# Interval estimation for systems with unknown input delays and gains

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***Abstract*—The estimation problem of a system with unknown time-delay and unknown input gains is considered. The interval observation technique is applied in order to obtain guaranteed interval of the system state. The proposed approach can be applied to linear delay systems and nonlinear time-delay systems in the output canonical form. The theoretical results are supported by numerical simulation and demonstration on example of thermic application.**

***Index Terms*—Interval observers, Quasi-monotone/Metzler/ co-operative systems, Time-delay systems, Uncertain input gains, Thermic application**

## I. INTRODUCTION

An unknown time-delay and an unknown input gains in models of control systems arise due to many reasons in various real-world applications. The time-delay may be related to transport delays (like in chemical, hydraulic or pneumatic systems) or computational delays (e.g. in digital controllers or communication networks) [1]. The uncertain gains are related with possible nonlinearities and identification errors. The problem of observer design for nonlinear systems with delays and uncertainty is rather complex [2]. Especially the observer synthesis is problematical for the cases when the model of a nonlinear delayed system contains parametric and signal uncertainties, or when the delay is time-varying or uncertain [3], [4], [5], [6], [7], [8], [9], [10], [11], [37]. An observer solution for this kind of problem is demanded, especially when the input gain of the system is also unknown.

Typically, in various applications the input delay and input gain are related. They both depend on the actuating system that may vary with time, environment or failures. An interval observer is expected to keep on providing a real-time estimation of the state variables in spite of these variations. Recent results[13], [12] have applied interval observers to the systems with unknown input delay and known parameters. The present work constitutes an extension to systems with unknown input delays and unknown (bounded) input gains.

The conventional observer in the absence of measurement noise and uncertainties has to converge to the exact value of the state of the estimated system (it gives a pointwise estimation of the state) [3]. In opposite, the interval observers evaluate at each time instant a set of admissible values for the state, consistently with the measured output (i.e. they provide an interval estimation) [14], [15], [16], which provides the guaranteed interval estimates of the system state in real-time. This property will simplify the control of transition processes with respect to system state. Such an interval estimation gives also a simultaneous estimation of the estimation error. An interval observer for time-delay systems can naturally deal with uncertain systems.

In some cases, the input gain has important impact in the design of observer which cannot be ignored. In this work, a new interval observer is presented, the case of time-varying uncertain delays is analyzed and the unknown input gains is additionally studied.

The paper is organized as follows. Some preliminaries are given is Section 2. The observer definition is given in Section 3, in the same section the observer design is performed. Last, in Section 4, an experimental example is considered: the results are applied to a thermic process, in which both transportation delay and electrothermic input gain depend on the external air flow.

## II. NOTATION AND DEFINITIONS

In the rest of the paper, the following definitions will be used:

- $\mathbb{R}$ is the Euclidean space ($\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$), $\mathcal{C}_\tau = C([-\tau, 0], \mathbb{R})$ is the set of continuous maps from $[-\tau, 0]$ into $\mathbb{R}$; $\mathcal{C}_{\tau+} = \{y \in \mathcal{C}_\tau :\ y(s) \in \mathbb{R}_+,\ s \in [-\tau, 0]\}$;
- $x_t$ is an element of $\mathcal{C}_\tau^n$ associated with a map $x_t : \mathbb{R} \to \mathbb{R}^n$ by $x_t(s) = x(t+s)$, for all $s \in [-\tau, 0]$;
- $|x|$ denotes the absolute value of $x \in \mathbb{R}$, $||x||$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$, $||\varphi|| = \sup_{t\in[-\tau,0]} ||\varphi(t)||$ for $\varphi \in \mathcal{C}_\tau^n$;
- for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}^p$ the symbol $||u||_{[t_0,t_1]}$ denotes its $L_\infty$ norm $||u||_{[t_0,t_1]} = ess\sup\{||u(t)||, t \in [t_0, t_1]\}$, or simply $||u||$ if $t_1 = +\infty$, the set of all such inputs $u \in \mathbb{R}^p$ with the property $||u|| < \infty$ will be denoted as $\mathcal{L}_\infty^p$ ;
- for a matrix $A \in \mathbb{R}^{n\times n}$ the vector of its eigenvalues is denoted as $\lambda(A)$;
- $E_n \in \mathbb{R}^n$ is stated for a vector with unit elements, $I_n$ and $0_n$ denote the identity and zero matrices of dimension $n \times n$ respectively;
- for two integers $n \leq N$ the symbol $\overline{n, N}$ denotes the sequence $n, n+1, \ldots, N-1, N$;
- $a\,\mathcal{R}\,b$ corresponds to an elementwise relation $\mathcal{R}$ ($a$ and $b$ are vectors or matrices): for example $a < b$ (vectors) means $\forall i :\ a_i < b_i$; for $\phi, \varphi \in \mathcal{C}_\tau^n$ the relation $\phi\,\mathcal{R}\,\varphi$ has

Si Chen and Zongxia Jiao are with University Beihang, 37, Avenue Zhichun, District Haidian, 100191, Beijing, Chine. Si Chen is with Ecole Centrale de Lille, Avenue Paul Langevin, 59651 Villeneuve d'Ascq, France. Denis Efimov and Jean-Pierre Richard are with Non-A project at Inria, Parc Scientifique de la Haute Borne, 40 avenue Halley, 59650 Villeneuve d'Ascq, France and with LAGIS UMR 8219, Ecole Centrale de Lille, Avenue Paul Langevin, BP 48, 59651 Villeneuve d'Ascq, France,

to be understood elementwise for all domain of definition of the functions, i.e. $\phi(s)\,\mathcal{R}\,\varphi(s)$ for all $s \in [-\tau, 0]$;
- denote $\underline{\mu}_{[\underline{\tau},\overline{\tau}]}h(t) = \inf_{s\in[\underline{\tau},\overline{\tau}]} h(t-s)$ and $\overline{\mu}_{[\underline{\tau},\overline{\tau}]}h(t) = \sup_{s\in[\underline{\tau},\overline{\tau}]} h(t-s)$ for a signal $h : \mathbb{R} \to \mathbb{R}$.

### A. Functional Differential Equation

A large number of processes can be modeled by a Functional Differential Equation (FDE):

$$\dot{x}(t) = f(t, x(t), x_t, d), \quad y(t) = h(t, x(t), x_t, d), \quad x_{t_0} = \varphi \in \mathcal{C}_\tau^n, \tag{1}$$

where $t \in \mathbb{R}$ is the time variable, $d \in \mathcal{S}_d$ is either a vector or a function representing disturbances or parameter uncertainties of the system, $\mathcal{S}_d \subset \mathcal{L}_\infty^q$ is a set of vectors or functions for which some bounds are usually supposed to be known, $x(t) \in \mathbb{R}^n$ is a vector of internal variables, $x_t \in \mathcal{C}_\tau^n$ and $\tau \in \mathbb{R}_+$ is the maximal delay, $y(t) \in \mathbb{R}^p$ is the output vector.

It is assumed that the system (1) has solutions (for example $f$ satisfies Carathéodory conditions, see [17]) defined over a maximal interval denoted by $\mathcal{I}_{(1)}(t_0, \varphi)$ where $t_0$ is the initial time and $\varphi$ is the initial function from $\mathcal{C}_\tau^n$.

### B. Comparison/cooperative systems

Following the Wazewski's contribution [18], which is probably one of the most important in this field concerning differential inequalities and giving necessary and sufficient hypotheses ensuring that the solution of $\dot{x} = f(t, x)$, with initial state $x_0$ at time $t_0$ and function $f$ satisfying the inequality $f(t, x) \le g(t, x)$ is overvalued by the solution of the so-called "comparison system" $\dot{z} = g(t, z)$, with initial state $z_0 \ge x_0$ at time $t_0$, or, in other words, the conditions on function $g$ that ensure $x(t) \le z(t)$ for $t \ge t_0$. These results were extended to many different classes of dynamical systems [19], [20], [21], [22], [23], [24]. Frequently these systems are also called monotone or cooperative [25]. Further in this subsection the exposition from [26] will be adopted.

Focusing on two systems:

$$\dot{x}(t) = f(t, x(t), x_t), \quad x(t) \in \mathbb{R}^n, \tag{2}$$

$$\dot{z}(t) = g(t, z(t), z_t), \quad z(t) \in \mathbb{R}^n, \tag{3}$$

the solutions of (3) with initial condition $\varphi_2$ and of (2) with initial condition $\varphi_1$ will be denoted as $z(t; t_0, \varphi_2)$ and $x(t; t_0, \varphi_1)$ respectively.

**Definition 1.** The system (3) is said to be a comparison system of (2) over $\Omega \subset \mathcal{C}_\tau^n$ if $\forall(\varphi_1, \varphi_2) \in \Omega^2$:

$$\mathcal{I} \neq \{t_0\},\ \mathcal{I} = \mathcal{I}_{(2)}(t_0, \varphi_1) \cap \mathcal{I}_{(3)}(t_0, \varphi_2),$$
$$\varphi_2 \ge \varphi_1 \Longrightarrow z(t; t_0, \varphi_2) \ge x(t; t_0, \varphi_1)\ \forall t \in \mathcal{I}.$$

Obviously, one can go beyond this concept to derive a qualitative analysis for positive solutions. For example, if $z(t; t_0, \varphi_2) \ge x(t; t_0, \varphi_1) \ge 0$ and if solution $z(t)$ converges to zero so does $x(t)$. A question naturally arises concerning the properties of the function $g$ ensuring that (3) is a comparison system of (2) over $\Omega$. For this, the following notion is required:

**Definition 2.** A functional

$$\begin{aligned} g \quad &: \quad \mathbb{R} \times \mathbb{R}^n \times \mathcal{C}_\tau^n \to \mathbb{R}^n \\ (t, x, y) \quad &\mapsto \quad g(t, x, y) \end{aligned}$$

is quasi-monotone non-decreasing in $x$ iff:

$$\forall t \in \mathbb{R}, \forall y \in \mathcal{C}_\tau^n, \forall (x, x') \in \mathbb{R}^n \times \mathbb{R}^n \forall i \in \overline{1, n} :$$
$$(x_i = x'_i) \wedge (x \le x') \Rightarrow g_i(t, x, y) \le g_i(t, x', y),$$

is non-decreasing in $y$ iff:

$$\forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n, \forall (y, y') \in \mathcal{C}_\tau^n \times \mathcal{C}_\tau^n :$$
$$y \le y' \Rightarrow g(t, x, y) \le g(t, x, y'),$$

is mixed quasi-monotone non-decreasing in $x$, non-decreasing in $y$ iff:

$$\forall t \in \mathbb{R}, \forall (x, x') \in \mathbb{R}^n \times \mathbb{R}^n, \forall (y, y') \in \mathcal{C}_\tau^n \times \mathcal{C}_\tau^n \forall i \in \overline{1, n} :$$
$$(x_i = x'_i) \wedge (x \le x') \wedge (y \le y') \Rightarrow (g_i(t, x, y) \le g_i(t, x', y')).$$

*Remark* 1. The latter definition is a special case of mixed quasimonotonicity given in [27]. More general versions also exist (see [28], [29]) and additional conditions are sometimes given (see [18]).

The following results may be easily proven.

**Lemma 1.** *A functional $g : (t, x, y) \mapsto g(t, x, y)$ is quasi-monotone non-decreasing in $x$ and non-decreasing in $y$ iff it is mixed quasi-monotone non-decreasing in $x$, non-decreasing in $y$.*

**Lemma 2.** *If $g$ is continuously differentiable with respect to $x$ and $y$, and $\forall t \in \mathbb{R}$, $\forall x \in \mathbb{R}^n$, $\forall y \in \mathcal{C}_\tau^n$*

$$\forall i \neq j : \frac{\partial g_i}{\partial x_j} \ge 0,\ \forall (i, j) : \frac{\partial g_i}{\partial y_j} \ge 0, \tag{4}$$

*then $g(t, x, y)$ is mixed quasi-monotone non-decreasing in $x$, non-decreasing in $y$.*

The following theorem states a comparison principle for functional differential equations.

**Theorem 1.** *Assume that:*
*H1) $\forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n, \forall y \in \mathcal{C}_\tau^n : \quad f(t, x, y) \le g(t, x, y)$,*
*H2) $g(t, x, y)$ is mixed quasi-monotone non-decreasing in $x$, non-decreasing in $y$,*
*H3) $g(t, x, y)$ is sufficiently smooth for (3) to possess, for every $z_{t_0} \in \Omega \subset \mathcal{C}_\tau^n$ and for every $t_0 \in \mathbb{R}$, a unique solution $z(t)$ for all $t \ge t_0$.*

*Then:*
*C1) For any $x_{t_0} \in \Omega$, the inequality $x(t) \le z(t)$ holds for every $t \ge t_0$ whenever it is satisfied for $t \in [t_0 - \tau, t_0]$. In other words, (3) is a comparison system of (2) over $\Omega$.*
*C2) Moreover, if $\forall t \ge t_0 : 0 \le g(t, 0, \varphi_0)$ and $z_{t_0} \ge 0$, then $0 \le z(t)$.*

*Remark* 2. One can refine the definitions given above by considering local comparison system and thus obtain a local version of this theorem (see [30], [31]).

### C. Linear cooperative systems with delays

Consider a linear system with constant delays and time-varying input

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{N} A_i x(t-\tau_i) + b(t), \qquad (5)$$

where $x(t) \in \mathbb{R}^n$ is the state, $x_t \in \mathcal{C}_\tau^n$ for $\tau = \max_{1\le i\le N} \tau_i$ where $\tau_i \in \mathbb{R}_+$ are the delays; a piecewise continuous function $b \in \mathcal{L}_\infty^n$ is the input; the constant matrices $A_i$, $i = \overline{0, N}$ have appropriate dimensions. The matrix $A_0$ is called Metzler if all its off-diagonal elements are nonnegative. The matrices $A_i$ are called nonnegative if $A_i \ge 0$ (elementwise). The function $g(t, x, x_t) = A_0 x(t) + \sum_{i=1}^{N} A_i x(t-\tau_i) + b(t)$ is mixed quasi-monotone non-decreasing in $x$, non-decreasing in $x_t$ if $A_0$ is Metzler and $A_i$, $i = \overline{1, N}$ are nonnegative.

**Definition 3.** The system (5) is called cooperative (or nonnegative [32]) if $A_0$ is Metzler and $A_i$, $i = \overline{1, N}$ are nonnegative matrices.

The cooperative system (5) admits $x(t) \in \mathbb{R}_+^n$ for all $t \ge t_0$ provided that $x_{t_0} \in \mathcal{C}_{\tau+}^n$ and $b : \mathbb{R} \to \mathbb{R}_+^n$.

**Lemma 3.** [33], [20], [32] *A cooperative system (5) is asymptotically stable for $b(t) \equiv 0$ for all $\tau \in \mathbb{R}_+$ iff there are $p, q \in \mathbb{R}_+^n (p > 0$ and $q > 0)$ such that*

$$p^T \sum_{i=0}^{N} A_i + q^T = 0.$$

Under conditions of the above lemma the system has bounded solutions for $b \in \mathcal{L}_\infty^n$ with $b(t) \in \mathbb{R}_+^n$ for all $t \in \mathbb{R}$.

**Lemma 4.** [16] *Given the matrices $A \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. If there is a matrix $L \in \mathbb{R}^{n\times p}$ such that the matrices $A - LC$ and $R$ have the same eigenvalues, then there is a $P \in \mathbb{R}^{n\times n}$ such that $R = P(A - LC)P^{-1}$ provided that the pairs $(A - LC, e_1)$ and $(R, e_2)$ are observable for some $e_1 \in \mathbb{R}^{1\times n}$, $e_2 \in \mathbb{R}^{1\times n}$.*

This result was used in [16] to design interval observers for LTI systems with a Metzler matrix $R$ (in other words, the lemma establishes the conditions when the matrix $A - LC$ is similar to a Metzler matrix). The main difficulty is to prove the existence of a real matrix $P$, and to provide a constructive approach of its calculation. In [16] the matrix $P = O_R O_{A-LC}^{-1}$, where $O_{A-LC}$ and $O_R$ are the observability matrices of the pairs $(A - LC, e_1)$ and $(R, e_2)$ respectively. Another (more strict) condition is that the Sylvester equation $PA - RP = QC$, $Q = PL$ has a unique solution $P$ provided that the pair $(A, C)$ is observable (in this case there exists a matrix $L$ such that $\lambda(A) \ne \lambda(A - LC) = \lambda(R)$, that is equivalent to existence of a unique $P$). Note that if the matrix $A - LC$ has only real positive eigenvalues, then $R$ can be chosen as diagonal or Jordan representation of $A - LC$.

### D. Interval analysis

**Lemma 5.** *Given a matrix $A \in \mathbb{R}^{m\times n}$ define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ and $|A| = A^+ + A^-$. Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \le x \le \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m\times n}$ be a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x}. \qquad (6)$$

**Lemma 6.** *Let $\underline{A} \le A \le \overline{A}$ for some $\underline{A}, \overline{A}, A \in \mathbb{R}^{n\times n}$ and $\underline{x} \le x \le \overline{x}$ for $\underline{x}, \overline{x}, x \in \mathbb{R}^n$, then*

$$\begin{aligned} &\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \le Ax \\ &\le \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-. \end{aligned} \qquad (7)$$

## III. MAIN RESULT

In this section, a design of an interval observer for the system (5) with varied input gains and constant delays is given, i.e. let (5) be written in the form as follow:

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + A_1x(t-\tau_0) + (B_0 \\ &\quad +\delta B_0(t))u(t) + (B_1 + \delta B_1(t))u(t-\tau(t)), \\ y &= Cx + v(t), \end{aligned} \qquad (8)$$

where $x(t) \in \mathbb{R}^n$ is the state, $\tau_0$is the constant delay and $\tau(t)$ is the time-varying uncertain delay; $u(t) \in \mathbb{R}^m$ is the input; $B_0 + \delta B_0(t)$ is the input gains without delay, where $\delta B_0(t)$ is varied and uncertain, $B_1 + \delta B_1(t)$ is the input gains with time-delay ($\delta B_1(t)$ is varied and uncertain); $y \in \mathbb{R}^p$ is the output available for measurements with a noise $v \in \mathcal{L}_\infty^p$. The matrices $A_0$, $A_1$, $B_0$, $B_1$, $C$ and the matrix functions $\delta B_0(t)$, $\delta B_1(t)$ have appropriate dimensions.

**Assumption 1.** *Let*
- $x \in \mathcal{L}_\infty^n$ *with* $\underline{x}_0 \le x_{t_0} \le \overline{x}_0$ *for some* $\underline{x}_0, \overline{x}_0 \in \mathcal{C}_\tau^n$;
- $||v|| \le V$ *for a given* $V > 0$;
- $\tau_0 \in \mathbb{R}_+$ *are known,* $\underline{\tau} \le \tau(t) \le \overline{\tau}$;
- $\underline{\delta B_i(t)} \le \delta B_i(t) \le \overline{\delta B_i}(t)$ *for all* $t \ge t_0$ *for some known* $\underline{\delta B_i}, \overline{\delta B_i} \in \mathcal{L}_\infty^m i = 0, 1$.

In this assumption it is supposed that the state of the system (5) is bounded with an unknown upper bound, but with a specified admissible set for initial conditions $[\underline{x}_0, \overline{x}_0]$. The upper bound on the measurement noise amplitude $V$ as well as the constant delays $\tau_0$ are assumed to be given. All uncertainty of the system is collected in the gains $\delta B_0(t)$, $\delta B_1(t)$ and delay $\tau(t)$.

For the estimation purposes, the system (8) can be rewritten as follow:

$$\dot{x}(t) = (A_0 - L_0C)x(t) + (A_1 - L_1C)x(t-\tau_0) + g(t), \qquad (9)$$

where

$$\begin{aligned} g(t) &= L_0y - L_0V + L_1y(t-\tau_0) + L_1V(t-\tau_0) \\ &\quad + [B_0 + \delta B_0(t)]\,u(t) + [B_1 + \delta B_1(t)]\,u(t-\tau(t)). \end{aligned} \qquad (10)$$

Denote $|L_i| = L_i^+ + L_i^-$ for $i = 1, 0$ and note that $\delta B_i(t)u(t) = \sum_{k=1}^{m} \delta B_i^k(t)u_k(t)$, $i = 1, 0$, where $\delta B_i^k$ is the $k$th column of the matrix function $\delta B_i$.

**Proposition 1.** *Let Assumption 1 be satisfied, then*

$$\underline{g(t)} \le g(t) \le \overline{g(t)}$$

*where*

$$\begin{aligned}\underline{g(t)} &= L_0y+L_1y(t-\tau_0)-|L_0|E_PV-|L_1|E_PV \quad (11)\\&+B_0u(t)+\sum_{k=1}^{m}[\underline{\delta B_0^k}(t)u_k^+(t)-\overline{\delta B_0^k}(t)u_k^-(t)\\&+B_1^{k+}\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t)-B_1^{k-}\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t)\\&+\underline{\delta B_1^k}^+(t)\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^+(t)-\overline{\delta B_1^k}^+(t)\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^-(t)\\&-\underline{\delta B_1^k}^-(t)\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^+(t)+\overline{\delta B_1^k}^-(t)\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^-(t)],\end{aligned}$$

$$\begin{aligned}\overline{g(t)} &= L_0y+L_1y(t-\tau_0)+|L_0|E_PV+|L_1|E_PV \quad (12)\\&+B_0u(t)+\sum_{k=1}^{m}[\overline{\delta B_0^k}(t)u_k^+(t)-\underline{\delta B_0^k}(t)u_k^-(t)\\&+B_1^{k+}\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t)-B_1^{k-}\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t)\\&+\overline{\delta B_1^k}^+(t)\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^+(t)-\underline{\delta B_1^k}^+(t)\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^-(t)\\&-\overline{\delta B_1^k}^-(t)\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^+(t)+\underline{\delta B_1^k}^-(t)\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^-(t)].\end{aligned}$$

*Proof.* According to Assumption 1 and the definition of $|L_i|$, $-|L_0|E_pV \le L_0V \le |L_0|E_pV$, $-|L_1|E_pV \le L_1V(t-\tau_0) \le |L_1|E_pV$ for all $t>0$ . The term $L_0y+L_1y(t-\tau_0)+B_0u(t)$ is completely known. To this end, it is necessary to calculate the interval bounds for $\delta B_0(t)u(t)+[B_1+\delta B_1(t)]\,u(t-\tau(t))$, using (6) and (7) we obtain:

$$\begin{gathered}\underline{\delta B_0^k}u_k^+-\overline{\delta B_0^k}u_k^- \le \delta B_0^ku_k \le \overline{\delta B_0^k}u_k^+-\underline{\delta B_0^k}u_k^-,\\B_1^{k+}\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t)-B_1^{k-}\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t)\le B_1^ku_k(t-\tau(t))\\ \le B_1^{k+}\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t)-B_1^{k-}\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k(t),\\ \underline{\delta B_1^k}^+\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^+-\overline{\delta B_1^k}^+\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^--\underline{\delta B_1^k}^-\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^+\\+\overline{\delta B_1^k}^-\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^-\le \delta B_1^ku_k(t-\tau(t))\le\overline{\delta B_1^k}^+\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^+\\-\underline{\delta B_1^k}^+\overline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^--\overline{\delta B_1^k}^-\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^++\underline{\delta B_1^k}^-\underline{\mu}_{[\underline{\tau},\overline{\tau}]}u_k^-.\end{gathered}$$

The sum of these estimates provides the interval bounds $\underline{g(t)},\overline{g(t)}$ such that $\underline{g(t)}\le g(t)\le\overline{g(t)}$. ☐

Following [13], the equation (9) and Proposition 1 let us propose the interval observer in the following form

$$\dot{\underline{x}}(t)=(A_0-L_0C)\underline{x}(t)+(A_1-L_1C)\underline{x}(t-\tau_0)+\underline{g}(t), \quad (13)$$

$$\dot{\overline{x}}(t)=(A_0-L_0C)\overline{x}(t)+(A_1-L_1C)\overline{x}(t-\tau_0)+\overline{g}(t). \quad (14)$$

**Theorem 2.** *Let Assumption 1 be satisfied, the matrix $A_0-L_0C$ be Metzler and the matrix $A_1-L_1C$ be nonnegative. Then*

$$\underline{x}(t)\le x(t)\le\overline{x}(t)$$

*for all $t>0$ provided that $\underline{x}(0)\le x(0)\le\overline{x}(0)$. In addition, if there exist $p,q\in\mathbb{R}_+^{2n}$ ($p>0$ and $q>0$) such that*

$$p^T[A_0+A_1-(L_0+L_1)C]+q^T=0,$$

*then $\underline{x},\overline{x}\in\mathcal{L}_\infty^n$.*

*Proof.* Consider the estimation errors $\underline{\epsilon}=x-\underline{x}$ and $\overline{\epsilon}=x-\overline{x}$, whose dynamics can be written as follows

$$\begin{aligned}\dot{\underline{\epsilon}} &= (A_0-L_0C)\underline{\epsilon}+(A_1-L_1C)\underline{\epsilon}(t-\tau_0)+\underline{d}(t), \quad (15)\\ \dot{\overline{\epsilon}} &= (A_0-L_0C)\overline{\epsilon}+(A_1-L_1C)\overline{\epsilon}(t-\tau_0)+\overline{d}(t),\\ &\underline{d}(t)=g(t)-\underline{g}(t),\ \overline{d}(t)=\overline{g}(t)-g(t).\end{aligned}$$

Note that the initial conditions $\underline{\epsilon}(0),\ \overline{\epsilon}(0)\in\mathbb{R}_+^n$and the dynamics of the errors are cooperative. By definition of $\underline{g},\overline{g}$ the signals $\underline{d},\overline{d}\in\mathbb{R}_+^{n-p}$. Thus $\underline{\epsilon}(t),\ \overline{\epsilon}(t)\in\mathbb{R}_+^n$for all $t>0$ provided that $\underline{\epsilon}(0),\overline{\epsilon}(0)\in\mathcal{C}_{\tau_0}^{n-p}$, the last relation is satisfied by the definition of $\underline{x}_0$ and $\overline{x}_0$. To prove that the errors $\underline{\epsilon}(t),\ \overline{\epsilon}(t)$ are bounded, as in [32], consider for (15) the Lyapunov functional $V:\mathcal{C}_{\tau+}^n\to\mathbb{R}_+$ defined as

$$V(\varphi)=p^T\varphi(0)+\int_{-\tau_0}^{0}p^T(A_1-L_1C)\varphi(s)ds.$$

Let us stress that for any $\varphi\in\mathcal{C}_{\tau_0+}^n$ the functional $V$ is positive definite and radially unbounded, its derivative for $\underline{\epsilon}$ takes the form (for $\overline{\epsilon}$ the analysis is the same):

$$\begin{aligned}\dot V &= p^T[(A_0-L_0C)\underline{\epsilon}(t)+(A_1-L_1C)\underline{\epsilon}(t-\tau_0)+\underline{d}(t)]\\&\quad+p^T(A_1-L_1C)[\underline{\epsilon}(t)-\underline{\epsilon}(t-\tau_0)]\\&= p^T[(A_0+A_1-(L_0+L_1)C)\underline{\epsilon}(t)+\underline{d}(t)]\\&\le -q^T\underline{\epsilon}(t)+p^T\underline{d}(t).\end{aligned}$$

Thus for $\underline{d}=0$ the system is globally asymptotically stable, and since $\underline{d}\in\mathcal{L}_\infty^{n-p}$ (by construction and Assumption 1) one finds that the error $\underline{\epsilon}$ is bounded (see [34] or [35] for the proof that in fact the system is input-to-state stable). ☐

The performance of the proposed interval observer will be shown on example of thermic process.

## IV. EXAMPLE

In this section, a model of thermic application is considered:

$$\begin{gathered}r^2\ddot\theta+2r\dot\theta+\theta=ku\,(t-\tau(t))+u_0,\\ y=\theta.\end{gathered}$$

The system used is of the second order, where $\theta\in\mathbb{R}$ is the temperature, $u\in\mathbb{R}$ is the input voltage, $\tau(t)>0$ is the time-delay, $r>0$ is the time constant, $k>0$ is the input gains, $u_0>0$ is a constant input value, $y\in\mathbb{R}$ is the output, which is the measured temperature.

The identification procedure, applied to the platform available at LAGIS laboratory, provided the interval estimates for the model parameters as below:

$$\begin{gathered}0.213=\underline{r}\le r\le\overline{r}=0.256,\\0.443=\underline{k}\le k\le\overline{k}=0.830,\\0.050=\underline{\tau}\le\tau\le\overline{\tau}=0.090.\end{gathered}$$

To conclude, the dispersion of the parameter $r$ values is rather narrow, that is why the median value $r=0.241$ will be used next for modeling, while the deviations of $k$ and $\tau$ cannot be neglected. Therefore, it is easy to verify that for $x=[\theta\ \dot\theta]$ this system can be written as (8) for

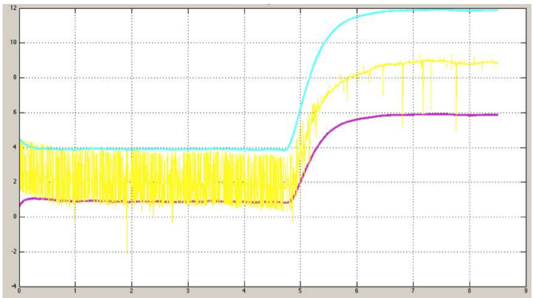

Figure 1. Interval estimation of the output for the thermic model (the temperature)

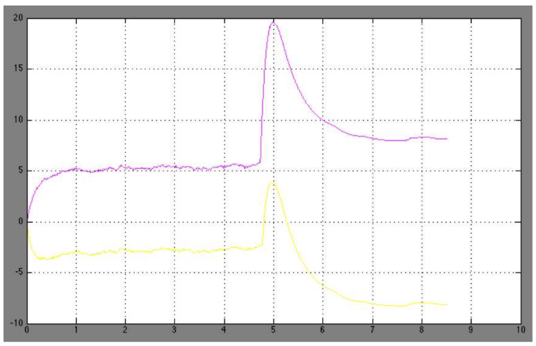


Figure 2. Interval estimation of derivative of temperature for the thermic model

$$A_0 = \begin{bmatrix} 0 & 1 \\ -r^{-2} & -2r^{-1} \end{bmatrix},\ A_1 = 0,\ B_0 = \delta B_0 = 0,$$

$$B_1 = \begin{bmatrix} 0 \\ r^{-2}k_0 \end{bmatrix},\ C = [1\ 0],$$

$$\begin{bmatrix} 0 \\ \frac{k_0-\delta k}{r^2} \end{bmatrix} = \underline{\delta B_1} \le \delta B_1(t) \le \overline{\delta B_1} = \begin{bmatrix} 0 \\ \frac{k_0+\delta k}{r^2} \end{bmatrix},$$

$$k_0 = \frac{\overline{k}+\underline{k}}{2},\ \delta k = \frac{\overline{k}-\underline{k}}{2}.$$

Assumption 1 is satisfied. Selecting the poles $[-5, -6]$ for the closed loop estimation dynamics $A_0 - L_0C$, we obtain

$$L = \begin{bmatrix} 2.701 \\ -17.217 \end{bmatrix}$$

and

$$A_0 - L_0C = \begin{bmatrix} -2.701 & 1 \\ 0 & -8.299 \end{bmatrix},$$

which is Metzler. The results of simulation for the interval observer 13, 14 are presented in figures 1 and 2.

## V. CONCLUSION

An approach for interval estimation of systems with uncertain time-varying delays and uncertain input gains is presented. Numerical experiments performed for a thermic process confirm the efficiency of the proposed method. The classical observers without delay in general is not very precise when applied to systems with time-varying delay. Overcoming this obstacle, that is, determining interval observers that give a satisfactory estimation result when a small delay is present, is the main objective in this work. In addition, the appearance of two unknown parameters in the same time makes it more difficult to define the suitable interval estimation. In this work, the problem of unknown input gain of the system, which cannot be ignored when the practical problem is applied, is also considered.

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