# Links between homotopy theory and type theory

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In the recent history of computer verified proof, we observe the conjunction of two domains that are seemingly extraordinarily distant: homotopy theory and type theory.

Type theory [6] is the realm of the motto "proof as programs, propositions as types", also often referred to as the *Curry-Howard isomorphism*. In this approach, proofs are objects of the logical discourse like any other object (number, program, statement) and their type is the proposition that they prove. For instance, a bounded integer can be encoded as a pair of an integer and a proof that this integer is within the prescribed bounds. This novelty also brings new problems: while we understand perfectly that there are several numbers in the same type of natural numbers, it is not immediately clear what we should do with several (distinct) proofs of the same statement. For instance, what should we do with two instances $(3, p_1)$ and $(3, p_2)$ describing the value 3 bounded by 4, where $p_1$ and $p_2$ are distinct proofs that 3 is smaller than 4.

The question of proof unicity (also called *proof irrelevance*) is especially important for proofs of equality between two values. This was studied at the end of the 1990s by Hoffmann and Streicher [5], and the conclusion was twofold. First, the unicity of proofs of equality is not a consequence of type theory's inherent foundations so that this property should be added as an axiom (which they named axiom K) if it is desired for all types; second, the type of equalities between elements of a given type can be endowed with a structure already known in mathematics as a *groupoid* structure. At about the same time, Hedberg [4] showed that proofs of equalities are unique in any type where equality is decidable. This result is put to efficient use in *Mathematical Components* [2].

Meanwhile, mathematics is also traversing crises of its own, first because some mathematical facts are now established with the help of intensive computations performed mechanically [3], second because some areas of mathematics reach a level of internal complexity where too few experts are competent to exert a critical eye. Bringing in computer verified proofs is a means to improve the confidence one can have in proofs that are otherwise refereed by too few people.

Homotopy theory is one of the topics where computer verified proof is being tested. Homotopy theory is about paths between objects, and it is well know that collections of paths also respect a groupoid structure. It thus feels natural to attempt to use directly the equality types of type theory to represent the paths used in homotopy theory. It turns out that this experiment works surprisingly well [1, 8]. For example, homotopy theory naturally considers higher-dimension paths (paths between paths) but similarly type theory is very well suited to consider equalities between equality proofs. Most of the notions around homotopies can be described around equality types and the whole system seems to provide a nice context to perform *synthetic homotopy theory* in the same spirit that Euclid's, Hilbert's, or Tarski's axiom systems can be used to perform *synthetic geometry*.

Type theory was already trying to pose as a new foundation for mathematics, but the question of giving a status to several proofs of the same statement was an unresolved issue. Insights from homotopy theory help clarifying this question: multiplicity of proofs should be encouraged, even though the collection of types where unicity of equality proofs is guaranteed should be given a specific status, as it simply corresponds to the sets of traditional mathematics. For these sets where unicity of equality proofs are guaranteed, one should be able to benefit from most of the machinery already developed in Mathematical Components for types with decidable equality.

New experiments are made possible in this homotopy type theory and new questions arise. In turn, these questions lead to the proposal of new extensions [7]. One such extension is the univalence axiom, which simply states that two homotopically equivalent types should be considered equal; another extension is the concept of higher inductive type, where inductive types can be equiped with paths between constructors. These extensions bring questions of consistency and computability that are still being studied.

## References


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