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Raffaele Mosca

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Raffaele Mosca

Dipartimento di Scienze, Universitá degli Studi "G. d'Annunzio", Pescara 65127, Italy.
`r.mosca@unich.it`

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We prove that on the class of (P_6,diamond)-free graphs the Maximum-Weight Independent Set problem and the Minimum-Weight Independent Dominating Set problem can be solved in polynomial time.

Keywords: Maximum-Weight Independent Set; Minimum-Weight Independent Dominating Set; polynomial-time algorithm.

1 Introduction

An *independent set* (or a *stable set*) in a graph G is a subset of pairwise nonadjacent vertices of G. An independent set of G is *maximal* if it is not properly contained in any other independent set of G.

The Maximum-Weight Independent Set (*WIS*) problem is the following: Given a graph $G = (V, E)$ and a weight function w on V, determine an independent set of G of maximum weight. Let $\alpha_w(G)$ denote the maximum weight of an independent set of G. The WIS problem reduces to the *IS* problem if all vertices v have the same weight $w(v) = 1$.

The WIS problem is NP-hard [34] and remains difficult for cubic graphs [27] and for planar graphs [26], while it can be efficiently solved for various graph classes which include perfect graphs [33] (and the class of perfect graphs includes the chordal graphs), $K_{1,3}$-free graphs [2, 37, 40, 42, 45], and $2K_2$-free graphs [21, 22, 38].

The Minimum-Weight Independent Dominating Set (*WID*) problem is the following: Given a graph $G = (V, E)$ and a weight function w on V, determine a maximal independent set of G of minimum weight. Let $\iota_w(G)$ denote the minimum weight of a maximal independent set of G. The WID problem reduces to the *ID* problem if all vertices v have the same weight $w(v) = 1$.

The WID problem is NP-hard [28] and remains difficult for chordal graphs [18] and for $2P_3$-free perfect graphs [46], while it can be efficiently solved for various graph classes which include permutation graphs [15], locally independent well-dominated graphs [47], and $2K_2$-free graphs [21, 22, 38].

Both WIS and WID remain difficult for triangle-free graphs [43]. Also, for both IS and ID, the class of P_5-free graphs is the unique minimal class, defined by forbidding a single connected subgraph, for which the computational complexity is an open question (see [1, 3, 7]).

On the other hand, several papers introduced structural properties on graphs containing no long induced paths (see e.g. [5, 6, 19, 39]), often applied to design efficient algorithms for solving various NP-hard problems, including WIS or WID, in subclasses of such graphs: concerning subclasses of P_5-free graphs, see e.g. [4, 9, 10, 11, 13, 16, 17, 23, 29, 31, 36]; concerning subclasses of P_6-free graphs, see e.g. [9, 24, 30, 35, 41, 44].

Let us focus on two such graph subclasses which involve triangle-free graphs as well.

The class of (P_5,diamond)-free graphs: a recent paper [8] shows that such graphs have bounded clique-width and that a corresponding clique-width expression can be constructed in $\mathcal{O}(n+m)$ time, which implies that a large class of NP-hard problems including WIS and WID can be solved for such graphs in $\mathcal{O}(n+m)$ time.

The class of (P_6,triangle)-free graphs: a recent paper [14] shows that such graphs have bounded clique-width and that a corresponding clique-width expression can be constructed in $\mathcal{O}(n^2)$ time, which implies that a large class of NP-hard problems including WIS and WID can be solved for such graphs in $\mathcal{O}(n^2)$ time.

This paper introduces a proof that WIS and WID can be solved for (P_6,diamond)-free graphs in $\mathcal{O}(n^7)$ time.

2 Notation and preliminaries

For any missing notation or reference, let us refer to [12].

Throughout this paper let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. For every $u \in V$, let $N(u) = \{v \in V : uv \in E\}$ be the set of *neighbors* of u. Let U, W be two subsets of V. Let $N(U) = \{v \in V \setminus U$: there exists $u \in U$ such that $uv \in E\}$. Let $N_W(U) = N(U) \cap W$. Let us say that U has a *join* (a *co-join*, respectively) with W, if each vertex in U is adjacent (is nonadjacent) to each vertex in W. Let $v \in V$. Let us say that: v *contacts* U if v is adjacent to some vertex of U; v is *universal* for U if v is adjacent to each vertex of U; v is *partial* to U if v contacts U but is not universal for U. Then let us say that U, with $\emptyset \subset U \subset V$, is a *module* of G $-$ often called *homogeneous set* in the literature $-$ if no vertex of $V \setminus U$ is partial to U.

Let $G[U]$ denote the subgraph of G induced by the vertex subset U. For any graph F, G is F*-free* if G contains no induced subgraph isomorphic to F.

A *component of* G is the vertex set of a maximal connected subgraph of G. A component of G is $trivial$ if it is a singleton, and $nontrivial$ otherwise.

Concerning WIS and WID, algorithmically an easy reduction works if the graph is disconnected: that is, if G has components $V_1, \ldots, V_k$, then $\alpha_w(G) = \alpha_w(G[V_1]) + \ldots + \alpha_w(G[V_k])$ and $\iota_w(G) = \iota_w(G[V_1]) + \ldots + \iota_w(G[V_k])$.

A path P_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j < k$. A cycle C_k has vertices $v_1, v_2, \ldots, v_k$ and edges $v_j v_{j+1}$ for $1 \leq j \leq k$ (subscript addition taken modulo k). A $triangle$ is a graph of three vertices a, b, c and edges ab, ac, bc. A $diamond$ is a graph of four vertices a, b, c, d and edges ab, ac, ad, bc, bd.

A $clique$ of G is a set of pairwise adjacent vertices of G. Notice that each component of G is a clique if and only if G is P_3-free.

A graph is $chordal$ if it contains no induced C_k, $k \geq 4$.

For chordal graphs, WIS and ID can be efficiently solved (see [25] and [20], respectively), while WID remains NP-hard on them [18].

In [32] the authors proved that distance-hereditary graphs have bounded clique-width, and that a corresponding clique-width expression can be constructed in $\mathcal{O}(n+m)$ time. Since chordal diamond-free graphs are distance-hereditary, a direct consequence is:

Theorem 1 ([32]) *Both the WIS and the WID problems can be solved for chordal diamond-free graphs in $\mathcal{O}(n+m)$ time.* $\square$

In [14] the authors proved that (P_6,triangle)-free graphs have bounded clique-width, and that a corresponding clique-width expression can be constructed in $\mathcal{O}(n^2)$ time. A direct consequence is:

Theorem 2 ([14]) *Both the WIS and the WID problems can be solved for (P_6,triangle)-free graphs in $\mathcal{O}(n^2)$ time.* $\square$

Obviously, the WIS (or WID) problem on a graph G with vertex weight function w can be reduced to the same problem on subgraphs $G[V \setminus N(v)]$ for every $v \in V$ in the following way:

$$\alpha_w(G) = \max\{\alpha_w(G[V \setminus N(v)]) \mid v \in V\}$$

$$\iota_w(G) = \min\{\iota_w(G[V \setminus N(v)]) \mid v \in V\}$$

Thus, whenever WIS (or WID) is solvable in time T for every subgraph $G[V \setminus N(v)]$ of G with $v \in V$, then it is solvable for G in time nT, plus $\mathcal{O}(n^3)$ additional steps to generate those subgraphs.

Let us conclude with an observation which will be often used later.

Observation 1 *Let $G = (V, E)$ be a graph, and $U \subseteq V$ with $|U| = k$. If one can solve WIS (or WID) for each induced subgraph of $G[V \setminus U]$ in time T, then one can solve WIS (or WID) for G in time $2^k(T + n^2)$.*

Proof: Let $I(U)$ be the family of independent sets of $G[U]$. Then to solve WIS (or WID) for G one can consider WIS (or WID) for $|I(U)|$ subgraphs of G, i.e., for $G[V \setminus U]$ and for $G[I \cup (V \setminus (N(I) \cup U))]$ for every $I \in I(U)$. Since $|I(U)| \leq 2^k$, the assertion follows. $\square$

Remark: The results of the next section are introduced only for WIS, by meaning that they hold for WID as well (by interchanging WIS with WID, and α with ι).

3 Independent sets in (P_6,diamond)-free graphs

Let us introduce a preliminary result.

Definition 1 *A graph $G = (V, E)$ is* green *if there exists a partition $\{X, Y\}$ of V (with X or Y possibly empty) such that:*

(i) $G[X]$ and $G[Y]$ are P_3-free;

(ii) each component of $G[Y]$ is a module of G;

(iii) each vertex in Y is adjacent to at most one vertex in each component of $G[X]$.

Notice that every P_3-free graph is green.

Lemma 1 *One can solve WIS for every green P_6-free graph in $O(n^3)$ time.*

Proof: Let $G = (V, E)$ be a green P_6-free graph. Assume without loss of generality that G is connected. Let $\{X, Y\}$ be a partition of V according to Definition 1. In particular, to our aim, by ii one can assume that Y is an independent set. In fact, one can contract each component K of $G[Y]$ into a singleton u with $w(u) = \alpha_w(G[K])$: that can be done in polynomial time since each component of $G[Y]$ is a clique.

Let W be the family of nontrivial components of $G[X]$. In particular, to our aim (similarly to above), one can assume that in each component $Q \in W$ at most one vertex is nonadjacent to any vertex in Y.

Claim 3.1 *There exists $y^* \in Y$ such that y^* contacts every element of W.*

Proof: For any $y \in Y$, let $J(y) = \{Q \in W : y \text{ contacts } Q\}$. Let $y^* \in Y$ be such that $|J(y^*)| \geq |J(y)|$ for every $y \in Y$. We state that this vertex y^* is a proper choice for Claim 3.1. Assume for a contradiction that there exists a component $Q \in W$ such that y^* does not contact Q. Since G is connected, one can select $y \in Y$ belonging to a shortest path from y^* to Q, such that y contacts Q. By definition of W and by iii, Q contains two (adjacent) vertices q_1 and q_2 such that y is adjacent to q_1 and nonadjacent to q_2. Then, since G is P_6-free, y contacts all the elements of W which are contacted by y^*. This implies $|J(y^*)| < |J(y)|$, a contradiction. □

Claim 3.2 *There exists at most one element of W of cardinality greater than 2.*

Proof: Assume for a contradiction that there exist two elements of W, say $\tilde{Q}$ and Q, with $|\tilde{Q}| \geq 3$ and $|Q| \geq 3$. Let $y^* \in Y$ according to Claim 3.1. Let $\tilde{q} \in \tilde{Q}$ and $q \in Q$ be adjacent to y^*. Since $|\tilde{Q}| \geq 3$, there exists $a \in Y$ adjacent to $q_a \in \tilde{Q}$, with $q_a \neq \tilde{q}$. Then a is adjacent to q, otherwise $a, q_a, \tilde{q}, y^*, q$ and any vertex in Q nonadjacent to a induce a P_6. Then, since $|Q| \geq 3$, there exists $b \in Y$ adjacent to $q_b \in Q$, with $q_b \neq q$. By symmetry, one has that b is adjacent to $\tilde{q}$. Since $|Q| \geq 3$, there exists $q' \in Q$ such that $q' \neq q$ and q' is nonadjacent to b. Then $q', q_b, b, \tilde{q}, q_a, a$ induce a P_6, a contradiction. □

If $|Q| \leq 2$ for every $Q \in W$, then by iii G is triangle-free. Otherwise, by Claim 3.2 there exists at most one element, say $\tilde{Q}$, of W of cardinality greater than 2. One can solve WIS in G by solving WIS in $G[V \setminus \tilde{Q}]$ and in $G[V \setminus N(\tilde{q})]$ for every $\tilde{q} \in \tilde{Q}$. Since such graphs are triangle-free, the lemma follows by Theorem 2. □

3.1 Deleting C_6's in (P_6,diamond)-free graphs

Throughout this subsection assume that $G = (V, E)$ is a (P_6,diamond)-free graph containing a 6-cycle C, say with vertices v_i and edges $v_i v_{i+1}$, $i \in \{1, \ldots, 6\}$ (subscript addition taken modulo 6). Let $N(C)$ be the set of vertices from $V \setminus C$ which are adjacent to some vertex in C. For any subset S of C, let M_S be the set formed by those vertices in $N(C)$ which are adjacent to each vertex in S and are nonadjacent to each vertex in $C \setminus S$. In particular, let us write M_1 for $S = \{v_1\}$, $M_{1,2}$ for $S = \{v_1, v_2\}$, and so on. Then let $Z(k)$ denote the set of vertices of $N(C)$ having exactly k neighbors in C.

Since G is (P_6, diamond)-free: $Z(1) = Z(5) = Z(6) = \emptyset$; each vertex in $Z(2)$ belongs to some of the sets $M_{i,i+2}$ or $M_{i,i+3}$, $i \in \{1, \ldots, 6\}$ (subscript addition taken modulo 6); each vertex in $Z(3)$ belongs to some of the sets $M_{i,i+2,i+4}$ or $M_{i,i+2,i+3}$ or $M_{i,i+3,i+4}$, $i \in \{1, \ldots, 6\}$ (subscript addition taken modulo 6).

Lemma 2 *Every component of $G[Z(0)]$ is green.*

Proof: Let K be a component of $G[Z(0)]$. Since G is connected, there exists $v \in V \setminus K$ which contacts K. If v is universal for K, then $G[K]$ is P_3-free (since G is diamond-free). Then let us assume that v is partial to K, and prove that $G[K]$ is green. Let us write $X = K \cap N(v)$, and $Y = K \setminus N(v)$. Since G is diamond-free, $G[X]$ is P_3-free. Let T be a component of $G[Y]$. Then T is a module of $G[K]$: otherwise, for any $x \in X$ partial to T, one has that two adjacent vertices in T together with x, v and two vertices in C would induce a P_6 (since G is (P_6,diamond)-free, v is the endpoint of an induced P_3 involving two vertices in C). Then T is a clique (since G is diamond-free), i.e., $G[Y]$ is P_3-free. Furthermore, each vertex in Y is adjacent to at most one vertex in each component of X, otherwise a diamond arises involving v. Then the lemma follows. □

Let us fix any vertex of C, say v_2.
Let us prove that WIS can be solved for $G[V \setminus N(v_2)]$ in $\mathcal{O}(n^6)$ time.
A partition of $V \setminus N(v_2)$ is given by $\{\{v_2, v_4, v_5, v_6\}, M_{1,3,4,6}, M_{1,3,5}, M_{1,3,4}, M_{1,4,5}, M_{3,5,6}, M_{3,6,1}, M_{4,6,1}, M_{6,3,4}, M_{1,3}, M_{1,4}, M_{1,5}, M_{3,5}, M_{3,6}, M_{4,6}, Z(0)\}$.
Since G is diamond-free, the sets $M_{1,3,4,6}, M_{1,3,4}, M_{1,4,5}, M_{3,5,6}, M_{3,6,1}, M_{4,6,1}, M_{6,3,4}$ have cardinality at most one. Then, by Observation 1, to our aim it is sufficient to prove that WIS can be solved for each induced subgraph of $G[U]$ in polynomial time, where a partition of U is given by $\{M_{1,3,5}, M_{1,3}, M_{1,4}, M_{1,5}, M_{3,5}, M_{3,6}, M_{4,6}, Z(0)\}$.
Since G is diamond-free: $M_{1,3,5}, M_{1,3}, M_{1,4}, M_{1,5}, M_{3,5}, M_{3,6}, M_{4,6}$ are independent sets. Since G is (P_6,diamond)-free: $M_{1,3,5} \cup M_{1,3} \cup M_{1,5} \cup M_{3,5}$ is an independent set; $M_{1,3} \cup M_{1,5} \cup M_{3,5} \cup M_{4,6}$ has a co-join with $Z(0)$.
For any $W \subseteq U$, let us write $W^* = W \cap Z(0)$.
For any $W \subseteq U$, let us say that a component K of $G[W^*]$ is of:

type 1 if K is not a clique and there exists a vertex in $W \setminus Z(0)$ partial to K;

type 2 if K is a clique and there exists a vertex in $W \setminus Z(0)$ partial to K;

type 3 otherwise.

Let $T_1(W), T_2(W), T_3(W)$ respectively denote the union of components of $G[W^*]$ of type 1, 2, 3.
Let us fix a subset $W \subseteq U$.
Notice that $M_{1,3,5}$ is an independent set, since G is diamond-free.
Let us consider the following binary relation '$\leq_1$' on $M_{1,3,5}$: for any $a, b \in M_{1,3,5}$, $a \leq_1 b$ if $N_{T_1(W)}(a) \subseteq N_{T_1(W)}(b)$. It is immediate to verify that $(M_{1,3,5}, \leq_1)$ is a partially ordered set.

Lemma 3 *Let $y \in M_{1,3,5}$ be maximal for $(M_{1,3,5}, \leq_1)$. Then $G[T_1(W) \setminus N(y)]$ is P_3-free.*

Proof: First let us prove that y contacts every component Q of type 1 of $G[W^*]$. Assume for a contradiction that there exists a component Q_1 of type 1 of $G[W^*]$ such that y does not contact Q_1. By definition of component of type 1, there exists $y_1 \in W \setminus Z(0)$ partial to Q_1. Since G is P_6-free (also recall that $W \subseteq U$), $y_1 \in M_{1,3,5}$. By the maximality of y there exists a vertex $\tilde{q}$ in some component $\tilde{Q}$ of type 1 of $G[W^*]$ such that $\tilde{q}$ is adjacent to y and nonadjacent to y_1. Then $\tilde{q}, y, v_1, y_1$ and two adjacent vertices of Q_1 induce a P_6, a contradiction.

To conclude the proof of the lemma one has to show that if y contacts a component Q of type 1 of $G[W^*]$, then $G[Q \setminus N(y)]$ is P_3-free. This can be shown by applying the last part of the argument of Lemma 2. □

Let us consider the following binary relation '$\leq_2$' on $M_{1,3,5}$: for any $a, b \in M_{1,3,5}$, $a \leq_2 b$ if $N_{T_2(W)}(a) \subseteq N_{T_2(W)}(b)$. It is immediate to verify that $(M_{1,3,5}, \leq_2)$ is a partially ordered set.

Lemma 4 *Let $y \in M_{1,3,5}$ be maximal for $(M_{1,3,5}, \leq_2)$. If $T_1(W) = \emptyset$, then $G[(W \setminus N(y))^*]$ admits at most one component of type 2.*

Proof: Let Q_1 be a component of type 2 of $G[(W \setminus N(y))^*]$. By definition of component of type 2 there exists $y_1 \in (W \setminus N(y)) \setminus Z(0)$ partial to Q_1. Since G is P_6-free (also recall that $W \subseteq U$), $y_1 \in M_{1,3,5}$. Let $q_1 \in Q_1$ be adjacent to y_1; let $q' \in Q_1$ be nonadjacent to y_1. Since $T_1(W) = \emptyset$, there exists a component of type 2 of $G[W^*]$, say $\tilde{Q}_1$, such that $Q_1 \subseteq \tilde{Q}_1$. In particular, there exists $\tilde{q}_1 \in \tilde{Q}_1 \setminus Q_1$ adjacent to y and nonadjacent to y_1, otherwise, by the maximality of y there would exist a vertex $t \in W \setminus \tilde{Q}_1$ adjacent to y and nonadjacent to y_1, i.e., vertices q', q_1, y_1, v_1, y, t would induce a P_6.

Let us prove that Q_1 is the unique component of type 2 of $G[(W \setminus N(y))^*]$. Assume for a contradiction that there exists another component of type 2, say Q_2, of $G[(W \setminus N(y))^*]$. Notice that y_1 is nonadjacent to any vertex q_2 of Q_2, otherwise $q', \tilde{q}_1, y, v_1, y_1$ and q_2 would induce a P_6. Then let $y_2 \in M_{1,3,5}$ be partial to Q_2. By symmetry, y_2 is nonadjacent to any vertex of Q_1. Then a vertex of $Q_2, y_2, v_1, y_1, q_1, q'$ induce a P_6, a contradiction. □

Now, let us consider the following cases:

1. $T_1(U) = T_2(U) = \emptyset$.

   Then each component K of $G[Z(0)]$ is a module of $G[U]$. Then, to our aim, one can assume that K is a singleton. In fact, one can contract K into a singleton u with $w(u) = \alpha_w(G[K])$: that can be done in polynomial time by Lemmas 2 and 1. So in general, one can assume that $Z(0)$ is an independent set. One can solve WIS in $G[U]$ by solving WIS in $G[U \setminus M_{1,4}]$ and in $G[U \setminus N(y)]$, for every $y \in M_{1,4}$.

   That can be done in $\mathcal{O}(n^3)$ time. In fact, by the assumptions and by the above properties, one can verify that $G[U \setminus M_{1,4}]$ is triangle-free, and that $G[U \setminus N(y)]$ is triangle-free for every vertex $y \in M_{1,4}$ (in particular, no vertex of $M_{3,6} \setminus N(y)$ is adjacent to a vertex of $Z(0) \setminus N(y)$, otherwise a P_6 arises). Then the assertion follows by Theorem 2.

   For the other two cases we note that the existence of a component Q of type 1 or type 2 in $U \cap Z(0)$ implies the existence of a vertex $a \in M_{1,3,5}$ which is partial to Q, similarly to the proof above. So, $M_{1,3,5}$ is nonempty.

2. $T_1(U) = \emptyset, T_2(U) \neq \emptyset$.

   Based on $(M_{1,3,5}, \leq_2)$, the vertices $y_1, \ldots, y_h$ of $M_{1,3,5}$ can be totally ordered so that y_i is maximal for $(\{y_i, \ldots, y_h\}, \leq_2)$ for $i = 1, \ldots, h$. Then one can solve WIS in $G[U]$ by sequentially solving WIS in $G[U \setminus N(y_1)]$, in $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$, and in $G[U \setminus M_{1,3,5}]$.

   That can be done in $\mathcal{O}(n^5)$ time. In fact, let us first consider $G[U \setminus N(y_1)]$. If $G[(U \setminus N(y_1))^*]$ admits no component of type 2, then one can refer to Case 1. Otherwise, by Lemma 4, $G[(U \setminus N(y_1))^*]$ admits a unique component of type 2, say Q. Then one can solve WIS in $G[U \setminus N(y_1)]$ by solving WIS in $G[(U \setminus N(y_1)) \setminus Q]$ and in $G[(U \setminus N(y_1)) \setminus N(q)]$, for every $q \in Q$: since for each of such graphs $G[H]$ one has that $G[H^*]$ has no component of type 2, one can refer to Case 1 and to Lemmas 1 and 2. Now, let us consider $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$: by the mentioned total order, one can apply the argument applied for $G[U \setminus N(y_1)]$ in order to show that WIS can be solved for such graphs in polynomial time. Finally, let us consider $G[U \setminus M_{1,3,5}]$: since no vertex in $U \setminus M_{1,3,5}$ is partial to any component of $G[Z(0)]$ (otherwise a P_6 arises), one can refer to Case 1.

3. $T_1(U) \neq \emptyset$.

   Based on $(M_{1,3,5}, \leq_1)$, the vertices $y_1, \ldots, y_h$ of $M_{1,3,5}$ can be totally ordered so that y_i is maximal for $(\{y_i, \ldots, y_h\}, \leq_1)$ for $i = 1, \ldots, h$. Then one can solve WIS in $G[U]$ by sequentially solving WIS in $G[U \setminus N(y_1)]$, in $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$, and in $G[U \setminus M_{1,3,5}]$.

   That can be done in $\mathcal{O}(n^6)$ time. In fact, let us first consider $G[U \setminus N(y_1)]$. By Lemma 3, $G[T_1(W) \setminus N(y_1)]$ is P_3-free. Then $G[(U \setminus N(y_1))^*]$ admits no component of type 1. Then one can refer to Case 2 and to Lemmas 1 and 2. Now, let us consider $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$: by the mentioned total order, one can apply the argument applied for $G[U \setminus N(y_1)]$ in order to show that WIS can be solved for such graphs in polynomial time. Finally, let us consider $G[U \setminus M_{1,3,5}]$: since no vertex in $U \setminus M_{1,3,5}$ is partial to any component of $G[Z(0)]$ (otherwise a P_6 arises), one can refer to Case 1.

Let us summarize the above argument as follows.

Theorem 3 *Let $G = (V, E)$ be a (P_6,diamond)-free graph containing a 6-cycle C. Then one can solve WIS for $G[V \setminus N(c)]$ in $\mathcal{O}(n^6)$ time, for any vertex c of C.* □

3.2 Deleting C_5's in (P_6,diamond,C_6)-free graphs

Throughout this subsection assume that $G = (V, E)$ is a (P_6,diamond,C_6)-free graph containing a 5-cycle C, say with vertices v_i and edges $v_i v_{i+1}$, $i \in \{1, \ldots, 5\}$ (subscript addition taken modulo 5). Let $N(C)$ be the set of vertices from $V \setminus C$ which are adjacent to some vertex in C. For any subset S of C, let M_S be the set formed by those vertices in $N(C)$ which are adjacent to each vertex in S and are nonadjacent to each vertex in $C \setminus S$. In particular, let us write M_1 for $S = \{v_1\}$, $M_{1,2}$ for $S = \{v_1, v_2\}$, and so on. Then let $Z(k)$ denote the set of vertices of $N(C)$ having exactly k neighbors in C.

Since G is (P_6, diamond)-free: $Z(4) = Z(5) = \emptyset$; each element of $Z(3)$ belongs to some of the sets $M_{i,i+2,i+3}$, $i \in \{1, \ldots, 5\}$ (subscript addition taken modulo 5).

Similarly to the previous subsection, one has the following fact.

Lemma 5 *Every component of $G[Z(0)]$ is green.* □

Lemma 6 *There exists a vertex c of C such that one of the following statements holds:*

(i) *$M_i \setminus N(c) = \emptyset$ for all $i \in \{2, \ldots, 5\}$, and $M_{i,i+1} \setminus N(c) = \emptyset$ for all $i \in \{1, \ldots, 5\}$ (subscript addition taken modulo 5);*

(ii) *$M_i \setminus N(c) = \emptyset$ for all $i \in \{1, \ldots, 5\}$, and $M_{i,i+1} \setminus N(c) = \emptyset$ for all $i \in \{1, \ldots, 4\}$ (subscript addition taken modulo 5).*

Proof: Since G is (P_6, C_6)-free, for all $i \in \{1, \ldots, 5\}$ (subscript addition taken modulo 5) one has that: if $M_i \neq \emptyset$, then $M_{i+2} = M_{i+3} = M_{i+1,i+2} = M_{i+3,i+4} = \emptyset$; if $M_{i,i+1} \neq \emptyset$, then $M_{i-1,i} = M_{i+1,i+2} = \emptyset$. This implies the lemma. □

Let us fix any vertex of C, say v_2, guaranteed by Lemma 6.

Let us prove that one can solve WIS for $G[V \setminus N(v_2)]$ in $\mathcal{O}(n^4)$ time.

A partition of $V \setminus N(v_2)$ is given by $\{\{v_2, v_4, v_5\}, M_{1,3,4}, M_{3,4,5}, M_{1,3}, M_{1,4}, M_{3,5}, M_1, Z(0)\}$.

Since G is diamond-free, the sets $M_{1,3,4}, M_{3,4,5}$ have cardinality at most one. Then, by Observation 1, to our aim it is sufficient to prove that WIS can be solved for each induced subgraph of $G[U]$ in polynomial time, where a partition of U is given by $\{M_{1,3}, M_{1,4}, M_{3,5}, M_1, Z(0)\}$.

Since G is diamond-free, $M_{1,3}, M_{1,4}, M_{3,5}$ are independent sets. Since G is P_6-free: M_1 has a co-join with $Z(0)$; each vertex in $M_{1,3} \cup M_{1,4} \cup M_{3,5}$ is not partial to any component of $G[Z(0)]$, i.e., each component of $G[Z(0)]$ is a module of $G[U]$. So by assertions similar to Lemmas 5 and 1, one can assume that $Z(0)$ is an independent set.

Now, let us consider the following cases, which are exhaustive by symmetry.

Case 1 statement i of Lemma 6 holds.

Case 1.1 $M_1 = \emptyset$. One can solve WIS in $G[U]$ by solving WIS in $G[U \setminus M_{3,5}]$ and in $G[U \setminus N(y)]$ for every $y \in M_{3,5}$. Since G is diamond-free, one can verify that such graphs are triangle-free. Then in this case one can solve WIS for $G[U]$ in $\mathcal{O}(n^3)$ time by Theorem 2.

Case 1.2 $M_1 \neq \emptyset$.

First assume that M_1 is a clique. One can solve WIS in $G[U]$ by solving WIS in $G[U \setminus M_1]$ and in $G[U \setminus N(y)]$ for every $y \in M_1$. Then, by referring to Case 1.1, in this case one can solve WIS for $G[U]$ in $\mathcal{O}(n^4)$ time.

Then assume that M_1 is not a clique, i.e., $G[M_1]$ is disconnected (since $G[M_1]$ is P_3-free). Then $M_{3,5}$ is partitioned into $\{M^0_{3,5}, M'_{3,5}\}$, where $M'_{3,5} = \{x \in M_{3,5} : x \text{ is universal for } M_1\}$, and $M^0_{3,5} = \{x \in M_{3,5} : x \text{ does not contact } M_1\}$ (in fact if $y, z \in M_1$ are nonadjacent and a vertex $x \in M_{3,5}$ is adjacent to y and nonadjacent to z, then v_4, v_3, x, y, v_1, z induce a P_6). One can solve WIS in $G[U]$ by solving WIS in $G[U \setminus M'_{3,5}]$ and in $G[U \setminus N(y)]$ for every $y \in M'_{3,5}$.

That can be done in $\mathcal{O}(n^4)$ time. Concerning graphs $G[U \setminus N(y)]$ for every $y \in M'_{3,5}$, one can refer to Case 1.1. Then let us consider $G[U \setminus M'_{3,5}]$. Notice that $M^0_{3,5}$ has a co-join with $Z(0)$, otherwise a P_6 arises involving a vertex of M_1. Then $U \setminus M'_{3,5}$ is partitioned into $\{X, Y\}$, where $X = M_1 \cup M_{1,3} \cup M_{1,4}$ (i.e., $G[X]$ is P_3-free) and $Y = M^0_{3,5} \cup Z(0)$ (i.e., Y is an independent set). Then each component of $G[U \setminus M'_{3,5}]$ is either P_3-free or green. Then the assertion follows by Lemma 1.

Case 2 statement ii of Lemma 6 holds.

One can solve WIS in $G[U]$ by solving WIS in $G[U \setminus M_{4,5}]$ and in $G[U \setminus N(y)]$ for every $y \in M_{4,5}$. Since G is diamond-free, $M_{4,5}$ is a clique. Then, by referring to Case 1.1 (3.2), in this case one can solve WIS for $G[U]$ in $\mathcal{O}(n^4)$ time.

Let us summarize the above argument as follows.

Theorem 4 *Let $G = (V, E)$ be a (P_6,diamond,C_6)-free graph containing a 5-cycle C.*

Then there exists a vertex c of C (which can be easily found) such that one can solve WIS for $G[V \setminus N(c)]$ in $\mathcal{O}(n^4)$ time. □

3.3 Deleting C_4's in (P_6,diamond,C_6, C_5)-free graphs

Throughout this subsection assume that $G = (V, E)$ is a (P_6,diamond,C_6, C_5)-free graph containing a 4-cycle C, say with vertices v_i and edges v_iv_{i+1}, $i \in \{1, \dots, 4\}$ (subscript addition taken modulo 4). Let $N(C)$ be the set of vertices from $V \setminus C$ which are adjacent to some vertex in C. For any subset S of C, let M_S be the set formed by those vertices in $N(C)$ which are adjacent to each vertex in S and are nonadjacent to each vertex in $C \setminus S$. In particular, let us write M_1 for $S = \{v_1\}$, $M_{1,2}$ for $S = \{v_1, v_2\}$, and so on. Then let $Z(k)$ denote the set of vertices of $N(C)$ having exactly k neighbors in C.

Since G is (P_6, diamond)-free: $Z(3) = Z(4) = \emptyset$.

Similarly to the previous subsection, one has the following fact.

Lemma 7 *Every component of $G[Z(0)]$ is green.* □

Let us fix any vertex of C, say v_2.

Let us prove that WIS can be solved for $G[V \setminus N(v_2)]$ in $\mathcal{O}(n^6)$ time.

A partition of $V \setminus N(v_2)$ is given by $\{\{v_2, v_4\}, M_{1,3}, M_{3,4}, M_{4,1}, M_1, M_3, M_4, Z(0)\}$. Then, by Observation 1, to our aim it is sufficient to prove that WIS can be solved for each induced subgraph of $G[U]$ in polynomial time, where a partition of U is given by $\{M_{1,3}, M_{3,4}, M_{4,1}, M_1, M_3, M_4, Z(0)\}$,

Let us introduce some preliminary definitions and lemmas.

Let us write:

$$\begin{aligned}
M_1^0 &= \{x \in M_1 : x \text{ does not contact } M_4\}\\
M_3^0 &= \{x \in M_3 : x \text{ does not contact } M_4\}\\
M_4^0 &= \{x \in M_4 : x \text{ does not contact } M_1 \cup M_3\}\\
X &= \{x \in M_1 : x \text{ contacts } M_4\} \cup \{x \in M_4 : x \text{ contacts } M_1\}\\
Y &= \{x \in M_3 : x \text{ contacts } M_4\} \cup \{x \in M_4 : x \text{ contacts } M_3\}
\end{aligned}$$

Let us write:

$$\begin{aligned}
Z_1 &= M_1^0 \cup \{z \in Z(0) : z \text{ contacts } M_1^0\}\\
Z_3 &= M_3^0 \cup \{z \in Z(0) : z \text{ contacts } M_3^0\}\\
Z_4 &= M_4^0 \cup \{z \in Z(0) : z \text{ contacts } M_4^0\}\\
Z_X &= X \cup \{z \in Z(0) : z \text{ contacts } X\}\\
Z_Y &= Y \cup \{z \in Z(0) : z \text{ contacts } Y\}\\
\tilde{Z} &= \{z \in Z(0) : z \text{ does not contact } M_1 \cup M_3 \cup M_4\}.
\end{aligned}$$

Lemma 8 *The following facts hold:*

(i) each pair of the sets $Z_1, Z_3, Z_4, Z_X, Z_Y, \tilde{Z}$ has a co-join;

(ii) each component of $G[Z_1 \cup Z_3 \cup Z_4 \cup \tilde{Z}]$ is green;

(iii) each component of $G[Z_X \cup Z_Y]$ is either P_3-free or bipartite.

Proof:

Proof of i. Since G is C_5-free, M_1 has a co-join with M_3. Then, since G is C_6-free, no vertex in M_4 can be adjacent to a vertex in M_1 and to a vertex in M_3 at the same time. It follows that each pair of the sets $M_1^0, M_3^0, M_4^0, X, Y$ has a co-join. Furthermore, since G is (P_6, C_6)-free, one can verify that two vertices chosen in two different sets $-$ among the mentioned sets $-$ cannot contact a component of $Z(0)$ at the same time. Finally, since G is P_6-free, if a vertex in $y \in M_1 \cup M_3 \cup M_4$ contacts a component K of $G[Z(0)]$, then y is universal for K. Then i follows.

Proof of ii. It is enough to deal with $G[Z_1]$, as the other subsets can be treated similarly. If $Z_1 = M_1^0$, then $G[Z_1]$ is P_3-free. Otherwise, since G is (P_6,diamond)-free, one can verify (similarly to the argument of Lemma 2) that each component of $G[Z_1]$ is green.

Proof of iii. It is enough to deal with $G[Z_1]$, as the other subsets can be treated similarly. If $Z_X = X$, then since G is (diamond,C_5)-free, each component of $G[Z_X]$ is bipartite. Otherwise, since G is P_6-free, each vertex in $Z(0)$ contacting a component K of $G[X]$ dominates K. Then, since G is diamond-free, each component of $G[Z_X]$ is a clique. $\square$

Lemma 9 *If $M_1 \neq \emptyset$ and $M_3 \neq \emptyset$, then:*

(i) $Z(0)$ *has a co-join with* $M_1 \cup M_3$*;*

(ii) $X = Y = \emptyset$.

Proof:

Proof of i. It follows since G is (P_6, C_6, C_5)-free.

Proof of ii. By symmetry, let us only prove that $X = \emptyset$. Assume for a contradiction that $X \neq \emptyset$. Then let $x_1 \in M_1$ be adjacent to $x_4 \in M_4$. Let $x_3 \in M_3$. By i of Lemma 8, x_1 and x_4 are nonadjacent to x_3. Then $x_4, x_1, v_1, v_2, v_3, x_3$ induce a P_6. $\square$

Lemma 10 *If a vertex $y \in M_{1,3}$ contacts a component K of $G[Z_X \cup Z_Y]$, then $K \setminus N(y)$ is either a clique or an independent set.*

Proof: By symmetry, let us consider only $G[Z_X]$. Let K be a component of $G[X]$. If $G[K]$ is P_3-free, then the assertion trivially follows. Then, by iii of Lemma 8, assume that $G[K]$ is bipartite. Let $y \in M_{1,3}$ contact K. Notice that y cannot be adjacent to two adjacent vertices of K, otherwise a diamond arises involving v_1. Then, to avoid a P_6, y is adjacent to all the vertices of a side of the bipartite graph, i.e., $K \setminus N(y)$ is an independent set. $\square$

Let us write $Z = Z_1 \cup Z_3 \cup Z_4 \cup Z_X \cup Z_Y \cup \tilde{Z}$. Then $\{M_{1,3}, Z\}$ is a partition of U.
For any $W \subseteq U$, let us write $W^* = W \cap Z$.
For any $W \subseteq U$, let us say that a component K of $G[W^*]$ is of:

type 1 if K is not a clique and there exists a vertex in $W \setminus Z$ partial to K;

type 2 if K is a clique and there exists a vertex in $W \setminus Z$ partial to K;

type 3 otherwise.

Let $T_1(W), T_2(W), T_3(W)$ respectively denote the union of components of $G[W^*]$ of type 1, 2, 3.
Let us fix a subset $W \subseteq U$.
Notice that $M_{1,3}$ is an independent set, since G is diamond-free.
Let us consider the following binary relation '$\leq_1$' on $M_{1,3}$: for any $a, b \in M_{1,3}$, $a \leq_1 b$ if $N_{T_1(W)}(a) \subseteq N_{T_1(W)}(b)$. It is immediate to verify that $(M_{1,3}, \leq_1)$ is a partially ordered set.

Lemma 11 *Let $y \in M_{1,3}$ be maximal for $(M_{1,3}, \leq_1)$. Then $G[T_1(W) \setminus N(y)]$ is P_3-free.*

Proof: If either $M_1 = \emptyset$ or $M_3 = \emptyset$, then one can apply an argument similar to that of Lemma 3, by considering also (i) of Lemma 8 and Lemma 10 − in detail, if $M_1 = \emptyset$ (if $M_3 = \emptyset$), then vertex v_1 (vertex v_3) is universal for $M_{1,3}$ and does not contact Z.

If $M_1 \neq \emptyset$ and $M_3 \neq \emptyset$, then one can apply an argument similar to that of Lemma 3, by considering Lemma 9 and the fact that no element of $M_{1,3}$ can be partial to a component of $G[M_1 \cup M_3]$ (since G is diamond-free). □

Let us consider the following binary relation '$\leq_2$' on $M_{1,3}$: for any $a, b \in M_{1,3}$, $a \leq_2 b$ if $N_{T_2(W)}(a) \subseteq N_{T_2(W)}(b)$. It is immediate to verify that $(M_{1,3}, \leq_2)$ is a partially ordered set.

Lemma 12 *Let $y \in M_{1,3}$ be maximal for $(M_{1,3}, \leq_2)$. If $T_1(W) = \emptyset$, then $G[(W \setminus N(y))^*]$ admits at most one component of type 2.*

Proof: One can apply the argument in the proof of Lemma 11, by considering Lemma 4 instead of Lemma 3. □

Now, let us consider the following cases.

Case 1 $M_{3,4} = M_{4,1} = \emptyset$.

Case 1.1 $T_1(U) = T_2(U) = \emptyset$.

Then each component K of $G[Z]$ is a module of $G[U]$. Then, to our aim, one can assume that K is a singleton. In fact, one can contract K into a singleton k with $w(k) = \alpha_w(G[K])$: that can be done in $\mathcal{O}(n^3)$ time by ii– iii of Lemma 8 and Lemma 1.

So in general, one can assume that Z is an independent set. Then $G[U]$ is bipartite. In this case one can solve WIS for $G[U]$ in time $\mathcal{O}(n^3)$ by Theorem 2.

Case 1.2 $T_1(U) = \emptyset, T_2(U) \neq \emptyset$.

Based on $(M_{1,3}, \leq_2)$, the vertices $y_1, \ldots, y_h$ of $M_{1,3}$ can be totally ordered so that y_i is maximal for $(\{y_i, \ldots, y_h\}, \leq_2)$ for $i = 1, \ldots, h$. Then one can solve WIS in $G[U]$ by sequentially solving WIS in $G[U \setminus N(y_1)]$, in $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$, and in $G[U \setminus M_{1,3}]$.

That can be done in $\mathcal{O}(n^5)$ time. In fact, let us first consider $G[U \setminus N(y_1)]$. If $G[(U \setminus N(y_1))^*]$ admits no component of type 2, then one can refer to Case 1.1. Otherwise, by Lemma 12, $G[(U \setminus N(y_1))^*]$ admits a unique component of type 2, say Q. Then one can solve WIS in $G[U \setminus N(y_1)]$ by solving WIS in $G[(U \setminus N(y_1)) \setminus Q)]$ and in $G[(U \setminus N(y_1)) \setminus N(q)]$, for every $q \in Q$: since for each of such graphs $G[H]$ one has that $G[H^*]$ has no component of type 2, one can refer to Case 1.1, to ii-iii of Lemma 8 and to Lemmas 7 and 1. Now, let us consider $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$: by the mentioned total order, one can apply the argument applied for $G[U \setminus N(y_1)]$ in order to show that WIS can be solved for such graphs in polynomial time. Finally, let us consider $G[U \setminus M_{1,3}] = G[Z]$: then one can refer to Case 1.1.

Case 1.3 $T_1(U) \neq \emptyset$.

Based on $(M_{1,3}, \leq_2)$, the vertices $y_1, \ldots, y_h$ of $M_{1,3}$ can be totally ordered so that y_i is maximal for $(\{y_i, \ldots, y_h\}, \leq_2)$ for $i = 1, \ldots, h$. Then one can solve WIS in $G[U]$ by sequentially solving WIS in $G[U \setminus N(y_1)]$, in $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$, and in $G[U \setminus M_{1,3}]$.

That can be done in $\mathcal{O}(n^6)$ time. In fact, let us first consider $G[U \setminus N(y_1)]$. By Lemma 11, $G[T_1(W) \setminus N(y_1)]$ is P_3-free. Then $G[(U \setminus N(y_1))^*]$ admits no component of type 1. Then one can refer to Case 1.2 and to Lemmas 7 and 1. Now, let us consider $G[(U \setminus \{y_1, \ldots, y_{i-1}\}) \setminus N(y_i))]$ for $i = 2, \ldots, h$: by the total order, one can apply the argument applied for $G[U \setminus N(y_1)]$ in order to show that WIS can be solved for such graphs in polynomial time. Finally, let us consider $G[U \setminus M_{1,3}] = G[Z]$: then one can refer to Case 1.1.

Case 2 $M_{3,4} \cup M_{4,1} \neq \emptyset$.

Since G is diamond-free, $M_{3,4}$ and $M_{4,1}$ are cliques. Since G is (P_6,diamond,C_6, C_5)-free, the following facts hold: $M_{3,4} \cup M_{4,1}$ has a co-join with $N(C) \setminus (N(v_2) \cup M_{3,4} \cup M_{4,1})$; a vertex in $M_{3,4} \cup M_{4,1}$ and a vertex in $N(C) \setminus (N(v_2) \cup M_{3,4} \cup M_{4,1})$ cannot contact a component of $G[Z(0)]$ at the same time; $M_{3,4}$ has a co-join with $M_{4,1}$.

First assume that $M_{3,4} \neq \emptyset$ and $M_{4,1} \neq \emptyset$. Then $M_{3,4} \cup M_{4,1}$ has a co-join with $Z(0)$ (otherwise a P_6 or a C_6 arises). In general, by the above facts, $M_{3,4} \cup M_{4,1}$ has a co-join with $U \setminus (M_{3,4} \cup M_{4,1})$. Then, since $M_{3,4}$ and $M_{4,1}$ are cliques, one can directly refer to Case 1.

Then assume that $M_{3,4} = \emptyset$. One can solve WIS for $G[U]$ by solving WIS in $G[U \setminus M_{4,1}]$ and in $G[U \setminus N(y)]$, for every $y \in M_{4,1}$. Since $M_{4,1}$ is a clique, one can directly refer to Case 1.

The case in which $M_{4,1} = \emptyset$ can be similarly treated, by symmetry.

Let us summarize the above argument as follows.

Theorem 5 *Let $G = (V, E)$ be a (P_6,diamond,C_6,C_5)-free graph containing a 4-cycle C. Then one can solve WIS for $G[V \setminus N(c)]$ in $\mathcal{O}(n^6)$ time, for any vertex c of C.* □

3.4 A solution for WIS and WID in (P_6,diamond)-free graphs

In this subsection we formalize an efficient method for solving WIS (or WID) in (P_6,diamond)-free graphs. To this end, let us first summarize the results of the previous subsections in the following theorem.

Theorem 6 *Let $G = (V, E)$ be a connected (P_6,diamond)-free graph containing a C_6 or a C_5 or a C_4. Then there exists a vertex c (which can be easily found) such that one can solve WIS (or WID) for $G[V \setminus N(c)]$ in $\mathcal{O}(n^6)$ time.*

Proof: If G contains a C_6, then the assertion follows by Theorem 3. If G is C_6-free and contains a C_5, then the assertion follows by Theorem 4. If G is (C_6, C_5)-free and contains a C_4, then the assertion follows by Theorem 5. □

To prove that WIS (or WID) is solvable in polynomial time on the class of (P_6,diamond)-free graphs, it suffices to find a polynomial upper bound $p(n) = O(n^7)$ on the number of steps sufficient for any allowed input of order n. If G is chordal, then we are done by Theorem 1. Otherwise, there exists a sixth-degree polynomial $q(n)$ with the property that in any (P_6,diamond)-free non-chordal graph one can determine a vertex x such that WIS (or WID) can be solved on $G_1 = G[V \setminus N(x)]$ in $q(n)$ time. If $G_1' = G - x$ is not chordal, then again one can find a vertex x' such that the problem can be solved on $G_2 = G[V(G_1') \setminus N(x')]$ in $q(n)$ time, and so on. In this way we obtain some graphs $G_1, G_2, \dots, G_k$ with $k < n$, such that each G_i is either chordal or admits an efficient WIS (or WID) algorithm. Thus, the total running time is $O(n(q(n) + r(n)))$ where $r(n)$ is the time needed to check whether the current graph is chordal and if it is not, then to find a suitable vertex under the conditions of Theorem 6.

Now, by Theorems 1 and 6 one obtains:

Theorem 7 *Both the WIS and the WID problems can be solved for (P_6,diamond)-free graphs in $\mathcal{O}(n^7)$ time.* □

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