up to $\rho_M$. If there is no linear dependency relation between the $\rho_k$ then we increase both $r$ and $M$ and repeat the procedure. At some point, the parameter $r$ will exceed $r_f$ and $M$ will exceed the order of the Picard-Fuchs equation of $\oint R$. There, a relation will be found between the $\rho_k$ and it will give the Picard-Fuchs equation. It is possible that a relation is found before the condition $r \geqslant r_f$ is met: it gives of course a differential equation, but it need not be the minimal one.

Theorem 25 and its corollary allow for an interesting variant of this approach. As above, we loop over $r$. For a given value of $r$, the forms $\rho_0$, $\rho_1$, etc. are computed as in Algorithm 1 but using $[\,]_r$ instead of $[\,]_{\mathrm{GD}}$. Contrary to the previous approach, the number of $\rho_i$ we compute before moving to the next value of $r$ is not bounded *a priori*. Instead, we compute $\rho_0$, $\rho_1$, etc. as long as $\rho_k$ stays in $F_n T^{n+1}$. Since $F_n T^{n+1}$ is finite dimensional, we have the following alternative: either there exists a relation between the $\rho_k$, or there exists a $k$ such that $\rho_k$ is not in $F_n T^{n+1}$. In the first case, the relation gives a differential equation for $\oint R$. In the second case, we increase $r$ and start over the computation of the $\rho_k$'s. Corollary 24 assures that as soon as $r \geqslant r_f$, the second condition is never met, so a relation will eventually be found. Algorithm 2 details the procedure.

**Theorem 30.** *Algorithm 2 terminates and outputs an annihilating operator of $\oint R$.*

## 7. Implementation

Algorithm 2 has been implemented in the computer algebra system Magma [7], with $\mathbb{Q}(t)$ as base field $\mathbb{K}$, with the usual derivation.[5] To be able to treat large

[5]The implementation is available at `http://github.com/lairez/periods`.

*Algorithm 2.* Computation of annihilating operators of the periods of a rational function

> *Input* — $a/f^q$ a homogeneous rational function in $\mathbb{K}(\mathbf{x})$ of degree $-n-1$
> *Output* — $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ a differential equation for $\oint R$
> **procedure** PicardFuchs($a/f^q$)
>     **for** $r$ from 1 to $\infty$ **do**
>         $\rho_0 \leftarrow [a\,\omega]_r$ $\quad\triangleright$ Compute the subspaces $X_r^q$ as they are needed.
>         **for** $m$ from 0 to $\infty$ while $\deg \rho_m \leqslant n \deg f$ **do**
>             **if** $\operatorname{rank}_{\mathbb{K}}(\rho_0, \dots, \rho_m) = m+1$ **then**
>                 $\rho_{m+1} \leftarrow [\delta(\rho_m)]_r$
>             **else**
>                 compute $a_0, \dots, a_{m-1} \in \mathbb{K}$ such that $\sum_{k=0}^{m-1} a_k \rho_k = \rho_m$
>                 **return** $\delta^m - \sum_{k=0}^{m-1} a_k \delta$

examples—like the ones in Section 8—the coefficient swell makes it necessary to implement a randomized evaluation-interpolation scheme which splits a computation over $\mathbb{Q}(t)$ into several analogous computations over different finite fields. However it comes at a price: since we lack tight *a priori* bounds on the size of the output—order, degree, size of the coefficients—the reconstruction step is not certified to be correct, even though the probability of failure can be made arbitrarily small. There are also several ways to cross-check the result independently. The variant is described in §7.2. In the introduction, I mentionned the guessing method which allows, in some cases, to compute an annihilating operator of a given period but gives no guarantee about its correctness. The nature of the risk of failure is very different though. In the evaluation-interpolation method, the algorithm is randomized and the probability of failure can be made arbitrarily small. It is even less probable that the algorithm returns twice the same wrong result. It is not possible to fool the algorithm on purpose with a specific input. In the guessing method, we do not know how to evaluate the risk of failure and the algorithm is deterministic so an error will be repeated again and again. It is in principle possible to fool the method with input designed for this purpose.

When a risk of failure is not acceptable, it is possible to compute certificates which can be used *a posteriori* to prove that what has been computed is correct, see §7.3.

7.1. **Implementation of $[\,]_r$ using Gröbner bases.** Let $M$ be the module $\Omega^{n+1} \oplus \Omega^n$, that is the free module generated by $\omega$ and the $\xi_i$, recall the definitions in §2.1 A convenient way to implement the reduction $[\,]_r$ is to compute a reduced Gröbner basis[6] say $G$, of the submodule $P$ of $M$ generated by the $\partial_i f \omega - \xi_i$, that is $\mathrm{d}f \wedge \xi_i - \xi_i$. We choose on $M$ a monomial ordering, denoted $\succ$, such that for all multi-indices $I$ and $J$, and all integer $j$

$$|I| + 1 \geqslant |J| + N \Longrightarrow x^I \omega \succ x^J \xi_j. \tag{8}$$

For example, any position-over-term (POT) ordering with $\omega \succ \xi_0 \succ \xi_1 \succ \cdots$ is fine. But a term-over-position (TOP), with $\omega \succ \xi_0 \succ \xi_1 \succ \cdots$, extending a graded ordering on $A$ works as well. This gives some flexibility in the implementation.

[6]See [16, chap. 5] for details about Gröbner bases for modules, the division algorithm, etc.

Let $\operatorname{rem}_G$ denote the remainder on division by $G$. The condition (8) on the order is enough to ensure that $\succ$ behaves like an order eliminating $\omega$.

The reason is the following. If we give to $\omega$ the degree 1 and to each $\xi_i$ the degree $N$, then $P$ is a homogeneous submodule of $M$. Thus any reduced Gröbner basis $G$ of $P$, whatever the monomial order, contains only homogeneous elements and the remainder on division by $G$ of a homogeneous element of degree $d$ is homogeneous of degree $d$. In particular we have the

**Lemma 31.** *Let $\alpha$ be an element of $\Omega^{n+1}$. Then the coefficient of $\omega$ in $\operatorname{rem}_G \alpha$ is zero if and only if $\alpha \in \mathrm{d}f \wedge \Omega^n$. In this case $\alpha = \mathrm{d}f \wedge \operatorname{rem}_G \alpha$.*

*Proof.* By definition of $G$ there exist polynomials $c_i$ such that

$$\alpha = \operatorname{rem}_G(\alpha) + \sum_{i=0}^{n} c_i(\mathrm{d}f \wedge \xi_i - \xi_i).$$

If the coefficient of $\omega$ in $\operatorname{rem}_G(\alpha)$ is zero then $\operatorname{rem}_G(\alpha)$ is in $\Omega^n$. Identifying the components gives

$$\alpha = \mathrm{d}f \wedge \sum_{i=0}^{n} c_i \xi_i = \left(\sum_{i=0}^{n} c_i \partial_i f\right)\omega \quad \text{and} \quad \operatorname{rem}_G(\alpha) = \sum_i c_i \xi_i.$$

Conversely, assume that $\alpha = \mathrm{d}f \wedge \beta$, for some $\beta$ in $\Omega^n$. We may assume that $\alpha$ is homogeneous of degree $d$ and that $\beta$ is homogeneous of degree $d - N$. In particular $\alpha - \beta$ is in $P$ and $\operatorname{rem}_G(\alpha - \beta) = 0$, since $G$ is a Gröbner basis of $P$. By linearity $\operatorname{rem}_G(\alpha)$ equals $\operatorname{rem}_G(\beta)$.

For the grading introduced above, the element $\beta$ is homogeneous of degree $d - n$, thus so is $\operatorname{rem}_G(\beta)$. Furthermore, the leading monomial of $\operatorname{rem}_G(\beta)$, with respect to $\succ$, is at most the leading monomial of $\beta$, which has the form $x^I \xi_i$ with $|I| = d - N - n$. The claim follows since no monomial of the form $x^J \omega$ has degree $d - n$ (with the alternative grading) and is less than $x^I \xi_i$, thanks to hypothesis (8). □

In the same way we prove that

**Lemma 32.** *The intersection $G \cap \Omega^n$ is a Gröbner basis of* Syz*.*

Together with a Gröbner basis of $\mathrm{Syz}'$, this Gröbner basis can be used to compute a basis of $\mathcal{S}_q/\mathcal{S}'_q$ in the following way. Using the Gröbner bases, we compute the set

$$S \stackrel{\text{def}}{=} \{\operatorname{lm}(\alpha) \mid \alpha \in \mathcal{S}_q\} \setminus \left\{\operatorname{lm}(\alpha) \;\middle|\; \alpha \in \mathcal{S}'_q\right\}.$$

Then, for each element $\alpha$ of $S$ we pick an element of $\mathcal{S}_q$ whose leading monomial is $\alpha$. Those elements form a basis of $\mathcal{S}_q/\mathcal{S}'_q$.

Gröbner bases in the module $M$ can be *emulated* by Gröbner bases in the polynomial ring $A$ with two extra variables, say $u$ and $v$. Let $A'$ be $A[u, v]$, let $\omega'$ denote $u^{n+1}$ and $\xi'_i$ denote $u^{n-i}v^{i+1}$. Let $M'$ be the $A$-submodule of $A'$ generated by $\omega'$ and $\xi'_i$. Let $P'$ be the ideal of $A'$ generated by $\partial_i f \omega' - \xi'_i$ and all the monomials $u^p v^q$, with $p + q = n + 2$. Let $\varphi$ be the $A$-linear map from $M'$ to $M$ sending $\omega'$ to $\omega$ and $\xi'_i$ to $\xi_i$. Finally, let $G'$ be a Gröbner basis with respect to any graded monomial ordering $\succ'$, say the graded reverse lexicographic ordering, with $u \succ v \succ x_0 \succ \cdots \succ x_n$.

If $\succ$, the monomial ordering for $M$, is the TOP ordering proposed above, then we have $\varphi(\operatorname{rem}_{G'} \alpha) = \operatorname{rem}_G \varphi(\alpha)$, and the proof is left to the reader.

*Algorithm 3.* Computation of $[\,]_r$

*Input* — $\alpha$ an element of $T^{n+1}$ and $q$ an integer
*Output* — $\mathrm{red}_q^{\mathrm{GD}}(\alpha)$ as defined in §3
**procedure** RedStep($\alpha$, $q$)
  $\alpha' \leftarrow \alpha - \alpha_q$
  $\rho + \beta \leftarrow \mathrm{rem}_G(\alpha_q)$, with $\rho \in \Omega^{n+1}$ and $\beta \in \Omega^n$
  **return** $\alpha' + \rho + \mathrm{d}\beta$

*Input* — $r \geqslant 1$ and $q \geqslant 0$ integers
*Output* — a basis of $X_q^r$, as defined in §4.3
**procedure** BasisX($r$, $q$)
  **if** $r = 1$ **then**
    **return** $\left\{\mathrm{d}\beta \;\middle|\; \beta \in (\text{a basis of } \mathcal{S}_{q-1}/\mathcal{S}'_{q-1})\right\}$
  **else**
    $X \leftarrow \mathrm{BasisX}(r-1, q+1)$
    **return** $\mathrm{Echelon}(\mathrm{BasisX}(1,q) \cup \{\mathrm{RedStep}(\alpha, q) \in X \mid \deg \alpha = qN\})$

*Input* — $\alpha$ an element of $T^{n+1}$, $r$ a positive integer
*Output* — $[\alpha]'_r$ as defined in §4.3
**procedure** Reduction($\alpha$, $r$)
  $q \leftarrow \deg \alpha / N$ and $\alpha' \leftarrow \alpha - \alpha_q$
  $\rho \leftarrow \mathrm{rem}(\mathrm{RedStep}(\alpha_q, q), \mathrm{BasisX}(r, q))$
  **return** $\rho_q + \mathrm{Reduction}(\alpha' + \rho_{q-1}, r)$

The computation of $X_q^r$ and $[\,]_r$ is detailed in Algorithm 3. The function Echelon takes as input a finite subset $S$ of $T^{n+1}$ and outputs a basis in echelon form of $\mathrm{Vect}(S)$, with respect to the monomial order $\succ$: that is, a basis $B$ of $\mathrm{Vect}(S)$ such that for all element $b$ of $B$, the leading monomial of $b$ does not appear with a non-zero coefficient in the other elements of $B$.

7.2. **Evaluation and interpolation scheme.** Let $h(t) = p/q$ be an element of $\mathbb{Q}(t)$ such that $q$ is a monic polynomial. Let $d$ be the maximum of $\deg p$ and $\deg q$, and $M$ be the maximum of the absolute values of numerators and denominators of the coefficients of $p$ and $q$. Given distinct primes $p_1, \dots, p_n$, distinct rational numbers $u_1, \dots, u_m$ and the evaluations $a_{i,j} \equiv h(u_j) \pmod{p_i}$, the fraction $h$ can be reconstructed given that no $p_i$ divides the denominator of some coefficient of $q$, no $u_j$ annihilates $q$, $\prod_{i=1}^m p_i > 2M$ and $m > 2d$. To do so, we first compute $a_i$ in $\mathbb{F}_{p_i}(t)$ such that $a_i \equiv h \pmod{p_i}$, using Cauchy interpolation [26, §5.8]. Then, by the Chinese remainder theorem, we compute $A$ such that $A \equiv h \pmod{\prod_i p_i}$. And then, using rational reconstruction [26, §5.10] to each coefficient of $A$, we recover $h$. Without *a priori* bounds on $h$, it is still possible to try to reconstruct it with the method above. Assume that we obtain a result $h'$, and let $M'$ and $d'$ be the analogues of $M$ and $d$ for $h'$. Under randomness assumptions, the bigger $\prod_{i=1}^m p_i - 2M'$ and $m - 2d'$ are, the higher is the probability that $h' = h$.

Any algorithm which inputs and outputs elements of $\mathbb{Q}(t)$ and which performs only field operations—addition, multiplication, negation, constant one, zero test, inversion—in $\mathbb{Q}(t)$ can be turned into a randomized evaluation-interpolation algorithm, simply by evaluating the input at $t = u$ and reducing it in $\mathbb{F}_p$, for several $p$ and $u$, and proceeding to the computation over $\mathbb{F}_p$. Indeed, the execution of the algorithm requires a finite number of operations, either field operations, which commute with $\nu$, or zero test. For generic values of $p$ and $u$, these tests yield the same result on evaluated or unevaluated data. For specific values of $p$ and $u$, a non-zero quantity can be evaluated to zero, so the computation over $\mathbb{F}_p$ may fail or return a result which is not the evaluation of the result of the computation over $\mathbb{Q}(t)$. It is important to be able to test that in order to exclude bad evaluations because the reconstruction process does not handle possibly wrong evaluations.

The number of evaluation points $(p, u)$ is chosen, *a priori* or on-the-fly, so that the reconstruction of the outputs is possible with high probability of success. If *a priori* bounds on the output are known it may be possible to certify the result. If no bounds are known, then the evaluation-interpolation algorithm may return a false result, but the probability of this event can be made arbitrarily small. This evaluation-interpolation approach is classical in computer algebra for avoiding the problem of coefficient swell.

Algorithm 2 depends on the derivation $\delta$, which is not a field operation, so the conversion to an evaluation-interpolation algorithm is not completely straightforward.

7.2.1. *Principle.* Let $u$ be in $\mathbb{Q}$ and $p$ be a prime number. Let $\nu$ be the partial function $\mathbb{Q}(t) \to \mathbb{F}_p$, which consists in evaluating $t$ in $u$ and reducing modulo $p$. The function $\nu$ is extended coefficient-wise to $\mathbb{Q}(t)[\mathbf{x}]$, $\Omega$, matrices, etc.

Let $f$ be a polynomial in $\mathbb{Z}[t][\mathbf{x}]$, and $\nu(f)$ be its evaluation in $\mathbb{F}_p[\mathbf{x}]$. We can consider the reductions $[\,]_r$ associated to $f$, but also the *evaluated* reduction, denoted $[\,]_r^\nu$, associated to $\nu(f)$, over $\mathbb{F}_p$. Given $\alpha \in T^{n+1}$, and for generic values of $p$ and $u$, the evaluations $\nu(\alpha)$ and $\nu([\alpha]_r)$ are defined and $\nu([\alpha]_r) = [\nu(\alpha)]_r^\nu$. However, the value of $\nu(\delta a)$ for some form $a$ cannot be deduced from $\nu(a)$, so Algorithm 2 requires an adaptation to fit into an evaluation-interpolation scheme.

As in Section 6, let $R = a/f^q$ be a rational function in $\mathbb{Q}(t)$, homogeneous of degree $-n-1$ with respect to the variables $\mathbf{x}$. Let $\alpha$ be $a\omega$. Once the value of $r$ is fixed, Algorithm 2 computes the terms of the sequence $(\rho_i)_{i\in\mathbb{N}}$, defined by $\rho_0 = [\alpha]_r$ and $\rho_{i+1} = [\delta(\rho_i)]_r$, until it finds a linear dependency relation between the $\rho_i$. For a prime $p$ and an evaluation point $u$, can we compute $\nu(\rho_i)$ using only operations in $\mathbb{F}_p$? The answer seems to be negative, but there are two ways to circumvent this issue.

The first one is to define $\rho_i$ to be $[\delta^i(\alpha)]_r$. With this definition, the principle and the halting condition $\deg \rho_i \leqslant nN$ of Algorithm 2 remain valid. And given $\nu(\delta^i(\alpha))$, which is certainly easy to compute, it is possible in this case to compute $\nu(\rho_i)$ using only operations in $\mathbb{F}_p$. This approach is feasible but it becomes terrible if $i$ reaches high values: indeed, the degree of $\delta^i(\alpha)$ is $\deg \alpha + iN$.

Another approach is to compute the matrix of the linear map, say $m$, such that

$$\rho_{i+1} = \rho_i^\delta + m(\rho_i),$$

where $\rho_i^\delta$ denotes the component-wise differentiation of $\rho_i$, as opposed to $\delta(\rho_i)$ which is $\rho_i^\delta - f^\delta \rho_i$. Such a linear map exists and its matrix in a certain basis can be computed by evaluation-interpolation.

7.2.2. *The matrix of $\delta$.* Let $J_r$ be the image $[T^{n+1}]_r$ of the reduction map $[\,]_r$. By construction, the reduction $[\,]_r$ is idempotent, that is $[\alpha]_r = \alpha$ for all $\alpha \in J_r$. The evaluation-interpolation algorithm relies on the following property of the reduction map $[\,]_r$:

*Algorithm 4.* Computation of annihilating operators of the periods of a rational function, randomized evaluation-interpolation method

*Input* — $R = a/f^q$ a rational function in $\mathbb{Q}(t)(\mathbf{x})$, homogeneous of degree $-n-1$ w.r.t. $\mathbf{x}$
*Output* — $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ an annihilating operator of $\oint R$, with high probability
**procedure** PicardFuchs($a/f^q$)
  **loop**
    $p \leftarrow$ random prime number
    Compute $\mathcal{M}$, $\rho_0$ and $\operatorname{Mat}_{\mathcal{M}} m$, as defined in §7.2.2, over $\mathbb{F}_p(t)$ by repeated evaluation of $t$ and rational interpolation.
    Compute $\rho_0, \rho_1, \ldots$ over $\mathbb{F}_p(t)$, with $\rho_{i+1} = \rho_i^\delta - m(\rho_i)$, until finding a relation $\rho_n + \sum_{i=0}^{n-1} a_i\rho_i = 0$ over $\mathbb{F}_p(t)$.
    Using the Chinese remainder theorem and computations modulo previous values of $p$, try to lift the $a_i$ in $\mathbb{Q}(t)$.
    **if** possible **then**
      **return** the lifting.

**Proposition 33.** *The space $J_r$ is stable under component-wise differentiation.*

*Sketch of the proof.* This is a consequence of the fact that $J_r$ is generated by monomials. More precisely, let $E$ be the, finite of infinite, minimal sequence $(b_0, \ldots)$ of monomials of $T^{n+1}$ which generates $T^{n+1}/\ker[\,]_r$; minimal with respect to the lexicographic order on sequences of monomials, where the monomials are compared with $\prec$. Then $E$ is a basis of $J_r$ containing only monomials. □

As a consequence $[\delta(\rho)]_r = \rho^\delta - [f^\delta\rho]_r$, for all $\rho \in J_r$.

Let $\mathcal{M}$ be the least set of monomials of $T^{n+1}$ such that $\operatorname{Vect}\mathcal{M}$ contains $\rho_0$ and is stable under the map $m : \rho \mapsto [f^\delta\rho]_r$, and let $B$ be the matrix in $\mathbb{Q}(t)^{\mathcal{M}\times\mathcal{M}}$ of the map $m_{|\operatorname{Vect}\mathcal{M}}$ in the basis $\mathcal{M}$. For generic values of $p$ and $u$, the basis $\mathcal{M}$, the matrix $\nu(B)$ and $\nu(\rho_0)$ are all computable using only operations in $\mathbb{F}_p$, once given $\nu(f)$, $\nu(f^\delta)$ and $\nu(\alpha)$. Once $\mathcal{M}$, $B$ and $\rho_0$ are reconstructed over $\mathbb{Q}(t)$, the $\rho_i$ are easily computed with $\rho_{i+1} = \rho_i^\delta - m(\rho_i)$, and the minimal operator $\mathcal{L} = \sum_i a_i(t)\delta^i$ such that $\sum_i a_i(t)\rho_i = 0$ can be deduced. It seems to be a good idea to reconstruct $B$ and $\rho_0$ over $\mathbb{F}_p(t)$ and compute $\mathcal{L}$ modulo $p$, and only then to use several moduli to reconstruct $\mathcal{L}$ over $\mathbb{Q}(t)$. The full procedure is summarized by Algorithm 4.

7.2.3. *Estimation of the probability of success.* Let $\mathcal{M}$, $\rho_0$ and $A = \operatorname{Mat}_{\mathcal{M}} m$ as in §7.2.2, computed over $\mathbb{Q}(t)$. For some $u$ in $\mathbb{Q}$ and some prime $p$, let $\mathcal{M}'$, $\rho_0'$ and $A'$ be the analogues computed over $\mathbb{F}_p$. It is not hard to check that $\nu(\ker[\,]_r)$ equals $\ker[\,]_r^\nu$, where $\nu(\ker[\,]_r)$ is the set of all $\alpha$ in $\ker[\,]_r$ such that $\nu(\alpha)$ is defined. Let $\alpha$ be an element of $T^{n+1}$, whose coefficients are polynomials in $t$ with integer coefficients. Do we have $\nu([\alpha]_r) = [\nu(\alpha)]_r^\nu$? The fact that $J_r$ is generated by monomials implies that $[\alpha]_r$ equals $\operatorname{rem}(\alpha, \ker[\,]_r)$, and that $[\nu(\alpha)]_r^\nu$ equals $\operatorname{rem}(\alpha, \ker[\,]_r^\nu)$. The equality is equivalent to $\nu(\operatorname{rem}(\alpha, \ker[\,]_r)) = \operatorname{rem}(\alpha, \nu(\ker[\,]_r))$. A sufficient condition is that the set $L$ of leading monomials of elements of $\ker[\,]_r$ equals the set $L'$ of leading monomials of $\nu(\ker[\,]_r)$. Since $\mathcal{M}$ (resp. $\mathcal{M}'$) is the complement of $L$ (resp. $L'$) in the set of all monomials of $T^{n+1}$, we obtain

**Lemma 34.** *If $\mathcal{M} = \mathcal{M}'$ then $A' = \nu(A)$ and $\rho_0' = \nu(\rho_0)$.*

Let $P$ be the probability that $\mathcal{M}' = \mathcal{M}$. Assume for simplicity that $\deg \alpha \leqslant nN$ and that $J_r$ is included in $F_n T^{n+1}$. Let $V$ be the subspace $\ker[\,]_r \cap F_{n+1}T^{n+1}$ and let $\mathcal{B}$ be an echelonized basis of $V$, formed by elements of $T^{n+1}$ whose coefficients are in $\mathbb{Z}[t]$. For the above equalities to hold, it is enough that for all $b$ in $\mathcal{B}$, the evaluation $\nu(\operatorname{lc} b)$ of the leading coefficient of $b$ is not zero.

Under the assumption, somewhat excessive, that for random $p$ and $u$ the $\nu(\operatorname{lc} b)$, with $b \in \mathcal{B}$ are independent and uniformly distributed in $\mathbb{F}_p$, the probability $P$ equals $(1 - \frac{1}{p})^{\#\mathcal{B}}$. Of course $\#\mathcal{B} \leqslant \dim F_{n+1}T^{n+1}$ and

$$\dim F_{n+1}T^{n+1} = \sum_{q=0}^{n+1} \binom{qN-1}{n} \leqslant \frac{(n+3/2)^{n+1}N^n}{(n+1)!}.$$

So that

$$P \geqslant \left(1 - \frac{1}{p}\right)^{\frac{5}{4}e^n N^n} \geqslant \exp\left(-\frac{5e^n N^n}{2p}\right). \tag{9}$$

So we will choose $p$ significantly bigger than $e^n N^n$ to have $P \ll 1$. The set $\mathcal{M}$ is not computed, so it is not possible to compare it with $\mathcal{M}'$. However, we can compare the different $\mathcal{M}'$ obtained for different values of $p$ and $u$. Typically, most of them will be mutually equal—and hopefully equal to $\mathcal{M}$—and a few will differ. We simply drop the pairs $(p, u)$ giving degenerated specialisation $\mathcal{M}'$.

7.3. **Computing partial certificates.** Recall that if $\mathcal{L} \in \mathbb{K}\langle\delta\rangle$ is an annihilating operator of $\oint a/f$, a certifate for $\mathcal{L}$ is a sequence $C_0, \dots, C_n$ of rational functions in $\mathbb{K}[\mathbf{x}, \frac{1}{f}]$ such that

$$\mathcal{L}(a/f) = \sum_{i=0}^{n} \partial_i C_i.$$

As already mentioned, a certificate is desirable because is allows to check *a posteriori* in a simple way that $\mathcal{L}$ annihilates $\oint a/f$, idependently of the algorithm used to obtain $\mathcal{L}$. However, a certificate is typically huge [9, Rem. 11] and computing a one is necessarily very costly. A compromise is possible: we may compute a certificate for each reduction $\rho_k$, as a $\beta_k \in T^n$ such that

$$\rho_k = \begin{cases} \alpha + D_f \beta_0 & \text{if } k = 0 \\ \delta(\rho_{k-1}) + D_f \beta_k & \text{if } k \geqslant 1. \end{cases} \tag{10}$$

Thus, to check that the output $\mathcal{L} = \sum_{k=0}^{n} a_k \delta^k$ of Algorithm 4 annihilates $\oint a/f$, it is enough to check Equation 10 for $k \leqslant r$ and to check that $\sum_k a_k \rho_k = 0$. The first checks imply that $\rho_k \equiv \delta^k \alpha$ modulo $D_f(T^n)$, and the last one implies that $\mathcal{L}(\alpha) \in D_f(T^n)$, and thus that $\mathcal{L}$ annihilates $\oint a/f$. Since the $\rho_k$'s are in $F_n T^{n+1}$, the $\beta_k$ are in $F_{n+r}T^n$ which ensures that their size is kept reasonnable.

It is possible to modify Algorithm 4 to compute these certificates $\beta_k$. With the notations of §7.2.2, it amounts to compute $\beta_0 \in T^n$ such that $\alpha = \rho_0 + D_f \beta_0$, and to compute some $\gamma_\mu \in T^n$, for $\mu \in \mathcal{M}$, such that $m(\mu) = f^\delta \mu + D_f(\gamma_\mu)$. Since $\rho_{k-1} = \delta(\rho_k) + m(\rho_k)$, it is possible to compute the $\beta_k$'s as linear combinations of the $\gamma_\mu$'s.

In the evaluation-interpolation scheme, it is possible to compute $\beta_0$ and the $\gamma_\mu$'s over $\mathbb{F}_p$, to reconstruct them over $\mathbb{F}_p(t)$, then to compute the $\beta_k$'s over $\mathbb{F}_p(t)$ and to reconstruct them over $\mathbb{Q}(t)$. Of course, it comes at an additional cost but a preliminary implementation seems to show that this cost is reasonnable.

## 8. Application to periods arising from mirror symmetry

Batyrev and Kreuzer [4] have recently constructed a family of 210 smooth Calabi–Yau varieties of dimension three with Hodge number $h^{1,1}$ equal to one. Their method is based on toric varieties of reflexive polytopes. To each variety is associated a one-parameter mirror family of varieties and we look for the Picard-Fuchs equation of a distinguished principal period. This computation is the first step toward the computation of other important invariants, like, mirror maps, *instanton* numbers, etc[7]. The 210 varieties gather together into 68 different classes of diffeomorphic manifolds [4, table 3]. The principal periods associated to diffeomorphic varieties need not coincide but they are typically expected to differ only by a rational change of variable.

In concrete terms, we look for a differential equation satisfied by periods of rational integrals in the form

$$F(t) \stackrel{\text{def}}{=} \oint_\gamma \frac{1}{1 - tg(x_1, \ldots, x_4)} \frac{\mathrm{d}x_1}{x_1} \frac{\mathrm{d}x_2}{x_2} \frac{\mathrm{d}x_3}{x_3} \frac{\mathrm{d}x_4}{x_4}, \tag{11}$$

where $g$ is a Laurent polynomial and the integral is taken over the cycle $\gamma$ defined by $|x_i| = \varepsilon$, with $\varepsilon$ a small positive real number. Here $g$ is $\sum_v x^v$, where the sum ranges over the vertices of a reflexive lattice polytope. For the 210 polytopes under consideration, Batyrev and Kreuzer claim that $F(t)$ satisfies a linear differential equation of order 4, as a consequence of $h^{1,1}$ being 1. Moreover, this differential equation should have maximally unipotent monodromy at $t = 0$.

A power series expansion of the integrand with respect to $t$ shows that

$$F(t) = \sum_n \mathrm{ct}(g^n) t^n, \tag{12}$$

where $\mathrm{ct}(g^n)$ stands for the constant term of $f^n$. Batyrev and Kreuzer have computed Picard-Fuchs operators for topologies #37, #40 and #43–68 of their list. They used the *guessing* method presented in the introduction: they computed the power series expansion of $F(t)$, using equation (12), until they reached a degree $d$ such that they could find a non-zero solution to the equation

$$\left( \sum_{i=0}^{4} \sum_{j=0}^{d} a_{i,j} t^j \theta^i \right) \cdot F(t) = \mathcal{O}(t^{5(d+1)+1}).$$

The issue with this technique is not the reconstruction step which can be done efficiently—with respect to the size of the computed operator—but the computation of the power series expansion: the number of monomials in $g^k$ is $\Theta(k^4)$, so the computation of $N$ terms of $F(t)$ with this technique take $\Theta(N^5)$ operations in $\mathbb{Z}$, and we may add an order of magnitude to reflect the binary complexity.

Metelitsyn [36] computed four more equations for topologies #24, #38, #39 and #41. His method is also guessing, with modular evaluation techniques, but he managed to improve the space complexity, not the time complexity though, in the power expansion step and he provided an implementation optimized with Gpu programming. Moreover, Almkvist [1] reports that Straten, Metelitsyn and Schömer

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[7] For an introduction to the topic, see [15; 5].

have computed one operator for the topology #17. To the best of my knowledge, no other computation succeeded in the remaining topologies (#1–16, #18-23 , #25–36, #42).

With the implementation described in Section 7, I have been able to compute a differential equation for the 136 remaining integrals, associated to 35 different topologies.[8]

8.1. **Minimal equation and crosschecking.** The equations obtained from the algorithm are not always minimal, for two reasons. Firstly they were obtained with $r = 2$ but a higher value might have caught a lower order equation. Secondly, the algorithm computes an annihilating operator of all the periods of a given rational function; a period associated to a given cycle may satisfy a lower order equation.

Nevertheless, once any differential equation $\mathcal{L}$ for $F(t)$ is obtained, it is easy to compute efficiently thousands of terms of its power series expansion: the relation $\mathcal{L}(F) = 0$ translates into a linear recurrence relation on the coefficients of the power series expansion and the initial conditions are given by Equation (12). Thus we may try to reconstruct the minimal equation $\mathcal{L}_0$. By contrast to the guessing method, the reconstructed equation $\mathcal{L}_0$ can be proven correct: it is enough to check that it is a right divisor of $\mathcal{L}$, and that it annihilates the first few terms[9] of $F(t)$. If the power series expansion does not reveal a lower order differential equation, we may conjecture that $\mathcal{L}$ is minimal. Proving it may be done using methods by van Hoeij [30], see §8.2.2 for an example.

Since Algorithm 4 is randomized, it is desirable to have criteria to crosscheck the result. The Picard-Fuchs equations of periods of rational integrals are known to have strong arithmetic properties: regular singularities with rational exponents and nilpotent $p$-curvature for all prime $p$, with a finite number of exceptions [31]. Checking these properties is a good confirmation of the correctness of the output: these properties are so strong that a bad reconstruction would most probably break them. In addition, the computation of many terms of the power series expansion of $F(t)$ using a annihilating operator $\mathcal{L}$ can also be used as a crosschecking: if the coefficients computed are all integers, as expected in view of Equation (12), this is also strong indication that the operator is indeed correct.

8.2. **Description of the results.** In depth treatment is a work in progress with Jean-Marie Maillard. This section presents two examples.[10]

8.2.1. *Topology #42, polytope v25.59.* For the period (11) with

$$g = wxyz + wxy + \frac{1}{wxy} + wxz + \frac{1}{wxz} + \frac{wy}{z} + \frac{z}{wy} + wy + \frac{1}{wy} + \frac{1}{wz} + w + \frac{1}{w} + \frac{xz}{y} + \frac{y}{xz} + \frac{1}{xy} + xz + \frac{1}{xz} + x + \frac{1}{x} + \frac{z}{y} + \frac{y}{z} + y + \frac{1}{y} + z + \frac{1}{z},$$

[8]The results are available at `http://pierre.lairez.fr/supp/periods`.

[9]Up to the maximal integral root of the indicial polynomial at zero of the right quotient of $\mathcal{L}$ by $\mathcal{L}_0$.

[10]There are two numberings. The first one, used in Table 3 of [4], numbers the 68 different topologies, ordered by increasing $h^{1,2}$ number, covering the 210 smooth Calabi-Yau threefolds with Picard number 1. The second one, used in the database `http://hep.itp.tuwien.ac.at/~kreuzer/math/0802`, numbers in the form v$x.y$ the 198849 reflexive 4D polytopes satisfying an extra property. The letter $x$ indicates the number of vertices.

where the first few terms of the power series expansion are

$$F(t) = 1 + 22t^2 + 204t^3 + 3474t^4 + 57000t^5 + 1031080t^6 + 19368720t^7 + \mathcal{O}(t^8).$$

I have computed the following Picard-Fuchs equation

$$\begin{aligned}
&t^3(7t+1)^2(25t-1)^2(2t+1)^3(101t+43)^3(3t+1)^3\partial^4 \\
&\quad + 2t^2(7t+1)(25t-1)(2t+1)^2(101t+43)^2(3t+1)^2(848400t^5 \\
&\qquad + 1012956t^4 + 413041t^3 + 62473t^2 + 1819t - 129)\partial^3 \\
&+ t(7t+1)(25t-1)(2t+1)(101t+43)(3t+1)(4627173600t^8 + 10573386192t^7 \\
&\quad + 10004988192t^6 + 5027593832t^5 + 1423146511t^4 + 219009622t^3 \\
&\qquad + 15394840t^2 + 182234t - 12943)\partial^2 \\
&+ (7t+1)(25t-1)(2t+1)(101t+43)(3t+1)(6169564800t^8 + 13061530080t^7 \\
&\quad + 11311205016t^6 + 5112706620t^5 + 1268815538t^4 + 164341135t^3 \\
&\qquad + 9051543t^2 + 74605t - 1849)\partial \\
&+ 8t(7t+1)(25t-1)(2t+1)(101t+43)(3t+1)(192798900t^6 + 375787872t^5 \\
&\quad + 294032949t^4 + 116697469t^3 + 24254991t^2 + 2406495t + 81356),
\end{aligned}$$

or, with $\theta = t\partial$, in a form which highlights the maximally unipotent monodromy,

$$\begin{aligned}
&1849\theta^4 - 43t\theta(142\theta^3 + 890\theta^2 + 574\theta + 129) \\
&\quad - t^2(6472690\theta^4 + 24418180\theta^3 + 35385030\theta^2 + 24239530\theta + 650848) \\
&\quad - t^3(72000000\theta^4 + 344239080\theta^3 + 653378980\theta^2 + 573793290\theta + 19251960) \\
&\quad - t^4(376107650\theta^4 + 2200299640\theta^3 + 4997812640\theta^2 + 5113935450\theta + 194039928) \\
&\quad - 2t^5(\theta+1)(549781210\theta^3 + 3247373700\theta^2 + 6650662260\theta + 466789876) \\
&\quad - t^6(\theta+2)(\theta+1)(1851815470\theta^2 + 9159314250\theta + 1176131796) \\
&\quad - 1212t^7(1389790\theta + 413408)(\theta+3)(\theta+2)(\theta+1) \\
&\quad - 64266300t^8(\theta+4)(\theta+3)(\theta+2)(\theta+1).
\end{aligned}$$

This equation satisfies the conditions given by Almkvist, Enckevort, Straten, and Zudilin [2] and it is not in their database [44]. The computation took 80 seconds and 30 megabytes of memory on a laptop.

Note that formula (11), and homogeneization, give a rational function $a/f$ with $f$ of degree 8 with respect to the integration variables. The change of variables which maps $x$ to $1/x$ and $w$ to $w/y$ lowers this degree down to 5. This improves dramatically the computation time. This kind of monomial substitution can be found by random trials and errors. Among the substitutions that lead to degree 5, some are better than others in terms of computation time; but this seems hard to predict.

8.2.2. *Topology #27, polytope v23.289.* For the period (11) with

$$\begin{aligned}
f = \frac{1}{w} + w + \frac{1}{x} + \frac{w}{x} + x + \frac{x}{w} + \frac{1}{y} + \frac{w}{y} + \frac{1}{xy} + \frac{w}{xy} + y + \frac{y}{w} + \frac{xy}{w} \\
+ \frac{1}{z} + \frac{w}{z} + \frac{x}{z} + \frac{1}{yz} + \frac{w}{yz} + \frac{w}{xyz} + z + \frac{z}{w} + \frac{z}{x} + \frac{z}{wx},
\end{aligned}$$

where the first few terms of the power series expansion are

$$F(t) = 1 + 18t^2 + 138t^3 + 2070t^4 + 29040t^5 + 452610t^6 + 7308000t^7 + \mathcal{O}(t^8),$$

I have computed an annihilating operator of order 6 and degree 29, let us denote it $\mathcal{L}_6$, which is too large to be reproduced here. The operator is not of order 4 and has not maximally unipotent monodromy. Is it the minimal equation of $F(t)$? Van Hoeij has proved[11] that if $\mathcal{L}_6$ admits a right factor of order 4 then the degree of the coefficients of this factor is at most 88. Thus, admitting that $\mathcal{L}_6$ is indeed an annihilating operator of $F(t)$, if the minimal annihilating operator of $F(t)$ has order 4, it would have degree at most 88. Zero being the only solution to the system of linear equations

$$\sum_{i=0}^{4}\sum_{j=0}^{88} a_{i,j} t^j f^{(i)}(t) = \mathcal{O}(t^{405}),$$

where the unknowns are the $a_{i,j}$, this shows that the minimal annihilating operator of $F(t)$ is not of order 4. The argument holds for orders 1, 2, 3 and 5 with respective degree bounds 10, 16, 45 and 125. This is rather surprising since it contradicts the claims of Batyrev and Kreuzer. The topology #17, polytope v18.16766, shows the same behavior with a minimal equation of order 6. This has been first reported by Almkvist [1], referring to a computation by Straten, Metelitsyn and Schömer. As Almkvist wrote about topology #17, “this example leaves some doubts about the reflexive polytopes.” I can only corroborate. The remaining operators have not been studied in depth yet, but it seems that only one of the 137 newly computed periods has a minimal equation of order 4.

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