Resolution except for minimal singularities II. The case of four variables

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II. THE CASE OF FOUR VARIABLES

EDWARD BIERSTONE, PIERRE LAIREZ, AND PIERRE D. MILMAN

Abstract. In this sequel to [4], we find the smallest class of singularities in four variables with which we necessarily end up if we resolve singularities except for normal crossings. The main new feature is a characterization of singularities in four variables which occur as limits of triple normal crossings singularities, and which cannot be eliminated by a birational morphism that avoids blowing up normal crossings singularities. This result develops the philsophy of [4], that the desingularization invariant together with natural geometric information can be used to compute local normal forms of singularities.

Contents

1. Introduction

This article is a sequel to [4]. We find the smallest class of singularities in four variables with which we necessarily end up if we resolve singularities except for normal crossings. The main feature beyond the techniques of [4] is a characterization of singularities in four variables which occur as limits of triple normal crossings singularities, and which

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cannot be eliminated by a birational morphism that avoids blowing up the triple normal crossings singularities (Theorem 1.7). The latter result develops the philsophy of [4], that the desingularization invariant of [2] together with natural geometric information can be used to compute local normal forms of singularities.

The reader is referred to [4] for the background and techniques of this article. Throughout the paper, an *algebraic variety* means a scheme of finite type over a field $\underline{k}$, and $\operatorname{char}\underline{k} = 0$.

Definitions 1.1. We say that X has *normal crossings* at a point a if, locally at a, X can be embedded in a smooth variety Z with local étale coordinates $(x_1, \dots, x_n)$ at a in which X is defined by a monomial equation

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0 \tag{1.1}$$

(where the α_i are nonnegative integers). We will say that X has *normal crossings of order* k (or nc$k$) at a if precisely k exponents α_i are nonzero in (1.1). A singularity $xy = 0$ is called *double normal crossings* (nc2), a singularity $xyz = 0$ *triple normal crossings* (nc3), etc. Let X^{nc} denote the locus of points of X having only normal crossings singularities. (X^{nc} includes all smooth points of X.)

A variety X has normal crossings at a if and only if it can be defined at a by a monomial equation with respect to formal coordinates, after a finite extension of the ground field $\underline{k}$.

Definition 1.2. Let $\mathcal{S}$ denote the following class of singularities in four variables (w, x, y, z):

$$\begin{array}{rl}
xy = 0 & \text{nc2} \\
xyz = 0 & \text{nc3} \\
xyzw = 0 & \text{nc4} \\
z^2 + xy^2 = 0 & \textit{pinch point } \text{pp} \\
z^2 + (y + 2x^2)(y - x^2)^2 = 0 & \textit{degenerate pinch point } \text{dpp} \\
x(z^2 + wy^2) = 0 & \textit{product } \text{prod} \\
z^3 + wy^3 + w^2x^3 - 3wxyz = 0 & \textit{cyclic point } \text{cp3}
\end{array}$$

In other words, $\mathcal{S}$ is the class of singularities that can be written in local étale coordinates (or in formal variables after finite field extension) as one of the *normal forms* in the preceding table.

Let $X^{\mathcal{S}}$ denote the locus of points of X having only singularities in $\mathcal{S}$. In other words, if X is an algebraic variety of dimension three, then $X^{\mathcal{S}}$ is the locus of points a of X such that either a is a smooth point or a has a neighbourhood U where $X|_U$ admits an embedding $X|_U \hookrightarrow Z$

in a smooth 4-dimensional variety Z, and X has a singularity in $\mathcal{S}$ at a, with respect to suitable local étale coordinates of Z.

All singularities in $\mathcal{S}$ are hypersurface singularities. We say that X is a *hypersurface* if, locally, X can be defined by a principal ideal on a smooth variety. (We say that X is an *embedded hypersurface* if $X \hookrightarrow Z$, where Z is smooth and X is defined by a principal ideal on Z.)

Theorem 1.3. *Let X denote a reduced variety of pure dimension* 3. *Then there is a morphism $\sigma : X' \to X$ given by a finite sequence of admissible blowings-up*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t = X' , \tag{1.2}$$

such that

(a) $X' = (X')^{\mathcal{S}}$,
(b) *σ is an isomorphism over X^{nc}.*

Moreover, the morphism $\sigma = \sigma_X$ (or the entire blowing-up sequence (1.2)*) can be realized in a way that is functorial with respect to étale morphisms.*

See [4, Rem. 1.15] on minimality of the class $\mathcal{S}$.

An *admissible* blowing-up means a blowing-up σ with centre C which is smooth and has only simple normal crossings with respect to the exceptional divisor. The latter condition means that, with respect to a suitable local embedding of X in a smooth variety Z and the induced blowing-up sequence of Z, there are regular coordinates (i.e., regular parameters) $(x_1, \ldots, x_n)$ at any point of C, in which C is a coordinate subspace and each component of the exceptional divisor is a coordinate hyperplane $(x_i = 0)$, for some i.

Remark 1.4. Theorem 1.3 can be reduced to the case of a hypersurface using the strong desingularization algorithm of [2, 3]. The algorithm involves blowing up with smooth centres in the maximum strata of the Hilbert-Samuel function. The latter determines the local embedding dimension, so the algorithm first eliminates points of embedding codimension > 1 without modifying normal crossings points (or points with singularities in $\mathcal{S}$).

Theorem 1.5. *Let $\mathcal{S}'$ denote the class of singularities $\mathcal{S}$ together with the following singularity:*

$$z^2 + y(wy + x^2)^2 = 0 \quad \text{exceptional singularity exc} \tag{1.3}$$

Let X denote a reduced variety of pure dimension 3. *Then there is a morphism $\sigma : X' \to X$ given by a finite sequence of admissible blowings-up* (1.2) *such that*

(a) $X' = (X')^{\mathcal{S}'}$,

(b) *σ is an isomorphism over* $X^{\mathcal{S}'}$.

Moreover, the morphism $\sigma = \sigma_X$ (or the entire blowing-up sequence (1.2)*) can be realized in a way that is functorial with respect to étale morphisms.*

Remark 1.6. *Exceptional singularity.* The exceptional singularity is a limit of dpp singularities that cannot be eliminated without blowing up the dpp-locus:

The equation (1.3) defines an embedded hypersurface $X \hookrightarrow Z := \mathbb{A}^4_{(w,x,y,z)}$. Outside the origin, X has only smooth points, 2nc singularities (when $z = wy + x^2 = 0$, $y \neq 0$), and degenerate pinch points dpp (when $z = y = wy + x^2 = 0$, $w \neq 0$). Any birational morphism $Z' \to Z$ (where Z' is smooth) which eliminates the exceptional singularity at 0 and is an isomorphism over the complement of 0, factors through the blowing-up of 0.

Let $\sigma : Z' \to Z$ denote the blowing-up of 0 and let X' be the strict transform of X by σ. In the chart of Z' with coordinates (w, x, y, z) in which σ is given by (w, wx, wy, wz), X' is defined by the equation $z^2 + w^3 y(y + x^2)^2 = 0$. After a "cleaning" blowing-up (centre $(z = w = 0)$; see [4, Sect. 2]), we get

$$z^2 + wy(y + x^2)^2 = 0. \tag{1.4}$$

The hypersurface (1.4) has quadratic cone singularities when $z = y = w = 0$, $x \neq 0$, pinch points pp when $z = w = y + x^2 = 0$, $y \neq 0$, and degenerate pinch points dpp when $z = y = y + x^2 = 0$, $w \neq 0$. Any birational morphism that preserves singularities in $\mathcal{S}$ factors through the blowing-up either of $(z = y = w = 0)$ or of 0. The former leads to another exceptional singularity, while the latter leads to another singularity of type (1.4) after cleaning.

Theorems 1.3 and 1.5 will be proved in Section 4. Our proofs also give normal forms or local models for the singularites of the total transform of X, corresponding to $\mathcal{S}$ or $\mathcal{S}'$. (Equivalently, they give local models for the *transform* of a divisor $D \subset Z$ ($\dim Z = 4$), where the transform is defined as the support of the birational transform plus the exceptional divisor). See [4, Sect. 1] and Section 4 below.

The results in this article form part of the subject of Pierre Lairez's Mémoire de Magistère at the Ecole Normale Supérieure. The authors are grateful to Franklin Vera Pacheco for many important comments.

1.1. **Limits of triple normal crossings points.** Our proofs of Theorems 1.3 and 1.5 are based on using the desingularization invariant of

[2] as a tool for computing and simplifying local normal forms. In [4, Sects. 1,5], we try to provide a working knowledge of the desingularization algorithm and the invariant as they are used here, for a reader not necessarily familiar with a complete proof of resolution of singularities. The reader is referred to the latter for more details of the notions below.

Suppose that $X \hookrightarrow Z$ is an embedded hypersurface, where Z is smooth. Let $\mathrm{inv} = \mathrm{inv}_X$ denote the desingularization invariant for X. We recall that inv is defined iteratively on the strict transform X_{j+1} of $X = X_0$ for any finite sequence of inv-*admissible* blowings-up

$$Z = Z_0 \xleftarrow{\sigma_1} Z_1 \longleftarrow \cdots \xleftarrow{\sigma_{j+1}} Z_{j+1}\,. \tag{1.5}$$

(A blowing-up is inv-*admissible* if it is admissible and inv is constant on each component of its centre.) In particular, $\mathrm{inv}(a)$, where $a \in X_{j+1}$ depends not only on X_{j+1} but also on the *history* of blowings-up (1.5).

Let $a \in X_j$. then $\mathrm{inv}(a)$ has the form

$$\mathrm{inv}(a) = (\nu_1(a), s_1(a), \ldots, \nu_t(a), s_t(a), \nu_{t+1}(a))\,, \tag{1.6}$$

where $\nu_k(a)$ is a positive rational number (called a *residual multiplicity*) if $k \leq t$, each $s_k(a)$ is a nonnegative integer (which counts certain components of the exceptional divisor), and $\nu_{t+1}(a)$ is either 0 or ∞. The successive pairs $(\nu_k(a), s_k(a))$ are defined inductively over *maximal contact* subvarieties of increasing codimension.

It is easy to see that, in *year zero* (i.e., if $j = 0$), then $\mathrm{inv}(a) = (2, 0, 1, 0, \infty)$ if and only if X has a double normal crossings singularity $z^2 + y^2 = 0$ at a. Following are some other year-zero hypersurface examples:

$$\begin{array}{rcl}
x = 0 & \text{smooth} & \mathrm{inv}(0) = \mathrm{inv}(\mathrm{nc}1) := (1, 0, \infty) \\
x_1 x_2 \cdots x_k = 0 & \mathrm{nc}k & \mathrm{inv}(0) = \mathrm{inv}(\mathrm{nc}k) := (k, 0, 1, 0, \ldots, 1, 0, \infty) \\
z^2 + xy^2 = 0 & \mathrm{pp} & \mathrm{inv}(0) = \mathrm{inv}(\mathrm{pp}) := (2, 0, 3/2, 0, 1, 0, \infty)
\end{array}$$

(where, for nc$k$, there are $k-1$ pairs $(1, 0)$). For $k \geq 3$, nc$k$ is not characterized by the value of inv; for example the singularity $x_1^k + x_2^k + \cdots + x_k^k = 0$ also has $\mathrm{inv}(0) = (k, 0, 1, 0, \ldots, 1, 0, \infty)$ with $k-1$ pairs $(1, 0)$.

Let $X \hookrightarrow Z$ denote an embedded hypersurface, where $\dim Z = 4$. As above, if a is a triple normal crossings point of X, then $\mathrm{inv}(a) = \mathrm{inv}(\mathrm{nc}3) := (3, 0, 1, 0, 1, 0, \infty)$ (this is "year zero"). Consider the desingularization algorithm applied to $X \subset Z$. In Theorem 1.7 following, we provide normal forms for the singularities which can occur at special points of a component C of the locus

$$(\mathrm{inv} = \mathrm{inv}(\mathrm{nc3})) \tag{1.7}$$

in the strict transform X_{j_0}, for any year j_0 of the resolution history (1.5), assuming that the generic point of C is nc3. Theorem 1.9 below provides normal forms (in $\mathcal{S}$) for the singularities we get by applying *cleaning* blowings-up to simplify the preceding singularities. (See [4, Sects. 1,2].)

The locus $(\mathrm{inv} = \mathrm{inv}(\mathrm{nc3})) \subset X_{j_0}$ is a smooth curve. Let $a \in C$, where C is a component of $(\mathrm{inv} = \mathrm{inv}(\mathrm{nc3}))$ which is generically nc3. Using the Weierstrass preparation theorem, in suitable étale coordinates (w, x, y, z) at $a = 0$, we can write the equation of X_{j_0} in Z_{j_0} as $f(w, x, y, z) = 0$, where f is nc3 on $(x = y = z = 0,\ w \neq 0)$, and

$$f(w, x, y, z) = z^3 + A(w, x, y)z^2 + B(w, x, y)z + C(w, x, y), . \tag{1.8}$$

and we can assume that $A = 0$, by "completing the cube".

Theorem 1.7. *Let $X \hookrightarrow Z$ denote an embedded hypersurface, where Z is a smooth algebraic variety of pure dimension four. Assume that the ground field $\underline{k}$ is algebraically closed. Consider a finite sequence of* inv*-admissible blowings-up* (1.5). *Let $a \in X_{j_0}$, for some $j = j_0$, and let $f = 0$ be a local defining equation for X_{j_0} at a. Suppose that Z_{j_0} has a regular coordinate system (w, x, y, z) at $a = 0$ such that:*

(i) *$w = 0$ is a local equation for the exceptional divisor (if the latter contains a);*
(ii) *X_{j_0} is* nc3 *at every nonzero point of the w-axis $(z = y = x = 0)$;*
(iii) $\mathrm{inv}(a) = \mathrm{inv}(\mathrm{nc3}) := (3, 0, 1, 0, 1, 0, \infty)$.

Then:

(1) *f has three analytic branches at $a = 0$ (i.e., three factors of order 1 in a suitable étale neighbourhood) if and only if*

$$f(w, x, y, z) = z\,(z + w^\alpha x)\left(z + w^\alpha\left(x\xi + w^\beta y\right)\right),$$

where $\xi = \xi(w, x, y)$, after a suitable étale coordinate change preserving $(w = 0)$.

On the other hand, suppose that f does not have three analytic branches at a. Then:

(2) *The following are equivalent:*
   (a) *f has two analytic branches at $a = 0$;*
   (b) *$f(w^2, x, y, z)$ has three analytic branches at 0;*

(c) *after an étale coordinate change preserving* $(w = 0)$, *we can write* $f(w,x,y,z)$ *either as*

$$(z + w^{\alpha}x)\left(z^2 + w^{2\alpha+1}\left(x\xi + w^{\beta}y\right)^2\right),$$

where $\xi = \xi(w,x,y)$, *or as*

$$\left(z + w^{\alpha}\left(y\eta + w^{\beta}x\right)\right)\left(z^2 + w^{2\alpha+1}y^2\right),$$

where $\eta = \eta(w,x,y)$.

(3) *The following are equivalent:*

(a) f *is analytically irreducible at* $a = 0$;

(b) $f(w^3,x,y,z)$ *has three analytic branches at* 0;

(c) *after an étale coordinate change preserving* $(w = 0)$, *we can write* $f(w,x,y,z)$ *as*

$$z^3 - 3w^{\beta}y\left(y\eta + w^{\gamma}x\right)z + w^{\alpha}y^3 + w^{3\beta-\alpha}\left(y\eta + w^{\gamma}x\right)^3,$$

where $\eta = \eta(w,x,y)$, $2\alpha < 3\beta$ *and* α *is not divisible by* 3.

Remarks 1.8. (1) Given that $\mathrm{inv}(a) = \mathrm{inv}(\mathrm{nc3})$, $a \in X_{j_0}$, and that X_{j_0} is generically nc3 on the component of $(\mathrm{inv} = \mathrm{inv}(\mathrm{nc3}))$ containing a, then we can choose coordinates satisfying the hypotheses of the theorem.

(2) The condition (b) in item (2) or (3) of Theorem 1.7 is reminiscent of the Abhyankar–Jung Theorem (cf. [1]). Note that the implication (a) $\Rightarrow$ (b) in (2) or (3) is not true if we weaken the hypothesis (iii) by assuming only that f has order 3 at a. For example, $f(w,x,y,z) = (z+x)\left(z^2 + (w+y)y^2\right)$ does not have three analytic branches after substituting w^2 for w.

Theorem 1.9. *With the hypotheses of Theorem 1.7, assume in addition that* $\mathrm{inv}(\mathrm{nc3})$ *is the maximum value of* inv *on* X_{j_0}. *Then there is a morphism* $\sigma : Z' \to Z_{j_0}$ *given by a finite sequence of admissible blowings-up of* $X_{j_0} \subset Z_{j_0}$ *with centres in the exceptional divisor, such that* $\sigma^{-1}(a)$ *intersects the strict transform of* $(\mathrm{inv} = \mathrm{inv}(\mathrm{nc3}))$ *in a single point* a', *and* X' *is defined at* a' *by one of the following equations in* $\mathcal{S}$, *according to the corresponding case of Theorem 1.7:*

(1) $xyz = 0$ nc3;

(2) $x(z^2 + wy^2) = 0$ prod;

(3) $z^3 + wy^3 + w^2x^3 - 3wxyz = 0$ cp3.

Theorem 1.9 follows from Theorem 1.7 by applying the cleaning lemma [4, Sect. 2] to the normal forms in the latter. The cleaning lemma is applied in exactly the same way as in [4, §§4.2, 4.3], so we only give an example (1.10 below) and refer the reader to [4] for details of the lemma and its use. The cleaning lemma will be used in the same way again in several other steps of the proofs of Theorems 1.3 and 1.5 in Section 4 below.

Example 1.10. As an illustration, we show how the cleaning lemma is used to simplify the normal form of Theorem 1.7(1) in order to obtain the equation for nc3 in Theorem 1.9(1) (cf. [4, Thm. 3.4]). In Theorem 1.7(1), $(z = 0)$ and $(z = x = 0)$ are successive maximal contact subspaces (see [4, §5.4]). A first application of the cleaning lemma involves desingularization of the monomial marked ideal $(w^\beta, 1)$ on $(z = x = 0)$: A blowing-up of $(z = x = w = 0)$ has the effect of reducing β by 1. By β such cleaning blowings-up, we reduce f to $z(z + w^\alpha x)(z + w^\alpha(x\xi' + y))$. A second application of the cleaning lemma involves resolving the monomial marked ideal $(w^\alpha, 1)$ on $(z = 0)$ to reduce f to $z(z+x)(z+x\xi'+y)$, and thus to xyz after a coordinate change.

Basic properties of cyclic singularities are presented in Section 2 following. Theorem 1.7 will be proved in Section 3.

2. Cyclic singularities

Let X denote a hypersurface of dimension $n - 1$; i.e., X is defined locally by an equation in n variables. Then n-fold normal crossings singularities nc$n$ of X are isolated, and the locus of $(n-1)$-fold normal crossings points is a smooth curve. A *cyclic singularity* or *cyclic point* of *order* $n - 1$, denoted cp$(n - 1)$, is an irreducible singularity that occurs as a limit of nc$(n - 1)$ points of a hypersurface in n variables, and which cannot be eliminated without blowing up nc$(n - 1)$ points.

The cyclic singularity cp$k$ of order k is related to the action of the cyclic group $\mathbb{Z}_k$ of order k on $\mathbb{C}^k$ by permutation of coordinates. In §2.1 following, we define cp3, which is needed for this article, but it will be clear how to generalize the construction to arbitrary k. The reader should check that cp2 = pp.

2.1. **Cyclic points of order** 3**.** Consider the action of $\mathbb{Z}_3$ on $\mathbb{C}^3$ by permutation of coordinates. $\mathbb{Z}^3$ is generated by the cyclic permutation $\rho = (1, 2, 3)$; in terms of the coordinates (X, Y, Z) of $\mathbb{C}^3$, $\rho(X, Y, Z) = (Z, X, Y)$.

The matrices in any finite abelian subgroup of the general linear group $\mathrm{GL}(k, \mathbb{C})$ can be diagonalized simultaneously [6, Prop. 2.7.2]. A diagonalization of the action of $\mathbb{Z}_3$ on $\mathbb{C}^3$ is given by

$$\begin{aligned} y_0 &= \frac{1}{3}(X+Y+Z) \\ y_1 &= \frac{1}{3}(X+\epsilon Y+\epsilon^2 Z) \\ y_2 &= \frac{1}{3}(X+\epsilon^2 Y+\epsilon Z) \end{aligned} \tag{2.1}$$

(the discrete Fourier transform); in other words,

$$y_i \circ \rho = \epsilon^i y_i, \quad i = 0, 1, 2,$$

where ϵ denotes the cube root of unity $\epsilon = e^{2\pi i/3}$.

It is easy to write a set of generators of the algebra of invariant polynomials for the diagonalized action. Following is a set of basic invariants for the action of $\mathbb{Z}_3$ above:

$$y_0,\ y_1y_2,\ y_1^3,\ y_2^3. \tag{2.2}$$

Consider the inverse linear transformation of (2.1):

$$\begin{aligned} X &= y_0 + y_1 + y_2 \\ Y &= y_0 + \epsilon^2 y_1 + \epsilon y_2 \\ Z &= y_0 + \epsilon y_1 + \epsilon^2 y_2. \end{aligned} \tag{2.3}$$

Let $\Phi(y_0, y_1, y_2)$ denote the polynomial XYZ obtained from (2.3). Then $\Phi(y_0, y_1, y_2)$ is invariant with respect to the action of $\mathbb{Z}_3$, so it is a polynomial in the basic invariants (2.2). Therefore $\Phi(z, w^{1/3}y, w^{2/3}x)$ is a polynomial in (x, y, z, w).

Definition 2.1. The *cyclic singularity* cp3 of *order* 3 is defined by

$$\Phi(z, w^{1/3}y, w^{2/3}x) = 0;$$

in other words, from (2.3), by

$$z^3 + wy^3 + w^2x^3 - 3wxyz = 0. \tag{2.4}$$

2.2. **Singularities in a neighbourhood of a cyclic point.** Consider the hypersurface $X \subset \mathbb{A}^4$ defined by (2.4). Then X has a cp3 singularity at the origin. When $w \neq 0$, X is nc3 along the w-axis, and has only nc2 singularities outside the w-axis.

On the other hand, $\operatorname{Sing} X \cap (w = 0)$ is the nonzero x-axis. We will show that X has degenerate pinch points along the nonzero x-axis.

For $x \neq 0$, write

$$\eta = \frac{y}{2x}, \quad \zeta = \frac{z}{x},$$

so that (2.4) can be rewritten as

$$w^2 + 2(4\eta^3 - 3\eta\zeta)w + \zeta^3 = 0,$$

or, after completing the square, as

$$\left(w + 4\eta^3 - 3\eta\zeta\right)^2 + \zeta^3 - \left(4\eta^3 - 3\eta\zeta\right)^2 = 0.$$

Now,

$$\zeta^3 - \left(4\eta^3 - 3\eta\zeta\right)^2 = \left((\zeta - 3\eta^2) - \eta^2\right)^2 \left((\zeta - 3\eta^2) + 2\eta^2\right).$$

In other words, if we make a change of coordinates

$$\begin{aligned} y' &= \frac{1}{2}\left(\frac{y}{x}\right), \\ z' &= \frac{z}{x} - \frac{3}{4}\left(\frac{y}{x}\right)^2, \\ w' &= w + \frac{1}{2}\left(\frac{y}{x}\right)^3 - \frac{3}{2}\left(\frac{y}{x}\right)\left(\frac{z}{x}\right) \end{aligned}$$

when $x \neq 0$, then (2.4) can be rewritten (after dropping primes) as

$$w^2 + \left(z - y^2\right)^2 \left(z + 2y^2\right) = 0. \tag{2.5}$$

This equation defines a degenerate pinch point when $y = z = w = 0$; i.e., (2.4) has a degenerate pinch point when $y = z = w = 0$, $x \neq 0$, as claimed. We also see that a cyclic point cp3 (2.4) becomes dpp after blowing up the nc3-axis ($x = y = z = 0$).

Note that the hypersurface ($w = 0$) with respect to the coordinates of (2.4) becomes ($w + 5y^3 + 3yz = 0$) in the new coordinates used in (2.5).

3. Limits of triple normal crossings points

A proof of Theorem 1.7 will be given in this section. Item (1) of the theorem is proved in *Resolution except for minimal singularities I* (see [4, Lemma 3.4]), so we only have to prove (2) and (3).

3.1. **Normal forms.** The following lemma isolates parts of Theorem 1.7 (2),(3) that are proved in this subsection. The proof of the theorem is completed in §3.2.

Lemma 3.1. *With the hypotheses of Theorem 1.7, assume that f does not split (i.e., does not have three local analytic branches) at $a = 0$. Then we have the following conclusions. (The statements following are enumerated as in Theorem 1.7.)*

(2) *The following are equivalent:*

(b) *$f(v^2,x,y,z)$ has three analytic branches at* 0*;*

(c) *after an étale coordinate change preserving $(w=0)$, we can write $f(w,x,y,z)$ either as*

$$(z+w^{\alpha}x)\left(z^2+w^{2\alpha+1}\left(x\xi+w^{\beta}y\right)^2\right), \tag{3.1}$$

where $\xi=\xi(w,x,y)$, or as

$$\left(z+w^{\alpha}\left(y\eta+w^{\beta}x\right)\right)\left(z^2+w^{2\alpha+1}y^2\right), \tag{3.2}$$

where $\eta=\eta(w,x,y)$.

(3) *The following are equivalent:*

(b) *$f(v^3,x,y,z)$ has three analytic branches at* 0*;*

(c) *after an étale coordinate change preserving $(w=0)$, we can write $f(w,x,y,z)$ as*

$$z^3-3w^{\beta}y\left(y\eta+w^{\gamma}x\right)z+w^{\alpha}y^3+w^{3\beta-\alpha}\left(y\eta+w^{\gamma}x\right)^3, \tag{3.3}$$

where $\eta=\eta(w,x,y)$, $2\alpha<3\beta$ and α is not divisible by 3*. (In particular, f is irreducible.)*

Proof. In both (2) and (3), it is clear that (c) $\Rightarrow$ (b). So in each case we will assume (b) and prove (c). We can assume that

$$f(w,x,y,z)=z^3+B(w,x,y)z+C(w,x,y), \tag{3.4}$$

and that $V(z)=(z=0)$ is a maximal contact hypersurface at $a=0$.

(2) For any root $\phi(v,x,y)$ of $f(v^2,x,y,z)=0$, $\phi(-v,x,y)$ is also a root. Therefore $f(v^2,x,y,z)$ has the form

$$f(v^2,x,y,z)=(z+\phi(v,x,y))(z+\phi(-v,x,y))(z+2\chi(v^2,x,y)),$$

where

$$\chi(v^2,x,y)=-\frac{1}{2}(\phi(v,x,y)+\phi(-v,x,y)),$$

by (3.4).

Now, we can write

$$\frac{1}{2}(\phi(v,x,y)-\phi(-v,x,y))=v\psi(v^2,x,y).$$

Therefore,

$$\phi(v,x,y)=-\chi(v^2,x,y)+v\psi(v^2,x,y),$$

and

$$f(v^2)=(z+2\chi(v^2))(z-\chi(v^2)+v\psi(v^2))(z-\chi(v^2)-v\psi(v^2))$$

(where $f(v^2)$ means $f(v^2,x,y,z)$, etc.). Therefore, after a change of coordinates $z'=z-\chi(w,x,y)$, we can write f in the form

$$f(w,x,y,z')=(z'+\chi'(w,x,y))\left((z')^2-w\psi'(w,x,y)^2\right). \tag{3.5}$$

It follows from (3.5) that the first coefficient (marked) ideal is equivalent to $(\chi', 1) + (w(\psi')^2, 2)$ on $(z = 0)$. Since the third entry of $\mathrm{inv}(a) = (3, 0, 1, 0, 1, 0, \infty)$ is 1, this marked ideal has maximal order [4, §5.4] after factoring w (to the highest power possible) [4, §5.8]. Therefore, either χ' has order 1 or ψ' has order 1, after this division by a power of w. The first case leads to the normal form (3.1), while the second leads to (3.2). We provide the explicit calculation for the first case and leave the second to the reader.

In the first case, we can write $\chi' = w^m x$ after an étale coordinate change (modifying only x and y), and w^{2m} divides $w(\psi')^2$. Moreover ψ' has positive order in (x, y), so we can write ψ' as $w^m(x\xi(w, x, y) + y\eta(w, x, y))$. The first coefficient ideal above can then be written as $(w^{2m}(x^2, w(x\xi + y\eta)^2), 2)$. Now, $V(z, x) = (z = x = 0)$ defines a next maximal contact subspace, and the corresponding coefficient ideal is $(wy^2\eta^2, 2)$ on $(z = x = 0)$. Since the fifth entry of $\mathrm{inv}(a)$ is 1, this coefficient ideal has maximal order, so $y\eta(w, 0, y)$ has order 1 after factoring w, and we can write $\psi' = w^m(x\xi + w^n y)$, after an étale coordinate change. This gives the normal form (3.1).

(3) Since f does not split but $f(v^3, x, y, z)$ splits at a, it follows that the latter factors as

$$\begin{aligned} f(v^3, x, y, z) &= (z + \phi(v, x, y))(z + \phi(\epsilon v, x, y))(z + \phi(\epsilon^2 v, x, y)) \\ &= XYZ, \text{ say}, \end{aligned}$$

where $\epsilon = e^{2\pi i/3}$. Define y_0, y_1, y_2 by the formulas (2.1). Then

$$\begin{aligned} y_0 &= z \\ y_1 &= \frac{1}{3}\left(\phi(v, x, y) + \epsilon\phi(\epsilon v, x, y) + \epsilon^2\phi(\epsilon^2 v, x, y)\right) \\ y_2 &= \frac{1}{3}\left(\phi(v, x, y) + \epsilon^2\phi(\epsilon v, x, y) + \epsilon\phi(\epsilon^2 v, x, y)\right). \end{aligned}$$

Clearly, vy_1 and v^2y_2 are $\mathbb{Z}_3$-invariant (with respect to the action on the v-variable), so that

$$\begin{aligned} vy_1 &= \eta_1(v^3, x, y) \\ v^2y_2 &= \eta_2(v^3, x, y), \end{aligned}$$

and we can write

$$\begin{aligned} y_1 &= v^{3m+2}\zeta_1(v^3, x, y) \\ y_2 &= v^{3n+1}\zeta_2(v^3, x, y), \end{aligned}$$

where $m, n \geq 0$.

Now consider

$$\zeta_1 = \zeta_1(w,x,y) \qquad\qquad y_1 = w^{m+2/3}\zeta_1$$
$$\zeta_2 = \zeta_2(w,x,y) \qquad\qquad y_2 = w^{n+1/3}\zeta_2;$$

both ζ_1 and ζ_2 are in the ideal generated by x, y. By (2.3),

$$f(w,x,y,z) = z^3 + w^{3n+1}\zeta_2^3 + w^{3m+2}\zeta_1^3 - 3w^{m+n+1}\zeta_1\zeta_2 z.$$

Then the first coefficient ideal is equivalent to $((w^{3m+2}\zeta_1^3, w^{3n+1}\zeta_2^3), 3)$ (cf.[4, Ex. 5.13]). Set $\alpha = \min\{3m+2, 3n+1\}$. Let ζ_1' denote the ζ_i corresponding to α, and ζ_2' the other. Since $\mathrm{inv}(a) = (3,0,1,\ldots)$, ζ_1' has order 1 at a, and we can assume that $\zeta_1' = y$. The next coefficient ideal is given by $(w^p(\zeta_2')^3, 3)$, for the appropriate p, on the maximal contact subspace $(z = y = 0)$. Since $\mathrm{inv}(a) = (3,0,1,0,1,\ldots)$, it follows that $\zeta_2'|_{(y=0)}$ has order 1 after dividing by w as much as possible. By a further coordinate change, we get (3.3), where $\beta = m+n+1$. Note that (3.3) would split if α were divisible by 3. $\square$

3.2. **Splitting lemmas.** In this subsection, we complete the proof of Theorem 1.7. We use the notation of the latter. We can also assume that

$$f(w,x,y,z) = z^3 - 3B(w,x,y)z + C(w,x,y), \tag{3.6}$$

and that $V(z) = (z = 0)$ is a maximal contact hypersurface at $a = 0$. Set

$$\Delta := C^2 - 4B^3;$$

i.e., -27Δ is the discriminant of f as a polynomial in z. Then the first coefficient (marked) ideal is

$$I := \big((B^3, C^2), 6\big) = \big((C^2, \Delta), 6\big)\,.$$

Since $\mathrm{inv}(a) = (3,0,1,\ldots)$, we have $I = w^\gamma \tilde{I}$, where $\tilde{I}$ has order 6 at $a = 0$.

The coordinate system (w,x,y,z) induces an identification of the completed local ring $\widehat{\mathcal{O}}_{Z,a}$ with the formal power series ring $\underline{k}[\![w,x,y,z]\!]$. Let $\underline{k}(\!(w)\!)$ denote the field of fractions of $\underline{k}[\![w]\!]$, and let $\overline{\underline{k}(\!(w)\!)}$ denote the algebraic closure of $\underline{k}(\!(w)\!)$. Then $\overline{\underline{k}(\!(w)\!)}$ is the field of formal Puiseux series in w over $\underline{k}$; i.e., formal Laurent series over $\underline{k}$ in $w^{1/n}$, with finitely many negative exponents, where n ranges over the nonnegative integers. Set

$$R := \underline{k}[\![w,x,y]\!],$$
$$S := \overline{\underline{k}(\!(w)\!)}[\![x,y]\!].$$

Then f splits in $S[z]$; say,

$$f = (z+\phi_0)(z+\phi_1)(z+\phi_2).$$

Moreover, each ϕ_j belongs to the ideal (x, y) generated by x and y in S, by the normal crossings hypothesis (ii) in Theorem 1.7. Define

$$\eta_i := \frac{1}{3}\sum_{j=0}^{2} \epsilon^{ij}(z+\phi_j), \quad i = 0, 1, 2,$$

where $\epsilon = e^{2\pi i/3}$ (cf. (2.1)). Then $\eta_0 = z$ and

$$\begin{aligned} f &= \prod_{i=0}^{2}\left(z + \epsilon^i \eta_1 + \epsilon^{2i}\eta_2\right) \\ &= z^3 - 3\eta_1\eta_2 z + \eta_1^3 + \eta_2^3 \end{aligned} \tag{3.7}$$

in $S[z]$ (cf. (2.3)). In particular,

$$B = \eta_1\eta_2, \quad C = \eta_1^3 + \eta_2^3, \quad \Delta = \left(\eta_1^3 - \eta_2^3\right)^2$$

in S. The preceding notation will be used throughout this section.

Lemma 3.2. *Under the hypotheses of Theorem 1.7, Δ factors in a sufficiently small étale neighbourhood of a as*

$$\Delta = \Phi^2\Psi,$$

where Ψ is not in the ideal generated by x, y.

Proof. By the normal crossings hypothesis (ii) in Theorem 1.7, Δ is a square at the generic point of $(x = y = 0)$. The assertion follows. □

Given $\theta \in R$, let $\operatorname{ord}\theta$ denote the order of θ with respect to the maximal ideal (w, x, y), and let $\operatorname{ord}_{(x,y)}\theta$ denote the order with respect to the ideal (x, y). Thus, $\operatorname{ord}\theta > 0$ if and only if θ is not a unit in R, and $\operatorname{ord}_{(x,y)}\theta > 0$ if and only if θ is not a unit in S.

We will prove the following three lemmas, *all under the hypotheses of Theorem 1.7.*

Lemma 3.3. *Assume that Δ is a square in R. Then $f(v^3, x, y, z)$ splits at $a = 0$.*

Lemma 3.4. *Assume that* $\operatorname{ord} B^3 > \operatorname{ord} C^2$. *Then Δ is a square in R.*

Lemma 3.5. *Assume that* $\operatorname{ord} B^3 \leq \operatorname{ord} C^2$. *Then $f(v^2, x, y, z)$ splits at $a = 0$.*

Theorem 1.7 is an immediate consequence of the preceding three lemmas and Lemma 3.1. Proofs of Lemmas 3.3–3.5 follow. The latter is the most delicate.

Proof of Lemma 3.3. Write $\Delta = A^2 \in R$; we can take $A = \eta_1^3 - \eta_2^3$. Recall $I = (B^3, C^2) = (\Delta, C^2) = w^\gamma \tilde{I}$, as above. Then $\gamma = \min\{\mathrm{ord}_{(w)} A^2, \mathrm{ord}_{(w)} C^2\}$. Therefore, γ is even; say $\gamma = 2\alpha$.

We have $4B^3 = (C-A)(C+A)$.

We claim that $w^{-\alpha}C$ and $w^{-\alpha}A$ are relatively prime in R: It is easy to check they are relatively prime in S since $A = \eta_1^3 - \eta_2^3$, $C = \eta_1^3 + \eta_2^3$, and the ideal $(\eta_1, \eta_2) = (x, y)$ in S. Since $\tilde{I}$ has order 6, either $\mathrm{ord}\, w^{-\gamma}\Delta = \mathrm{ord}_{(x,y)} w^{-\gamma}\Delta$ or $\mathrm{ord}\, w^{-\gamma}C^2 = \mathrm{ord}_{(x,y)} w^{-\gamma}C^2$. In either case, we can use Lemma 3.6 following to conclude that $w^{-\alpha}C$, $w^{-\alpha}A$ are relatively prime in R.

Therefore, $w^{-\delta}(C-A) = 2w^{-\delta}\eta_2^3$ and $w^{-\delta}(C+A) = 2w^{-\delta}\eta_1^3$ are relatively prime in R, where δ denotes the largest power of w that divides $C-A$ and $C+A$. Moreover, their product $4w^{-2\delta}B^3$ is a cube times a power of w in R. Hence both η_1^3 and η_2^3 are cubes (times powers of w) in R. By (3.7), $f(v^3, x, y, z)$ splits in $\underline{k}[\![v, x, y]\!][z]$ and the result follows. □

Lemma 3.6. *Let $G \in R$. Suppose that $\mathrm{ord}\, G = \mathrm{ord}_{(x,y)} G$. Let $\theta \in R$ be a nonunit which divides G. Then θ is also a nonunit in S.*

Proof. Consider a decomposition of G into irreducible factors in R, $G = \prod \theta_i^{n_i}$, where the n_i are positive integers. For all i, $\mathrm{ord}\, \theta_i \geq \mathrm{ord}_{(x,y)} \theta_i$. By the hypothesis, $\sum n_i \mathrm{ord}\, \theta_i = \sum n_i \mathrm{ord}_{(x,y)} \theta_i$. Therefore, $\mathrm{ord}\, \theta_i = \mathrm{ord}_{(x,y)} \theta_i$, for all i. The result follows. □

Proof of Lemma 3.4. Since $\mathrm{ord}\, C^2 < \mathrm{ord}\, B^3$, $\mathrm{ord}\, C^2 = \gamma + 6$. Therefore, γ is even (say $\gamma = 2\alpha$), and $\Delta = w^{2\alpha}\widetilde{\Delta}$, where $\mathrm{ord}\, \widetilde{\Delta} = 6 = \mathrm{ord}_{(x,y)} \widetilde{\Delta}$. By Lemma 3.2, Δ is a square in R. □

Proof of Lemma 3.5. Since $\mathrm{ord}\, B^3 \leq \mathrm{ord}\, C^2$, $\mathrm{ord}\, B^3 = \gamma + 6$. Therefore, γ is divisible by 3; say $\gamma = 3\alpha$. Recall that $(z = 0)$ is a maximal contact hypersurface for f, $I = ((B^3, C^2), 6) = w^\gamma \tilde{I}$ is the associated coefficient ideal, and $\tilde{I} = (\tilde{I}, 6)$ is the companion ideal. (The latter has maximal order since $\mathrm{inv}(a) = (3, 0, 1, \ldots)$.) By the Weierstrass preparation theorem, we can assume that

$$B = w^\alpha u(y^2 - Px^2),$$

where $P = P(w, x)$ and u is a unit.

We will prove that P is a power of w times a unit. First note that P is a unit in S because $(\eta_1, \eta_2) = (x, y)$ in S, so the initial form of $B = \eta_1\eta_2$ in S is a non-degenerate quadratic form. In particular, P has a square root in S. Since S is a UFD, we can write

$$\eta_1 = u_1(y + \sqrt{P}x), \quad \eta_2 = u_2(y - \sqrt{P}x),$$

where u_1, u_2 are units of S.

Since $\tilde{I} \subset (y^2 + Px^2, 2)$, $(y = 0)$ is a maximal contact hypersurface for $\tilde{I}$ in $(z = 0)$. The associated coefficient ideal of $\tilde{I}$ is

$$J := \left(Px^2, 2\right) + \left(\left.\frac{\partial^2 C}{\partial y^2}\right|_{y=0}, 1 \right) + \left(\left.\frac{\partial C}{\partial y}\right|_{y=0}, 2 \right) + \left(C|_{y=0}\, 3 \right).$$

Since $C = u_1^3(y + \sqrt{P}x)^3 + u_2^3(y - \sqrt{P}x)^3$, a direct computation shows that the marked ideal J reduces to $(Px^2, 2)$.

Since $\mathrm{inv}(a) = (3, 0, 1, 0, 1, \ldots)$, the marked ideal J has order 1 after fully factoring w, i.e., P is a unit times a power w^β of w in R. Therefore we can assume that

$$B = w^\alpha(y^2 - w^\beta x^2),$$

and we can write

$$\eta_1 = u_1(y + w^{\beta/2}x), \quad \eta_2 = u_2(y - w^{\beta/2}x).$$

We now substitute $w = v^2$. So we can assume that $B = v^{2\alpha}(y^2 - v^{2\beta}x^2)$. We will prove that $f(v^2, x, y, z)$ splits.

First we note that it is enough to prove that $\Delta(v^2, x, y)$ is a square in $\underline{k}[\![v, x, y]\!]$: The latter implies that $f(v^6, x, y, z)$ splits, by Lemma 3.3, and therefore that either $f(v^2, x, y, z)$ or $f(v^3, x, y, z)$ splits, since f is a polynomial of degree 3 in z. Recall that $I = (B^3, C^2) = w^\gamma I'$, where γ is divisible by 3. Then C is divisible by $w^{3\delta}$, for some δ. If $f(v^3, x, y, z)$ splits, this would contradict (b) $\Rightarrow$ (c) in Lemma 3.1(3).

Let F denote the field of fractions of $R := \underline{k}[\![v, x, y]\!]$, M the field of fractions of $S := \overline{\underline{k}(\!(v)\!)}[\![x, y]\!]$, and L the subfield of M generated over F by η_1 and η_2 — i.e., the splitting field of $f = f(v^2, x, y, z)$. We consider the Galois group $\mathrm{Gal}_F L$ of L over F.

The Galois group $\mathrm{Gal}_F L$ is a subgroup of the symmetric group $\mathfrak{S}_3$, where we can view the latter as the group of the permutations of

$$\left\{\eta_1, \epsilon\eta_1, \epsilon^2\eta_1, \eta_2, \epsilon\eta_2, \epsilon^2\eta_2\right\}$$

preserving the expresions $\eta_1\eta_2$ and $\eta_1^3 + \eta_2^3$.

We will prove that $\mathrm{Gal}_F L$ has no element of order 2. Consider $\sigma \in \mathrm{Gal}_F L$. Now, $y \pm v^\beta x \in F$, so $\sigma\eta_1 = (\sigma u_1)(y + v^\beta x)$. By Lemma 3.7 following, σu_1 is a unit of S. Therefore, $\sigma\eta_1$ cannot be either η_2, $\epsilon\eta_2$ or $\epsilon^2\eta_2$, so that σ cannot be of order 2.

As a consequence, η_1^3 and η_2^3 are fixed by $\mathrm{Gal}_F L$; therefore, $\eta_1^3, \eta_2^3 \in F$. So Δ has a square root in F, namely, $\eta_1^3 - \eta_2^3$. Since R is a unique factorization domain and F is its field of fractions, Δ also has a square root in R. $\square$

Lemma 3.7. *Let $S^\times$ denote the set of units of S. Let σ be an automorphism of the field of fractions of S. Then $\sigma(S^\times) = S^\times$ and $\sigma S = S$.*

Proof. Let M denote the field of fractions of S. As subsets of M, the sets S and $S^\times$ admit the following characterizations, due to the fact that S is a formal power series ring over an algebraically closed field: $S^\times = \{f \in M : \text{ for all } n \in \mathbb{N}, \text{ there exists } y \in M \text{ such that } f = y^n\}$, and $S = \{f \in M : f \in S^\times \text{ or } 1 + f \in S^\times\}$. The lemma follows. □

4. Minimal singularities in four variables

In the section, we will prove Theorems 1.3 and 1.5 using Theorems 1.7 and 1.9.

Proof of Theorem 1.3. We can reduce to the case that X is an embedded hypersurface (see Remark 1.4). We then construct the blowing-up sequence in several steps. (It is possible also to find local normal forms for the minimal singularities of the total transform; see Remark 4.1 following the proof.) Let $\mathcal{Q}_0 := \{\text{nc4}\}$. (The notation $\{\text{nc4}\}$ means the set of points with nc4 singularities; see [4, Notation 1.16].)

(I) Following the desingularization algorithm, we can blow up with closed admissible centres lying over the complement of $\mathcal{Q}_0$ until the maximum value of the invariant over the complement of $\mathcal{Q}_0$ is $\mathrm{inv}(\text{nc3}) := (3, 0, 1, 0, 1, 0, \infty)$. The locus $(\mathrm{inv} = \mathrm{inv}(\text{nc3}))$ is a smooth curve, and its components where X is not generically nc3 are closed. So we can blow up these components. (For simplicity of notation, we use X to mean also its strict transform in any year of the blowing-up history, $\mathcal{Q}_0$ to mean the inverse image of $\mathcal{Q}_0$, etc.)

Now, by Theorem 1.9; i.e., by the cleaning lemma [4, Sect. 2] applied to the normal forms in Theorem 1.7, there is a further sequence of admissible blowings-up after which every point of the strict transform of the locus $(\mathrm{inv} = \mathrm{inv}(\text{nc3}))$ is of one of the following three kinds (where an asterisk in the table means that the exceptional divisor may or may not be present at the indicated point).

	singularity	exceptional divisor
nc3	$xyz = 0$	$(w = 0)^*$
prod	$x(z^2 + wy^2) = 0$	$(w = 0)$
cp3	$z^3 + wy^3 + w^2x^3 - 3wxyz = 0$	$(w = 0)$

The following table lists the singlarities which occur in small neighbourhoods of $\mathcal{Q}_0$ and the points in the preceding table. (The equations in the following table are in suitable étale coordinates at the indicated singular points, not necessarily with respect to the coodinates in the preceding table.)

		singularity	exceptional divisor
nc4	nc3		
	nc2		
nc3	nc2	$xy = 0$	$(w = 0)^*$
prod	nc3		
	nc2		
	nc2	$x(z^2 + w) = 0$	$(w = 0)$
	pp	$z^2 + wy^2 = 0$	$(w = 0)$
cp3	nc3		
	nc2		
	dpp		$(w = 0)$

The absence of an exceptional divisor in any row of the table means, of course, that the indicated singularity is outside the exceptional divisor shown in the preceding table. In the case of nc3 in this table, the exceptional divisor (if present) is transverse to smooth points in a small neighbourhood. In the case of prod, there are neighbouring smooth points $z^2 + w = 0$ with tangent exceptional divisor $(w = 0)$. In the last row of the table, the exceptional divisor $(w = 0)$ is with respect to the coordinates for the equation of cp3 in the first table above. In this case, the degenerate pinch points dpp occur along the nonzero x-axis, and the exceptional divisor has tangential contact (order 3) at smooth points.

(II) Let us say we are now in year j_1. Let $\mathcal{Q}_{j_1} := \{\text{nc4, nc3, prod, cp3}\} =$ closure of $\{\text{nc3}\}$. The points nc4, prod and cp3 are isolated.

Recall that, according to [4, Rmk. 4.4], we cannot, in general, reduce limits of degenerate pinch points to dpp by cleaning. In this step we will show, however, that, after additional blowings-up of points, limiting points of those components of the dpp locus that are adherent to $\{\text{cp3}\}$ can be cleaned up to give only dpp. At the same time, we will clean up the components adherent to $\{\text{prod}\}$ of the locus of nc2 points $x(z^2 + w) = 0$ with tangent exceptional divisor $(w = 0)$.

So we blow up $\{\text{prod, cp3}\}$. First consider the effect on a prod singularity $x(z^2 + wy^2) = 0$. After blowing up, we have a prod singularity with the same equation, at the origin of the chart with coordinates $(w, x/w, y/w, z/w)$ (the "w-chart"; for economy of notation we use (w, x, y, z) again to denote the new coordinates). At nonzero points of the y-axis in this chart, we have nc2 singularities $x(z^2+w)=0$ with exceptional divisor $(w=0)$, in suitable étale local coordinates. In the y-chart, with coordinates $(w/y, x/y, y, z/y)$ (which we again denote simply (w,x,y,z)), these points occur along the nonzero w-axis. At the origin of this chart, we have a singularity $x(z^2+wy)=0$ with exceptional divisor $(y=0)$. Let us blow up this point. Then in the new w-chart, we get $x(z^2+y)=0$ with exceptional divisor $(w=0)+(y=0)$.

Secondly, consider the effect of blowing up a cp3 point $z^3+wy^3+w^2x^3-3wxyz=0$, where $(w=0)$ is the exceptional divisor. In the w-chart, with coordinates $(w, x/w, y/w, z/w)$, we get a singularity of the same kind at the origin, with dpp singularities along the nonzero x-axis. (According to §2.2, these dpp singularities can be written $w^2+(z-y^2)^2(z+2y^2)=0$, with exceptional divisor $(w+5y^3+3yz=0)$, in suitable local coordinates.) These dpp points occur along the nonzero w-axis of the x-chart, with coordinates $(w/x, x, y/x, z/x)$. At the origin of this chart we have

$$z^3+wxy^3+w^2x^2-3wxyz=0,$$

with exceptional divisor $(x=0)$. Let us blow up this point twice. Then we get

$$z^3+w^3xy^3+x^2-3wxyz=0, \tag{4.1}$$

with exceptional divisor $(x=0)+(w=0)$ (and the dpp singularities still along the nonzero w-axis). Completing the square with respect to x, we can rewrite (4.1) as

$$\left(x+\frac{w^3y^3-3wyz}{2}\right)^2+z^3-\frac{(w^3y^3-3wyz)^2}{4}=0,$$

which is the same thing as

$$\begin{aligned}&\left(x+\frac{w^3y^3-3wyz}{2}\right)^2\\ &\quad+\left(z-3\left(\frac{wy}{2}\right)^2-\left(\frac{wy}{2}\right)^2\right)^2\left(z-3\left(\frac{wy}{2}\right)^2+2\left(\frac{wy}{2}\right)^2\right)=0.\end{aligned}$$

Therefore, after a change of variables, we have

$$x^2+(z-w^2y^2)^2(z+2w^2y^2)=0, \tag{4.2}$$

with exceptional divisor $(x+5w^3y^3+3wyz=0)+(w=0)$ (compare §2.2).

We can apply the cleaning lemma to (4.2): We first blow up $(x=z=w=0)$ twice to get

$$x^2+w^2(z-y^2)^2(z+2y^2)=0,$$

with exceptional divisor $(x + 5wy^3 + 3wyz = 0) + (w = 0)$. Then we blow up $(x = w = 0)$ to get a dpp

$$x^2 + (z - y^2)^2(z + 2y^2) = 0,$$

with exceptional divisor $(x+5y^3+3yz = 0)+(w = 0)$. (In particular, no new singularity types occur as limits of dpp points in a neighbourhood of {cp3}.) The centres of the blowings-up involved in the cleaning are isolated from {nc4, nc3, prod, cp3} = closure of {nc3}.

(III) Let us say we are now in year $j_2 \geq j_1$. Let $\mathcal{Q}_{j_2}$ denote the union of {nc4, nc3, prod, cp3} = closure of {nc3}, the adherent components of the dpp locus, and the adherent components of the locus of nc2 points $x(z^2+w) = 0$ with tangential exceptional divisor $(w = 0)$. Then $\mathcal{Q}_{j_2}$ is a closed subset of (the strict transform in year j_2 of) X. (Note that in the current year j_2, for each cp3 singularity $z^3 + wy^3 + w^2x^3 - 3wxyz = 0$, where $(w = 0)$ is the exceptional divisor, the adherent component of the dpp locus comprises the dpp points which lie in the indicated component $(w = 0)$ of the exceptional divisor. The purpose of step (II) was to guarantee that these dpp points together with the cp3 points form a closed subset. A similar remark applies to each prod point and the neighbouring nc2 points with tangential exceptional divisor.)

In some neighbourhood of $\mathcal{Q}_{j_2}$, all singular points outside $\mathcal{Q}_{j_2}$ are nc2, $xy = 0$, with exceptional divisor $(w = 0)$ if present (in suitable étale local coordinates). Moreover, all previous blowings-up are inv_1-admissible, so we can extend inv_1 in year j_2 to an invariant inv as usual (i.e., considering j_2 to be "year zero" for $\mathrm{inv}_{3/2}$). At a neighbourhing nc2 point a outside $\mathcal{Q}_{j_2}$, we have $\mathrm{inv}(a) = \mathrm{inv}(\mathrm{nc2}) := (2, 0, 1, 0, \infty)$ if a lies outside the exceptional divisor. If a is in the exceptional divisor, we have either $\mathrm{inv}(a) = (2, 0, 1, 1, 1, 0, \infty)$, or $\mathrm{inv}(a) = (2, 1, 1, 0, 1, 0, \infty)$. In either case, the exceptional divisor is transverse to the nc2-locus and therefore transverse to any maximal contact hypersurface.

We can blow up with closed admissible centres outside $\mathcal{Q}_{j_2}$ until the maximal value of inv is $\leq (2, 1, 1, 0, 1, 0, \infty)$. Then any component of the locus $(\mathrm{inv} = (2, 1, 1, 0, 1, 0, \infty))$ which is not generically nc2 with transverse exceptional divisor is separated from $\mathcal{Q}_{j_2}$, so we can also blow up these components. It is now not difficult to see that, at any singularity outside $\mathcal{Q}_{j_2}$ with $\mathrm{inv} = (2, 1, 1, 0, 1, 0, \infty)$, we can choose local étale coordinates (v, w, x, y) in which X and the "old" exceptional divisor (counted by $s_1 = 1$ in inv) are given (respectively) either by equations of the form

$$z^2 + w^\alpha y^2 = 0,$$
$$\zeta + w^\beta(\eta + w^\gamma v) = 0,$$

where $\alpha \le 2\beta$, ζ belongs to the ideal generated by z, η belongs to the ideal generated by y, and $(w = 0)$ is the "new" exceptional divisor, or by equations of the form

$$z^2 + w^\alpha(\nu + w^\gamma y)^2 = 0,$$
$$\zeta + w^\beta v = 0,$$

where $\alpha \ge 2\beta$, ζ is in the ideal generated by z, ν is in the ideal generated by v, and $(w = 0)$ is the new exceptional divisor. (Compare with [4, §1.2 and Lemma 4.2].) In both cases, by the cleaning lemma, we can blow up to get either nc2, $z^2 + y^2 = 0$, or pp, $z^2 + wy^2 = 0$, where $(v = 0)$ and perhaps $(w = 0)$ (for example, in the pp case) give the support of the exceptional divisor.

We now repeat essentially the same operations using $(2, 0, 1, 1, 1, 0, \infty)$ for the maximum value of inv outside $\mathcal{Q}_{j_2}$ and the strict transform of $(\mathrm{inv} = (2, 1, 1, 0, 1, 0, \infty))$ above, then again using inv(nc2) for the maximum value outside $\mathcal{Q}_{j_2}$ and the strict transforms of the previous two inv-loci. The points outside $\mathcal{Q}_{j_2}$ that have been cleaned up are all nc2 or pp, with exceptional divisor as indicated above.

(IV) We can now use the desingularization algorithm to resolve any singularities remaining outside the locus of points with singularities in $\mathcal{S}$, by admissible blowings-up. This completes the proof. □

Remark 4.1. The proof above provides normal forms for the total transform at every singular point of the strict transform. In order to get appropriate normal forms for the total transform at smooth points of the strict transform, we should continue as in [4, Rmk. 4.3]. The normal forms will include the possibility of tangential contact of a component of the exceptional divisor, of order 2 (for, example, in a neighbourhood of a pinch point) or order 3 (for example, in a neighbourhood of a degenerate pinch point).

Proof of Theorem 1.5. We begin as in Theorem 1.3, repeat steps (I) and (II) of the latter, and then continue as follows.

(III) Let us say we are now in year $j_2 \ge j_1$. Set $\mathcal{Q}'_{j_2} := \mathcal{Q}_{j_2} \cup \{\mathrm{exc}\}$, where $\mathcal{Q}_{j_2}$ is the subset defined in step (III) of Theorem 1.3 and $\{\mathrm{exc}\}$ denotes the set of exceptional singularities (1.3). We can blow up with closed admissible centres outside $\mathcal{Q}'_{j_2}$ until the maximum value of the inv over the complement of $\mathcal{Q}'_{j_2}$ is $\mathrm{inv}(\mathrm{dpp}) := (2, 0, 3/2, 0, 2, 0, \infty)$.

The locus (inv $=$ inv(dpp)) is a smooth curve. Each component of this curve either contains no dpp or is generically dpp (according as $\operatorname{Sing} X$ has codimension > 2 or $= 2$ at the generic point). Components with no dpp are closed and separated from $\mathcal{Q}'_{j_2}$; we can blow up to get rid of these components. So any remaining component of (inv $=$ inv(dpp)) is generically dpp.

By [4, Rmk. 4.4], at any point of (inv $=$ inv(dpp)), the (strict transform of) X is defined by an equation of the form

$$z^2 + w^\alpha \left(y + w^\beta x^2\right)^2 \left(y - 2w^\beta x^2\right) = 0,$$

in suitable étale coordinates (w, x, y, z), where w^α, w^β are monomials in the exceptional divisor $(w = 0)$. (The exceptional divisor has only one component at the given point, since any component is transverse to (inv $=$ inv(dpp).)

We can use the cleaning lemma to blow up to reduce to the case that $\beta = 0$ and $|\alpha| = 0$ or 1. (The centres of blowing up involved are closed in X and disjoint from $\mathcal{Q}'_{j_2} \cup \{\text{dpp}\}$.) If $\alpha = 0$, then we have a dpp. If $|\alpha| = 1$, then we can rewrite the equation as

$$z^2 + wy(y + x^2)^2 = 0. \tag{4.3}$$

(We recall that blowing up $(z = y = w = 0)$ results in an exceptional singularity $z^2 + y(wy + x^2) = 0$.)

(IV) Let us say we are now in year $j_3 \geq j_2$. Let $\mathcal{Q}'_{j_3}$ denote $\mathcal{Q}'_{j_2}$ together with all degenerate pinch points and their limits. (Limits of dpp outside $\mathcal{Q}'_{j_2}$ are either dpp or singularities of the form (4.3).) The blowings-up involved in (III) are inv_1-admissible, so we can extend inv_1 in year j_3 to an invariant inv as usual (i.e., considering j_3 to be year zero for $\text{inv}_{3/2}$; cf. step (III) in the proof of Theorem 1.3). We can blow up with closed admissible centres outside $\mathcal{Q}'_{j_3}$ until the maximum value of the inv over the complement of $\mathcal{Q}'_{j_3}$ is $\text{inv(pp)} := (2, 0, 3/2, 0, 1, 0, \infty)$.

We argue as in step (III). The locus (inv $=$ inv(pp)) is a smooth curve. Each component of this curve either contains no pp or is generically pp. Components with no pp are closed and separated from $\mathcal{Q}'_{j_3}$; we can blow up to get rid of these components. So any remaining component of (inv $=$ inv(pp)) is generically pp. At any point of (inv $=$ inv(pp)), X is defined by an equation of the form

$$z^2 + w^\alpha \left(y + w^\beta x\right)^2 \left(y - 2w^\beta x\right) = 0,$$

in suitable étale coordinates (w, x, y, z), where w^α, w^β are monomials in the exceptional divisor $(w = 0)$. We can proceed as in the proof of [4, Thm. 1.14], using the cleaning lemma, to reduce to the case that

$\alpha = \beta = 0$; i.e., to a pinch point with exceptional divisor transverse to the pp locus.

(V) Say we are now in year $j_4 \geq j_3$, and set $\mathcal{Q}'_{j_4} := \mathcal{Q}'_{j_3} \cup \{\mathrm{pp}\}$. We then repeat the argument of Step (III) in Theorem 1.3, successively using $\mathrm{inv} = (2, 1, 1, 0, 1, 0, \infty)$, $(2, 0, 1, 1, 1, 0, \infty)$, $(2, 0, 1, 0, \infty)$ as maximum value and then cleaning the corresponding locus. The points outside $\mathcal{Q}'_{j_4}$ that have been cleaned up are all either nc2, $z^2 + y^2 = 0$, or pp, $z^2 + uy^2 = 0$, with appropriate exceptional divisor.

(VI) Say we are now in year $j_5 \geq j_4$, and define $\mathcal{Q}'_{j_5}$ by adjoining the latter points to $\mathcal{Q}'_{j_4}$. Then $\mathcal{Q}'_{j_5}$ comprises singularities in $\mathcal{S}'$, together perhaps with singularities of type (4.3). Consider a singularity (4.3). At a nearby point a where $z = y = w = 0$, $y + x^2 \neq 0$, we can choose étale coordinates (w, x, y, z) in which we have $z^2 + wy = 0$ (i.e., a is a quadratic cone singularity) with exceptional divisor $(w = 0)$. So $\mathrm{inv}(a) = (2, 0, 1/2, 1, 1, 0, \infty)$.

We can blow up with closed admissible centres separated from $\mathcal{Q}'_{j_5}$ until the maximum value of inv in the complement of $\mathcal{Q}'_{j_5}$ is $(2, 0, 1/2, 1, 1, 0, \infty)$. Then the locus $(\mathrm{inv} = (2, 0, 1/2, 1, 1, 0, \infty))$ extends to a point (4.3) as above, as $z = y = w = 0$. So this locus, together with limiting points of the form (4.3) is a closed set that provides an admissible centre of blowing up. We blow up this locus. The effect is to convert singularities (4.3) to exceptional singularities exc.

(VII) We can now use the desingularization algorithm to resolve any singularities remaining outside the locus of points with singularities in $\mathcal{S}'$, by admissible blowings-up. This completes the proof. (Normal forms for the total transform can again be handled as in Remark 4.1.) $\square$

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