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# Self-complementing permutations of $k$ -uniform hypergraphs

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A  $k$ -uniform hypergraph  $H = (V; E)$  is said to be *self-complementary* whenever it is isomorphic with its complement  $\overline{H} = (V; \binom{V}{k} - E)$ . Every permutation  $\sigma$  of the set  $V$  such that  $\sigma(e)$  is an edge of  $\overline{H}$  if and only if  $e \in E$  is called *self-complementing*. 2-self-complementary hypergraphs are exactly self-complementary graphs introduced independently by Ringel (1963) and Sachs (1962).

For any positive integer  $n$  we denote by  $\lambda(n)$  the unique integer such that  $n = 2^{\lambda(n)}c$ , where  $c$  is odd.

In the paper we prove that a permutation  $\sigma$  of  $[1, n]$  with orbits  $O_1, \dots, O_m$  is a self-complementing permutation of a  $k$ -uniform hypergraph of order  $n$  if and only if there is an integer  $l \geq 0$  such that  $k = a2^l + s$ ,  $a$  is odd,  $0 \leq s < 2^l$  and the following two conditions hold:

- (i)  $n = b2^{l+1} + r$ ,  $r \in \{0, \dots, 2^l - 1 + s\}$ , and
- (ii)  $\sum_{i: \lambda(|O_i|) \leq l} |O_i| \leq r$ .

For  $k = 2$  this result is the very well known characterization of self-complementing permutation of graphs given by Ringel and Sachs.

**Keywords:** Self-complementing permutations,  $k$ -uniform hypergraphs

## 1 Introduction

Let  $V$  be a set of  $n$  elements. The set of all  $k$ -subsets of  $V$  is denoted by  $\binom{V}{k}$ . A  $k$ -uniform hypergraph  $H$  consists of a *vertex-set*  $V(H)$  and an *edge-set*  $E(H) \subseteq \binom{V(H)}{k}$ . Two  $k$ -uniform hypergraphs  $G$  and  $H$  are *isomorphic*, if there is a bijection  $\sigma : V(G) \rightarrow V(H)$  such that  $e \in E(G)$  if and only if  $\{\sigma(x) | x \in e\} \in E(H)$ . The complement of a  $k$ -uniform hypergraph  $H$  is the hypergraph  $\overline{H}$  such that  $V(\overline{H}) = V(H)$  and the edge set of which consists of all  $k$ -subsets of  $V(H)$  not in  $E(H)$  (in other words  $E(\overline{H}) = \binom{V(H)}{k} - E$ ). A  $k$ -uniform hypergraph  $H$  is called *self-complementary* (*s-c* for short) if it is isomorphic with its complement  $\overline{H}$ . Isomorphism of a  $k$ -uniform self-complementary hypergraph onto its complement is called *self-complementing permutation* (or *s-c permutation*).

The 2-uniform self-complementary hypergraphs are exactly self-complementary graphs. This class of graphs has been independently discovered by Ringel and Sachs who proved the following.

**Theorem 1 (Ringel (Rin63) and Sachs (Sac62))** *Let  $n$  be a positive integer. A permutation  $\sigma$  of  $[1, n]$  is a self-complementing permutation of a self-complementary graph of order  $n$  if and only if all the orbits of  $\sigma$  have their cardinalities congruent to 0 (mod 4) except, possibly, one orbit of cardinality 1.*

Observe that by Theorem 1 an s-c graph of order  $n$  exists if and only if  $n \equiv 0$  or  $n \equiv 1 \pmod{4}$  or, equivalently, whenever  $\binom{n}{2}$  is even. In (SW) we prove that a similar result is true for  $k$ -uniform hypergraphs.

**Theorem 2 ((SW))** *Let  $k$  and  $n$  be positive integers,  $k \leq n$ . A  $k$ -uniform self-complementary hypergraph of order  $n$  exists if and only if  $\binom{n}{k}$  is even.*

A simple criterion for evenness of  $\binom{n}{k}$  has been given in (Gla99) (and then rediscovered in (KHRM58)).

**Theorem 3 ((Gla99; KHRM58))** *Let  $k$  and  $n$  be positive integers,  $k = \sum_{i=0}^{+\infty} c_i 2^i$  and  $n = \sum_{i=0}^{+\infty} d_i 2^i$ , where  $c_i, d_i \in \{0, 1\}$  for every  $i$ .  $\binom{n}{k}$  is even if and only if there is  $i_0$  such that  $c_{i_0} = 1$  and  $d_{i_0} = 0$ .*

Theorem 3 asserts that  $\binom{n}{k}$  is even if and only if  $k$  has 1 in a certain binary place while  $n$  has 0 in the corresponding binary place. For example,  $\binom{27}{13}$  is even since  $13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$  and  $27 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$  (so we have  $c_2 = 1$  and  $d_2 = 0$ ).

Except for Theorem 1 which is a characterization of the self-complementing permutations for graphs, there are already two published results characterizing the permutations of  $k$ -uniform s-c hypergraphs for  $k > 2$ . Namely, Kocay in (Koc92) (see also (Pal73)) and Szymański in (Szy05) have characterized the s-c permutations of s-c  $k$ -uniform hypergraphs for, respectively,  $k = 3$  and  $k = 4$ . This work is a continuation of the work of (SW) and (Woj06). We generalize all the results mentioned above by giving a characterization of the s-c permutations of  $k$ -uniform hypergraphs for any integers  $k$  and  $n$ .

## 2 Result

Any positive integer  $n$  may be written in the form  $n = 2^l c$ , where  $c$  is an odd integer. Moreover,  $l$  and  $c$  are uniquely determined. We write then  $\lambda(n) = l$ . Note that in the binary expansion of  $n$ ,  $\lambda(n)$  is the index of the first 1-bit. For any set  $A$  we shall write  $\lambda(A)$  in place of  $\lambda(|A|)$ , for short.

In the proof of our main result we shall need the following lemma proved in (Woj06).

**Lemma 1** *Let  $k, m$  and  $n$  be positive integers, and let  $\sigma : V \rightarrow V$  be a permutation of a set  $V$ ,  $|V| = n$ , with orbits  $O_1, \dots, O_m$ .  $\sigma$  is a self-complementing permutation of a self-complementary  $k$ -uniform hypergraph, if and only if, for every  $p \in \{1, \dots, k\}$  and for every decomposition*

$$k = k_1 + \dots + k_p$$

*of  $k$  ( $k_j > 0$  for  $j = 1, \dots, p$ ), and for every subsequence of orbits*

$$O_{i_1}, \dots, O_{i_p}$$

*such that  $k_j \leq |O_{i_j}|$  for  $j = 1, \dots, p$ , there is a subscript  $j_0 \in \{1, \dots, p\}$  such that*

$$\lambda(k_{j_0}) < \lambda(O_{i_{j_0}})$$

Given any integer  $l \geq 0$ . If the binary expansion of  $k$  is 1-bit in position  $l$ , then  $k$  can be written in the form  $k = a_l 2^l + s_l$ , where  $a_l$  is odd and  $0 \leq s_l < 2^l$ .

**Theorem 4** *Let  $k$  and  $n$  be integers,  $k \leq n$ . A permutation  $\sigma$  of  $[1, n]$  with orbits  $O_1, \dots, O_m$  is a self-complementing permutation of a  $k$ -uniform hypergraph of order  $n$  if and only if there is a nonnegative integer  $l$  such that  $k = a_l 2^l + s_l$ , where  $a_l$  is odd and  $0 \leq s_l < 2^l$ , and the following two conditions hold:*

$$(i) \quad n = b_l 2^{l+1} + r_l, \quad r_l \in \{0, \dots, 2^l - 1 + s_l\}, \text{ and}$$

$$(ii) \quad \sum_{i: \lambda(O_i) \leq l} |O_i| \leq r_l.$$

**Proof:**

**Sufficiency.** By contradiction. Let  $n, k, l, a_l, b_l, s_l$  and  $r_l$  be integers verifying the conditions of the theorem, let  $\sigma$  be a permutation of  $[1, n]$  with orbits  $O_1, \dots, O_m$  verifying (ii), and let us suppose that  $\sigma$  is not a s-c permutation of any  $k$ -uniform s-c hypergraph of order  $n$ . Then, by Lemma 1, there is a decomposition of  $k = k_1 + \dots + k_t$  and a subsequence of orbits  $O_{i_1}, \dots, O_{i_t}$  such that

$$0 < k_j \leq |O_{i_j}| \tag{1}$$

and

$$\lambda(k_j) \geq \lambda(O_{i_j}) \tag{2}$$

for  $j = 1, \dots, t$ .

Since  $a_l$  is odd, we have  $k \equiv 2^l + s_l \pmod{2^{l+1}}$ . By (2),  $\sum_{j: \lambda(O_{i_j}) > l} k_j \equiv 0 \pmod{2^{l+1}}$ . Therefore

$$k = \sum_{j=1}^t k_j = \sum_{j: \lambda(O_{i_j}) > l} k_j + \sum_{j: \lambda(O_{i_j}) \leq l} k_j \equiv \sum_{j: \lambda(O_{i_j}) \leq l} k_j \pmod{2^{l+1}}$$

Hence, and by (1), (i) and (ii) we have  $\sum_{j: \lambda(O_{i_j}) \leq l} k_j \leq \sum_{j: \lambda(O_{i_j}) \leq l} |O_{i_j}| < 2^{l+1}$ , and therefore

$$2^l + s_l = \sum_{j: \lambda(O_{i_j}) \leq l} k_j \leq \sum_{j: \lambda(O_{i_j}) \leq l} |O_{i_j}| \leq r_l < 2^l + s_l$$

a contradiction.

**Necessity.** Let  $1 \leq k \leq n$  and let  $\sigma$  be a permutation of the set  $[1, n]$  with orbits  $O_1, \dots, O_m$ . Let us suppose that for every integer  $l$  such that  $k = a_l 2^l + s_l$ , where  $a_l$  is odd positive integer,  $0 \leq s_l < 2^l$ , and  $n = b_l 2^{l+1} + r_l$ ,  $0 \leq r_l < 2^{l+1}$  we have either

$$r_l \in \{2^l + s_l, \dots, 2^{l+1} - 1\}$$

or

$$r_l \in \{0, \dots, 2^l - 1 + s_l\} \quad \text{and} \quad \sum_{i: \lambda(O_i) \leq l} |O_i| > r_l$$

We shall prove that  $\sigma$  is not a s-c permutation of any s-c  $k$ -uniform hypergraph of order  $n$ . For this purpose we shall give two claims.

**Claim 1** For every nonnegative integer  $l$  such that  $k = a_l 2^l + s_l$ , where  $a_l$  is odd and  $0 \leq s_l < 2^l$ , we have

$$\sum_{i: \lambda(O_i) \leq l} |O_i| \geq 2^l + s_l$$

**Proof of Claim 1.** Let us write  $\sum_{i: \lambda(O_i) \leq l} |O_i|$  and  $\sum_{i: \lambda(O_i) > l} |O_i|$  in their binary forms:

$$\sum_{i: \lambda(O_i) \leq l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j$$

$$\sum_{i: \lambda(O_i) > l} |O_i| = \sum_{j=0}^{\infty} f_j 2^j$$

where  $e_j, f_j \in \{0, 1\}$  for every  $j$ . Observe that  $f_j = 0$  for  $j = 0, \dots, l$  and therefore

$$\sum_{j=0}^l e_j 2^j = r_l \tag{3}$$

We shall consider two cases.

Case 1.  $r_l \in \{0, \dots, 2^l + s_l - 1\}$  and  $\sum_{i: \lambda(O_i) \leq l} |O_i| > r_l$ .

We have  $n \geq 2^{l+1}$  (otherwise  $r_l = n = \sum_{i: \lambda(O_i) \leq l} |O_i|$ ).

Since  $\sum_{j=0}^{\infty} e_j 2^j > r_l$ , and by (3), we obtain  $\sum_{j=0}^{\infty} e_j 2^j \geq 2^{l+1} > 2^l + s_l$ .

Case 2.  $r_l \in \{2^l + s_l, \dots, 2^{l+1} - 1\}$ .

We have  $\sum_{i: \lambda(O_i) \leq l} |O_i| = \sum_{j=0}^{\infty} e_j 2^j \geq \sum_{j=0}^l e_j 2^j = r_l \geq 2^l + s_l$ , and the claim is proved.  $\square$

**Claim 2** Let  $\alpha_1, \dots, \alpha_q$  and  $\lambda_1, \dots, \lambda_q$  be integers such that  $0 < \alpha_i, 0 \leq \lambda_i \leq \lambda(\alpha_i)$  and  $\lambda_i \leq l$  for  $i = 1, \dots, q$  and  $\sum_{i=1}^q \alpha_i \geq 2^l$ . Then there are  $\beta_1, \dots, \beta_q$  such that for every  $i = 1, \dots, q$

$$0 \leq \beta_i \leq \alpha_i \tag{4}$$

and

$$\text{either } \beta_i = 0 \text{ or } \lambda(\beta_i) \geq \lambda_i \tag{5}$$

and

$$\sum_{i=1}^q \beta_i = 2^l \tag{6}$$

**Proof of Claim 2.** The existence of  $\beta_1, \dots, \beta_q$  verifying (4)-(5) and  $\sum_{i=1}^q \beta_i \leq 2^l$  is very easy. Indeed, it is immediate that  $\beta_1 = 2^{\lambda_1}, \beta_2 = \dots, \beta_q = 0$  is a sequence with the desired properties.

So let us suppose that  $\beta_1, \dots, \beta_q$  is a sequence verifying (4)-(5) and  $\sum_{i=1}^q \beta_i \leq 2^l$  such that  $\sum_{i=1}^q \beta_i$  is maximal. If  $\sum_{i=1}^q \beta_i = 2^l$  then the proof is complete. So let us suppose that  $\sum_{i=1}^q \beta_i < 2^l$ . Then there is  $i_0 \in \{1, \dots, q\}$  such that  $\beta_{i_0} < \alpha_{i_0}$ . Observe that  $\beta_{i_0} + 2^{\lambda_{i_0}} \leq \alpha_{i_0}$ . The sequence  $\bar{\beta}_1, \dots, \bar{\beta}_q$  defined by  $\bar{\beta}_{i_0} = \beta_{i_0} + 2^{\lambda_{i_0}}$  and  $\bar{\beta}_i = \beta_i$  for  $i \neq i_0$  also verifies (4)-(5) and  $\sum_{i=1}^q \bar{\beta}_i \leq 2^l$ , which contradicts the maximality of the sum  $\sum_{i=1}^q \beta_i$ , and the claim is proved.  $\square$

We shall use our claims to construct a decomposition of  $k$  in the form  $k = k_1 + \dots + k_m$  such that

- (1)  $k_1, \dots, k_m$  are nonnegative integers,
- (2)  $k_i \leq |O_i|$  for  $i = 1, \dots, m$ , and
- (3)  $\lambda(k_i) \geq \lambda(O_i)$  whenever  $k_i > 0$

By Lemma 1, this will imply that  $\sigma$  is not a s-c permutation of any  $k$ -uniform s-c hypergraph. Let us write  $k$  in its binary form:

$$k = 2^{l_t} + 2^{l_{t-1}} + \dots + 2^{l_1} + 2^{l_0}$$

where  $l_0 < l_1 < \dots < l_t$ .

By Claim 1,  $\sum_{i:\lambda(O_i) \leq l_0} |O_i| \geq 2^{l_0}$ . Hence, and by Claim 2, there are nonnegative integers  $k_1^{(0)}, k_2^{(0)}, \dots, k_m^{(0)}$  such that  $k_i^{(0)} = 0$  for  $i$  such that  $\lambda(O_i) > l_0$  and

$$\begin{aligned} k_i^{(0)} &\leq |O_i| \text{ for } i = 1, \dots, m \\ \lambda(k_i^{(0)}) &\geq \lambda(O_i) \text{ whenever } k_i^{(0)} > 0 \end{aligned}$$

and

$$\sum_{i=1}^m k_i^{(0)} = 2^{l_0}$$

Note that, for  $i = 1, \dots, m$ , we have  $\lambda(|O_i| - k_i^{(0)}) \geq \lambda(O_i)$ .

Let us suppose that we have already constructed  $k_1^{(j)}, \dots, k_m^{(j)}$ , ( $j \leq t$ ), such that  $k_i^{(j)} = 0$  for  $i$  such that  $\lambda(O_i) > l_j$  and

$$\begin{aligned} k_i^{(j)} &\leq |O_i| \text{ for } i = 1, \dots, m \\ \lambda(k_i^{(j)}) &\geq \lambda(O_i) \text{ whenever } k_i^{(j)} > 0 \\ \sum_{i=0}^m k_i^{(j)} &= 2^{l_j} + 2^{l_{j-1}} + \dots + 2^{l_0} \end{aligned}$$

and

$$\lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i)$$

If  $j = t$ , then we have already found a desired decomposition of  $k$ . If  $j < t$ , then, by Claim 1, we have

$$\sum_{i:\lambda(O_i) \leq l_{j+1}} (|O_i| - k_i^{(j)}) \geq 2^{l_{j+1}}.$$

$\lambda(|O_i| - k_i^{(j)}) \geq \lambda(O_i)$  for every  $i \in \{1, \dots, m\}$  such that  $|O_i| - k_i^{(j)} > 0$ . Hence, and by Claim 2, there are  $\beta_1, \dots, \beta_m$  such that  $\beta_i = 0$  for  $i$  such that  $\lambda(O_i) > l_{j+1}$  and

$$\begin{aligned} 0 \leq \beta_i &\leq |O_i| - k_i^{(j)} \text{ for } i = 1, \dots, m \\ \lambda(O_i) &\leq \lambda(\beta_i) \text{ for } i = 1, \dots, m \text{ whenever } \beta_i \neq 0 \\ \sum_{i=1}^m \beta_i &= 2^{l_{j+1}} \end{aligned}$$

Thus we may define for every  $i = 1, \dots, m$

$$k_i^{(j+1)} = k_i^{(j)} + \beta_i$$

to obtain the sequence  $(k_1^{(j+1)}, \dots, k_m^{(j+1)})$  verifying for every  $i \in \{1, \dots, m\}$

$$k_i^{(j+1)} = 0 \text{ for } i \text{ such that } \lambda(O_i) > l_{j+1}$$

$$k_i^{(j+1)} \leq |O_i|$$

$$\lambda(k_i^{(j+1)}) \geq \lambda(O_i) \text{ whenever } k_i^{(j+1)} > 0$$

and

$$\sum_{i=1}^m k_i^{(j+1)} = 2^{l_{j+1}} + 2^{l_j} + \dots + 2^{l_0}$$

It is clear that  $k = \sum_{i=1}^m k_i^{(t)}$  and the proof of Theorem 4 is complete.  $\square$

Theorem 4 implies very easily the following theorem first proved by Kocay.

**Corollary 1 (Kocay (Koc92))**  *$\sigma$  is a self-complementing permutation of a self-complementary 3-uniform hypergraph if and only if either all the orbits of  $\sigma$  have even cardinalities, or else, it has 1 or 2 fixed points and the all remaining orbits of  $\sigma$  have their cardinalities being multiples of 4.*

For  $k = 2^l$  Theorem 4 may be written as follows.

**Corollary 2** *Let  $l$  and  $n$  be nonnegative integers,  $2^l < n$ , and let  $0 \leq r < 2^{l+1}$  be such that  $n \equiv r \pmod{2^{l+1}}$ . A permutation  $\sigma$  of  $[1, n]$  with orbits  $O_1, \dots, O_m$  is a self-complementing permutation of a  $2^l$ -uniform self-complementary hypergraph if and only if*

$$(i) \ r \in \{0, \dots, 2^l - 1\} \text{ and}$$

$$(ii) \ \sum_{i: \lambda(O_i) \leq l} |O_i| \leq r.$$

Theorem 2 for  $l = 1$  (i.e. for graphs) is exactly Theorem 1, and for  $l = 2$  the following theorem proved by Szymański in (Szy05).

**Corollary 3** *A permutation  $\sigma$  is self-complementing permutation of a 4-uniform hypergraph of order  $n$  if and only if  $n \equiv r \pmod{8}$  with  $r = 0, 1, 2$  or  $3$ , and the sum of the cardinalities of orbits which are not multiples of 8 is at most 3.*

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## References

- [Gla99] J.W.L. Glaisher. On the residue of a binomial coefficient with respect to a prime modulus. *Quarterly Journal of Mathematics*, 30:150–156, 1899.
- [KHRM58] S.H. Kimball, T.R. Hatcher, J.A. Riley, and L. Moser. Solution to problem e1288: Odd binomial coefficients. *Amer. Math. Monthly*, 65:368–369, 1958.
- [Koc92] W. Kocay. Reconstructing graphs as subsumed graphs of hypergraphs, and some self-complementary triple systems. *Graphs and Combinatorics*, 8:259–276, 1992.
- [Pal73] E.M. Palmer. On the number of  $n$ -plexes. *Discrete Math.*, 6:377–390, 1973.
- [Rin63] G. Ringel. Selbstkomplementare graphen. *Arch. Math.*, 14:354–358, 1963.
- [Sac62] H. Sachs. Über selbstkomplementare graphen. *Publ. Math. Debrecen*, 9:270–288, 1962.
- [SW] A. Szymański and A.P. Wojda. A note on  $k$ -uniform self-complementary hypergraphs of given order. *submitted*.
- [Szy05] A. Szymański. A note on self-complementary 4-uniform hypergraphs. *Opuscula Mathematica*, 25/2:319–323, 2005.
- [Woj06] A.P. Wojda. Self-complementary hypergraphs. *Discussiones Mathematicae – Graph Theory*, 26:217–224, 2006.



