Comments on Finite-time Stability of Time-Delay Systems

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Abstract

Recently proposed conditions on finite-time stability in time-delay systems are revisited and it is shown that they are incorrect. General comments on possibility of finite-time convergence in time-delay systems and a necessary condition are given.

1. Introduction

The time-delay dynamical systems attract a lot of attention in different areas of practice Chiasson and Loiseau [3], Erneux [6]. Analysis of stability in these systems is complicated Richard [16], the most of stability conditions deal with linear time-delay models, where the necessary and sufficient conditions have been obtained for some special cases Gu et al. [7], Hale [8], Kolmanovsky and Nosov [11]. Prior to N.N. Krasovskii's papers on Lyapunov functionals and B.S. Razumikhin's papers on Lyapunov functions, L.E. El'sgol'ts (see El'sgol'ts and Norkin [5] and references therein) considered the stability problem of the solution $x(t) \equiv 0$ of time-delay systems by proving that the function $t \to V(x(t))$ is decreasing. Here V is some Lyapunov function. He showed that it is only possible in some rare special cases. Therefore, there are two generic methods for stability analysis in time-delay systems: Lyapunov-Krasovskii approach and Lyapunov-Razumikhin method. The former one is based on analysis of derivative for a functional, and it provides *qualitative* and *quantitative* estimates on the system convergence. The latter approach is based on derivative analysis of a function and, from the point of view of the convergence rate, it gives mainly a *qualitative* conclusion (stability/instability of a time-delay system can be detected without estimation of the convergence rate, see Theorem 1 below).

It is frequently important to quantify the rate of convergence in the system: exponential, asymptotic, finite-time or fixed-time (see the results obtained for ordinary differential equations in Roxin [17], Dorato [4], Moulay and Perruquetti [13], Nersesov et al. [14], Polyakov [15]). Frequently, the homogeneity theory is used to evaluate finite-time or fixed-time stability in the delay-free case Bhat and Bernstein [2], Andrieu et al. [1], Polyakov [15]: for example, if a system is globally attractive and homogeneous of negative degree, then it is finite-time stable. There is a recent interest to analysis of finite-time stability behavior for time-delay systems Karafyllis [9], Moulay et al. [12], Yang and Wang [18, 19]. The paper Karafyllis [9] proposes design of a control, which implicitly contains some prediction mechanisms and time-varying gains in order to compensate the delay influence on the system dynamics and guarantee a kind of finite-time stability for the closed-loop system. The main result of Moulay et al. [12] is given in Proposition 2 below: in order to establish finite-time stability for a functional differential equation it is necessary to find a Lyapunov-Krasovskii functional $V(\phi)$ whose derivative is upper bounded by a certain negative function of the functional $V(\phi)$ itself (it is a more restrictive condition than in the conventional Lyapunov-Krasovskii approach, where a function of $|\phi(0)|$ is required). Yang and Wang [18, 19] base their study on the Lyapunov-Razumikhin approach, they claim improvements over earlier results.

In this note, we argue that some key results in Yang and Wang [18, 19] are incorrect as stated, and we provide new insight on the features of finite-time stability for time-delay systems. The El'sgol'ts' arguments are recalled.

The outline of this work is as follows. The preliminary definitions and finite-time stability conditions for time-delay systems are given in Section 2. A counterexample to a key result in Yang and Wang [18, 19] is presented and discussed in Section 3. A necessary condition for finite-time stability for a class of time-delay systems and some supplementary comments are provided in Section 4. The result of Moulay et al. [12] is quoted in concluding Section 5.

2. Preliminaries

Consider an autonomous functional differential equation of the retarded type Kolmanovsky and Nosov [11]:

$$dx(t)/dt = f(x_t),\ t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $x_t \in C_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ (we denote by $C_{[-\tau,0]}$, $0 < \tau < +\infty$ the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm $||\phi|| = \sup_{-\tau \leq \varsigma \leq 0} |\phi(\varsigma)|$, where $|\cdot|$ is the standard Euclidean norm); $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is a continuous function, $f(0) = 0$. The representation (1) includes pointwise or distributed retarded systems. We assume that (1) has a solution $x(t, x_0)$ satisfying the initial condition $x_0 \in C_{[-\tau,0]}$, which is defined on some finite time interval $[-\tau, T)$ (we will use the notation $x(t)$ to reference $x(t, x_0)$ if the origin of x_0 is clear from the context).

The upper right-hand Dini derivative of a locally Lipschitz continuous functional $V : C_{[-\tau,0]} \to \mathbb{R}_+$ along the system (1) solutions is defined as follows for any $\phi \in C_{[-\tau,0]}$:

$$D^+V(\phi) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(\phi_h) - V(\phi)],$$

where $\phi_h \in C_{[-\tau,0]}$ for $0 < h < \tau$ is given by

$$\phi_h = \begin{cases} \phi(\theta + h), & \theta \in [-\tau, -h) \\ \phi(0) + f(\phi)(\theta + h), & \theta \in [-h, 0]. \end{cases}$$

For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ the upper directional Dini derivative is defined as follows:

$$D^+V[x_t(0)]f(x_t) = \lim_{h \to 0^+} \sup \frac{V[x_t(0) + hf(x_t)] - V[x_t(0)]}{h}.$$

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to the class $\mathcal{K}_\infty$ if it is also radially unbounded.

2.1. Stability definitions

Let Ω be an open subset of $C_{[-\tau,0]}$ containing 0.

Definition 1. Moulay et al. [12] The equilibrium $x = 0$ of (1) is said to be

(a) stable if there is $\sigma \in \mathcal{K}$ such that for any $x_0 \in \Omega$, $|x(t, x_0)| \leq \sigma(||x_0||)$ for all $t \geq 0$;

(*b*) asymptotically stable if it is stable and $\lim_{t\to+\infty}|x(t,x_0)|=0$ for any $x_0\in\Omega$;

(*c*) finite-time stable if it is stable and for any $x_0\in\Omega$ there exists $0\le T^{x_0}<+\infty$ such that $x(t,x_0)=0$ for all $t\ge T^{x_0}$. The functional $T_0(x_0)=\inf\{T^{x_0}\ge 0 : x(t,x_0)=0\ \forall t\ge T^{x_0}\}$ is called the settling time of the system (1).

If $\Omega=C_{[-\tau,0]}$, then the origin is called *globally* stable/asymptotically stable/finite-time stable.

For the forthcoming analysis we will need Lyapunov-Razumikhin theorem, which is given below (we have adapted to our case the formulation of Gu et al. [7], where time-dependent functional differential equations are considered).

Theorem 1. *Gu et al. [7] Let $\alpha_1,\alpha_2\in\mathcal{K}$ and $\eta:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous nondecreasing function. If there exists a Lipschitz continuous function $V:\mathbb{R}^n\to\mathbb{R}$ such that*

$$\alpha_1(|x|)\le V(x)\le\alpha_2(|x|)\quad\forall x\in\mathbb{R}^n$$

and the derivative of V along a solution $x(t)$ of (1) satisfies

$$D^+V[x(t)]f(x_t)\le-\eta(|x(t)|)\ if\ V[x(t+\theta)]\le V[x(t)]\tag{2}$$

for $\theta\in[-\tau,0]$, then (1) is stable at the origin.

If, in addition, $\eta\in\mathcal{K}$ and there exists a continuous nondecreasing function $p(s)>s$ for $s>0$ such that the condition (2) is strengthened to

$$D^+V[x(t)]f(x_t)\le-\eta(|x(t)|)\ if\ V[x(t+\theta)]\le p\{V[x(t)]\}$$

for $\theta\in[-\tau,0]$, then (1) is asymptotically stable at the origin.

If in addition $\alpha_1\in\mathcal{K}_\infty$, then (1) is globally asymptotically stable.

2.2. Sufficient conditions of finite-time stability in time-delay systems

The following is stated in Yang and Wang [19] as Lemma 1 (an extension of the Lyapunov-Razumikhin method for analysis of finite-time stability in (1)):

"*Consider the system (1) with $f(\phi)=F(\phi(0),\phi(-\tau))$, $\phi\in\Omega$, $F(0,0)=0$ and uniqueness of the solution in forward time. If there exist real numbers $\beta>1$, $k>0$, a Class-$\mathcal{K}$ function σ and a differentiable Lyapunov function, $V(x)$, of the system (1) such that*

$$\begin{aligned}\sigma(|x|)&\le V(x),\\ \dot V&\le-kV^{\beta^{-1}}(x),\ x\in\Omega\end{aligned}\tag{3}$$

hold along the trajectory of the system whenever $V[x(t+\theta)]\le V[x(t)]$ for $\theta\in[-\tau,0]$, then the system (1) is finite-time stable. If $\Omega=\mathbb{R}^n$ and σ is a Class-$\mathcal{K}_\infty$ function, then the origin is a globally finite-time stable equilibrium of the system

(1). *Furthermore, the settling time of the system* (1) *with respect to the initial condition* $\phi \in C_\delta$ *satisfies*

$$T_0(\phi) \leq \frac{\beta}{k(\beta-1)} V^{\frac{\beta-1}{\beta}}[\phi]$$

for all $t \geq 0$."

For completeness we are going to give the "proof" of this lemma from Yang and Wang [19]:

"Since $V(x)$ is a Lyapunov function for the system (1), applying Razumikhin Theorem Gu et al. [7], it is easy to know that the system (1) is asymptotically stable under the conditions of the lemma. Next, we need to prove that $T_0(\phi) < +\infty$. Based on Condition for $\dot{V}$, one can obtain $\int_{V(\phi)}^{0} \frac{dz}{z^{1/\beta}} \leq -k \int_0^t d\tau$, from which it follows that $T_0(\phi) \leq \frac{\beta}{k(\beta-1)} V^{\frac{\beta-1}{\beta}}[\phi]$ for all $t \geq 0$. Thus, the proof is completed."

In Yang and Wang [18] the same conclusion is obtained for the time-varying system (1) using a similar argumentation.

3. Comments on Yang and Wang [19]

We claim that Lemma 1 in Yang and Wang [19] is incorrect as stated. Indeed, as we can conclude from the result of Theorem 1, Lemma 1 in Yang and Wang [19] is based on the condition (2), which allows only stability (non asymptotic) to be concluded. However, even if we would ask the inequality (3) to be satisfied whenever $V[\phi(\theta)] \leq p\{V[\phi(0)]\}$ for all $\theta \in [-\tau, 0]$, where $p : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function such that $p(s) > s$ for all $s > 0$, then only asymptotic stability can be proven following Theorem 1. The claim about a finite-time convergence stays wrong since for a solution $x(t, x_0)$ with initial condition $x_0 \in \Omega$ the Lyapunov-Razumikhin condition (3) defines the rate of convergence of V only for the set of time instants

$$\mathrm{T}_{x_0}^{+} = \left\{ t \in \mathbb{R}_+ : \sup_{\theta \in [-\tau, 0]} V[x(t+\theta, x_0)] \leq p\{V[x(t, x_0)]\} \right\}$$

while for $t \in \mathrm{T}_{x_0}^{-}$ with

$$\mathrm{T}_{x_0}^{-} = \left\{ t \in \mathbb{R}_+ : \sup_{\theta \in [-\tau, 0]} V[x(t+\theta, x_0)] > p\{V[x(t, x_0)]\} \right\}$$

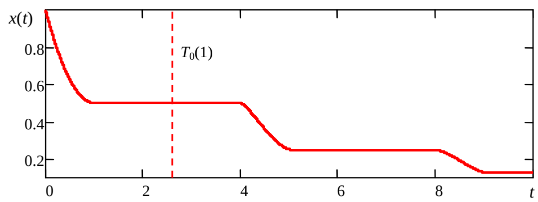

Figure 1: A trajectory of the counterexample (4)

there is no restriction on convergence rate of V. It is exactly the arguments used in the "proof" of Lemma 1 from Yang and Wang [19] given above: a finite-time rate of convergence is established for $t \in \mathrm{T}^+_{x_0}$ only, which is clearly not sufficient since, for an illustration, if $V[x(t)]$ is strictly decreasing then $\mathrm{T}^-_{x_0} = [0, +\infty)$.

For another illustration consider the following counterexample:

$$\dot{x}(t) = -|2x(t) - x(t-\tau)|^{0.5}\mathrm{sign}[2x(t) - x(t-\tau)]. \tag{4}$$

Using $V(x) = 0.5x^2$ we obtain

$$\dot{V} \leq -\sqrt{2-\sqrt{2}}|x|^{1.5} = -kV^{\beta^{-1}} \text{ if } V[x(t-\tau)] \leq 2V[x(t)]$$

for $\beta = 4/3$ and $k = \sqrt{2-\sqrt{2}}\,2^{3/4}$. Thus, by Theorem 1 the system is globally asymptotically stable, and from Yang and Wang [18, 19] since all conditions of Lemma 1 Yang and Wang [19] are also satisfied, one would conclude that the system is finite-time stable. Consequently, for the initial conditions $x_0 \in C_{[-\tau,0]}$ such that $|x_0(0)| \leq 1$ the settling time function would possess the estimate:

$$T_0(x_0) \leq 3.$$

Take $\tau > 3$ and for any $\delta \in [-1, 1] \setminus \{0\}$ select initial conditions

$$x_0(s) = \begin{cases} 2\delta, & s \in [-\tau, -\tau+3] \\ \frac{\delta}{3-\tau}s + \delta, & s \in (-\tau+3, 0] \end{cases}.$$

Obviously, $\dot{x}(t, x_0) = 0$ and $x(t, x_0) = x_0(0) = \delta \neq 0$ for all $t \in [0, T_0(x_0)]$, therefore *the given settling time estimate is invalid.* In addition, the results of the system simulation for $x_0(s) = 1$ for $s \in [-\tau, 0]$ are shown in Fig. 1 for $\tau = 4$ and the simulation step $h = 10^{-5}$. Clearly the system is not finite-time stable and the settling time estimate is wrong.

A peculiarity of the Lyapunov-Razumikhin conditions of stability is that they are delay-independent (see Theorem 1), thus the estimate on the settling time $T_0(x_0)$ obtained in Lemma 1 of Yang and Wang [19] is also delay independent. Consequently, it is possible to select delay value τ for a given initial conditions $x_0 \in C_{[-\tau,0]}$ such that $T_0(x_0) < \tau$, as it has been performed in the counterexample above. Obviously, convergence to zero independently on the part of initial conditions $x_0(t)$ with $t \in [T_0(x_0)-\tau, 0]$ is possible only under special restrictions on f in (1), which we are going to analyze in the next section.

4. About necessary conditions for finite-time stability in time-delay systems

Assume that the system (1) is finite-time stable for some $\tau > 0$ and the settling-time functional is continuous and $T_0(0) = 0$. Then there is $x_0 \in C_{[-\tau,0]}$ such that $T_0(x_0) \leq \tau$ and at the instant $T_0(x_0)$ the right-hand side of (1) is still dependent on the initial conditions x_0. Thus, without additional assumptions on the right-hand side $f(x_t)$ and its dependence on $x_t(0)$, or without skipping the continuity requirement of $T_0(x_0)$, an existence of finite-time stability phenomenon for time-delay systems is questionable.

In the remainder of this section we will consider

$$f(\phi) = F(\phi(0), \phi(-\tau))$$

for all $\phi \in \Omega$, where $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ is a continuous function. Our conjecture is that the condition $F(0, z) = 0$ for any $z \in \mathbb{R}^n$ is "necessary" for a finite-time convergence of (1) from Ω to the origin.

Remark 1. We would like to stress that requirement on continuity of F is crucial for the consideration below. Indeed in discontinuous time-delay systems the finite-time stability may be easily observed without such a restriction on F. For example, it is a simple exercise to check that the system

$$\dot{x}(t) = -(1 + |x(t-\tau)|)\mathrm{sign}[x(t)] + x(t-\tau)$$

is globally finite-time stable (take $V(x) = 0.5x^2$, $\dot{V} \leq -\sqrt{2V}$ and apply a variant of Proposition 2). In this discontinuous case $0 \in F(0, z)$ for any $z \in \mathbb{R}^n$.

Note that by the definition of finite-time convergence $x(t, x_0) = 0$ for all $t \geq T_0(x_0)$ and by the definition of $T_0(x_0)$ there is a non-empty set of time instants

$$\mathcal{T}_{x_0} = \{t \in [T_0(x_0) - \tau, T_0(x_0)] : x(t, x_0) \neq 0\}.$$

Proposition 1. *Let* (1) *be finite-time convergent in* Ω*, then*

$$\forall t \in \mathcal{T}_{x_0}: \ F[0, x(t, x_0)] = 0 \tag{5}$$

for any $x_0 \in \Omega$.

Proof. Take $x_0 \in \Omega$ and the corresponding settling time $T_0(x_0)$. Assume that the necessary condition (5) is not satisfied, its negation implies that for some $x_0 \in \Omega$ there exists $t' \in \mathcal{T}_{x_0}$ such that $F(0, x(t', x_0)) \neq 0$ (the measure of $\mathcal{T}_{x_0}$ is not zero since $x(t)$ and F both are continuous), then $\dot{x}(t' + \tau) \neq 0$ and $x(t, x_0) \neq 0$ for some $t \geq T_0(x_0)$ that is a contradiction. □

A simple, but not equivalent, way to check this condition in practice is to verify that

$$F(0, z) = 0 \tag{6}$$

for any $z \in \mathbb{R}^n$. In Kolmanovskii and Myshkis [10], a similar "necessary" condition has been indicated for the El'sgol'ts approach El'sgol'ts and Norkin [5]. Indeed, consider the case $n = 1$, take a Lyapunov function $V(x) = 0.5x^2$, which is a reasonable choice for the scalar case. We have $\dot{V}(t) = x(t)F[x(t), x(t-\tau)]$, if the inequality $\dot{V}(t) \leq 0$ is satisfied around the origin, then we necessarily obtain (6).

In Yang and Wang [18, 19] the restriction $F(0, 0) = 0$ has been imposed that is not sufficient.

5. Discussion

Finite-time stability can be presented in time-delay systems, but only under rather strong restrictions on the right-hand side of the system. The most general sufficient conditions are given in Moulay et al. [12]:

Proposition 2. *Moulay et al. [12] Let system* (1) *have unique solutions in forward time. If there exist a continuous functional* $V : \Omega \to \mathbb{R}_+$, $\epsilon > 0$ *and two functions* α, r *of class* $\mathcal{K}$ *such that* $\dot{z} = -r(z)$ *has a flow,* $\int_0^\epsilon \frac{dz}{r(z)} < +\infty$ *and for all* $\phi \in \Omega$

$$\alpha(|\phi(0)|) \leq V(\phi), \ D^+V(\phi) \leq -r[V(\phi)],$$

then the system (1) *is finite-time stable with a settling time functional* $T_0(\phi)$ *satisfying the inequality:*

$$T_0(\phi) \leq \int_0^{V(\phi)} \frac{dz}{r(z)}.$$

The following example has been given in Moulay et al. [12] for any $0 < \alpha < 1$:

$$\dot{x}(t) = -|x(t)|^\alpha \operatorname{sign}[x(t)]\{1 + x(t-\tau)^2\}. \tag{7}$$

Using the Lyapunov functional $V(\phi) = 0.5\phi(0)^2$ with $\dot{V}(\phi) \leq -2^{\frac{1+\alpha}{2}} V^{\frac{1+\alpha}{2}}(\phi)$ the finite-time stability has been established. Note that for this example the "necessary" condition (6) is satisfied.

The system (7) is an example, where the El'sgol'ts' arguments can be applied ($V(\phi) = 0.5\phi(0)^2$ is in fact a Lyapunov function). Note that the result of Lemma 1 in Yang and Wang [19] without the Razumikhin condition ("whenever...") is correct and in this case it is also a special case of Proposition 2. In such a reformulation the result extends El'sgol'ts and Norkin [5] to finite-time stability, as well as the finite-time stability results of Roxin [17] to time-delay systems.

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