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# The distribution of ascents of size $d$ or more in compositions

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A composition of a positive integer  $n$  is a finite sequence of positive integers  $a_1, a_2, \dots, a_k$  such that  $a_1 + a_2 + \dots + a_k = n$ . Let  $d$  be a fixed nonnegative integer. We say that we have an ascent of size  $d$  or more if  $a_{i+1} \geq a_i + d$ . We determine the mean, variance and limiting distribution of the number of ascents of size  $d$  or more in the set of compositions of  $n$ . We also study the average size of the greatest ascent over all compositions of  $n$ .

**Keywords:** compositions, distributions, generating functions, ascents

## 1 Introduction

A composition of a positive integer  $n$  is a finite sequence of positive integers  $a_1, a_2, \dots, a_k$  such that  $a_1 + a_2 + \dots + a_k = n$ . It is well known that there are  $2^{n-1}$  compositions of  $n$ . The compositions (denoted by  $a_1 a_2 a_3 \dots a_k$ ) for  $n = 1, 2, \dots, 5$  are:

$n$																
1	1															
2	11	2														
3	111	12	21	3												
4	1111	13	31	22	112	121	211	4								
5	11111	14	41	23	32	113	131	311	122	212	221	1112	1121	1211	2111	5

Let  $d \geq 0$  be a fixed integer. We say, that we have an ascent of size  $d$  or more, whenever  $a_{i+1} \geq a_i + d$ . For example, there are 3 ascents of size 2 or more that occur in the compositions of 5: 14, 113, and 131.

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In Sections 2, 3 and 4, respectively, we determine the mean, variance and asymptotic distribution of the number of ascents of size  $d$  or more in compositions of  $n$ . Ordinary ascents (the cases  $d = 0$  and  $d = 1$ ) have previously been studied by Carlitz (1) and more recently by Chinn, Heubach and Grimaldi in (2).

Finally in Section 5 we investigate the maximum value of  $d$  for which compositions of  $n$  can expect to have an ascent of size  $d$ . That is, we find the average value of the *largest ascent* that occurs in the compositions of  $n$ . For example, the compositions of 4 in the table have maximum ascents of sizes 0, 2, 0, 0, 1, 1, 0, 0, respectively, giving an average largest ascent size per composition of  $1/2$  when  $n = 4$ .

We note that the asymptotic expression for the maximum ascent size in Theorem 4 involves a fluctuating function of  $n$  of mean zero. A similar phenomenon has been observed when studying averages of certain other statistics for compositions, such as the largest part size (11) or the number of distinct part sizes (8), (7).

## 2 The average number of ascents of size $d$ or more in compositions

For fixed  $d \geq 0$  we wish to find the average number of ascents of size  $d$  or more per composition of  $n$ .

We use the “adding-the-slice” technique which was originally used by Flajolet and Prodinger in (4) and more recently, for example, by Knopfmacher and Prodinger in (9).

Let  $j$  be the value of the last component of the composition with  $k$  parts, i.e.  $a_k = j$ . We proceed from a composition with  $k$  parts to a composition with  $k + 1$  parts. We denote by  $f_k(z, u, v)$  the generating function where  $z$  marks the size  $n$ ,  $u$  the value of  $j$  and  $v$  the number of ascents of size  $d$  or more in compositions with  $k$  parts.

In moving from a composition with  $k$  parts to a composition with  $k + 1$  parts, where  $a_k = j$ , we have an ascent whenever the new last integer has any value from  $j + d$  onwards. This gives the following rule for adding a new part or “slice” to the end of the composition:

$$\begin{aligned} u^j &\longrightarrow zu + (zu)^2 + (zu)^3 + \dots + (zu)^{j+d-1} + v \{ (zu)^{j+d} + (zu)^{j+d+1} + \dots \} \\ &= zu \frac{1 - (zu)^{j+d-1}}{1 - zu} + v(zu)^{j+d} \frac{1}{1 - zu}. \end{aligned}$$

This implies that

$$\begin{aligned} f_{k+1}(z, u, v) &= \frac{zu}{1 - zu} f_k(z, 1, v) - \frac{(zu)^d}{1 - zu} f_k(z, zu, v) + \frac{v(zu)^d}{1 - zu} f_k(z, zu, v) \\ &= \frac{zu}{1 - zu} f_k(z, 1, v) - \frac{(1 - v)(zu)^d}{1 - zu} f_k(z, zu, v). \end{aligned} \tag{2.1}$$

Now define  $F(z, u, v) := \sum_{k \geq 1} f_k(z, u, v)$ . Then summing (2.1) over  $k \geq 1$  gives

$$F(z, u, v) - f_1(z, u, v) = \frac{zu}{1 - zu} F(z, 1, v) - \frac{(1 - v)(zu)^d}{1 - zu} F(z, zu, v),$$

so that

$$F(z, u, v) = \frac{zu}{1 - zu} F(z, 1, v) + \frac{zu}{1 - zu} - \frac{(1 - v)(zu)^d}{1 - zu} F(z, zu, v),$$

where we have used

$$f_1(z, u, v) = zu + (zu)^2 + (zu)^3 + \cdots = \frac{zu}{1 - zu}.$$

At this stage we iterate the recursion for  $F(z, u, v)$ .

$$\begin{aligned} F(z, u, v) &= \frac{zu}{1 - zu} F(z, 1, v) + \frac{zu}{1 - zu} - \frac{(1 - v)(zu)^d}{1 - zu} \times \\ &\quad \times \left\{ \frac{z^2 u}{1 - z^2 u} F(z, 1, v) + \frac{z^2 u}{1 - z^2 u} - \frac{(1 - v)(z^2 u)^d}{1 - z^2 u} F(z, z^2 u, v) \right\} \\ &= \left[ \frac{zu}{1 - zu} - \frac{(1 - v)z^2 u (zu)^d}{(1 - zu)(1 - z^2 u)} \right] [F(z, 1, v) + 1] + \frac{(1 - v)^2 (zu)^d (z^2 u)^d}{(1 - zu)(1 - z^2 u)} \times \\ &\quad \times \left\{ \frac{z^3 u}{1 - z^3 u} F(z, 1, v) + \frac{z^3 u}{1 - z^3 u} - \frac{(1 - v)(z^3 u)^d}{1 - z^3 u} F(z, z^3 u, v) \right\} \\ &= \left[ \frac{zu}{1 - zu} - \frac{(1 - v)z^2 u (zu)^d}{(1 - zu)(1 - z^2 u)} + \frac{(1 - v)^2 z^3 u (zu)^d (z^2 u)^d}{(1 - zu)(1 - z^2 u)(1 - z^3 u)} \right] [F(z, 1, v) + 1] \\ &\quad - \frac{(1 - v)^3 (zu)^d (z^2 u)^d (z^3 u)^d}{(1 - zu)(1 - z^2 u)(1 - z^3 u)} F(z, z^3 u, v). \end{aligned}$$

We keep iterating, noting that  $F(z, z^m u, v) \rightarrow 0$  as  $m \rightarrow \infty$  for  $|z| < \frac{1}{2}$  and  $u, v$  in a suitable small neighbourhood of 1, and put  $u = 1$  to obtain

$$F(z, 1, v) = \sum_{i \geq 1} \frac{(-1)^{i-1} z^i (1 - v)^{i-1} z^{d \binom{i}{2}}}{(1 - z)(1 - z^2) \cdots (1 - z^i)} [F(z, 1, v) + 1]. \quad (2.2)$$

By adding the term 1 for the empty composition we obtain the bivariate generating function for compositions according to the number of ascents of size  $d$  or more as

$$F(z, v) := 1 + F(z, 1, v) = \frac{1}{1 - \tau(z, v)}, \quad (2.3)$$

where

$$\tau(z, v) := \sum_{i \geq 1} \frac{(-1)^{i-1} z^i (1 - v)^{i-1} z^{d \binom{i}{2}}}{(1 - z)(1 - z^2) \cdots (1 - z^i)}. \quad (2.4)$$

The expected value of the number of ascents of size  $d$  or more is  $\left. \frac{[z^n] \frac{\partial F(z, v)}{\partial v}}{2^{n-1}} \right|_{v=1}$ . For this we shall need

$$\tau(z, 1) = \sum_{i \geq 1} \frac{(-1)^{i-1} z^i (1 - v)^{i-1} z^{d \binom{i}{2}}}{(1 - z)(1 - z^2) \cdots (1 - z^i)} \Bigg|_{v=1} = \frac{z}{1 - z}, \quad (2.5)$$

and

$$\frac{\partial \tau(z, v)}{\partial v} \Bigg|_{v=1} = \sum_{i \geq 2} \frac{(-1)^i z^i (i - 1) (1 - v)^{i-2} z^{d \binom{i}{2}}}{(1 - z)(1 - z^2) \cdots (1 - z^i)} \Bigg|_{v=1} = \frac{z^{2+d}}{(1 - z)(1 - z^2)}. \quad (2.6)$$

In particular, the generating function for all compositions is  $F(z, 1) = \frac{1}{1-\tau(z,1)} = \frac{1-z}{1-2z}$ .

Now

$$\begin{aligned} \left. \frac{\partial F(z, v)}{\partial v} \right|_{v=1} &= \left. \frac{\frac{\partial \tau(z, v)}{\partial v}}{(1 - \tau(z, v))^2} \right|_{v=1} = \frac{z^{2+d}}{(1+z)(1-2z)^2} \\ &= z^d \left[ \frac{1}{9(1+z)} + \frac{1}{6(1-2z)^2} - \frac{5}{18(1-2z)} \right]. \end{aligned}$$

So that

$$\begin{aligned} [z^n] \left. \frac{\partial F(z, v)}{\partial v} \right|_{v=1} &= \frac{(-1)^{n-d}}{9} + \frac{(n-d+1)2^{n-d}}{6} - \frac{5 \cdot 2^{n-d}}{18} \\ &= \frac{(-1)^{n-d}}{9} + \frac{(3n-3d-2)2^{n-d}}{18}. \end{aligned}$$

After dividing by  $2^{n-1}$ , the total number of compositions of  $n$ , we have

**Theorem 1** *The expected number of ascents of size  $d$  or more in the compositions of  $n$  is*

$$\mathbb{E}(n) := \frac{2^{-d}}{9}(3n-3d-2) + \frac{2}{9} \frac{(-1)^{n-d}}{2^n}, \text{ for } n \geq d.$$

Hence for fixed  $d$ , as  $n \rightarrow \infty$ ,

$$\mathbb{E}(n) = \frac{2^{-d}}{3} n + O(1).$$

Previously Chinn, Heubach and Grimaldi found the number of ascents for  $d = 0$  and  $d = 1$  in (2). The case  $d = 0$  corresponds to the number of rises plus the number of levels, whereas  $d = 1$  corresponds to the number of rises.

### 3 Variance of the number of ascents of size $d$ or more in compositions

To find the variance we first need to compute  $\left. \frac{\partial^2 F(z, v)}{\partial v^2} \right|_{v=1}$ . In addition to formulas (2.3) to (2.6) from Section 2 we require

$$\left. \frac{\partial^2 \tau(z, v)}{\partial v^2} \right|_{v=1} = \sum_{i \geq 3} \frac{(-1)^{i-1} z^i (i-1)(i-2)(1-v)^{i-3} z^{d \binom{i}{2}}}{(1-z)(1-z^2) \cdots (1-z^i)} \Big|_{v=1} = \frac{2z^{3(1+d)}}{(1-z)(1-z^2)(1-z^3)}.$$

Then

$$\left. \frac{\partial^2 F(z, v)}{\partial v^2} \right|_{v=1} = \frac{(1-\tau) \frac{\partial^2 \tau(z, v)}{\partial v^2} + 2\tau'^2}{(1-\tau)^3} \Big|_{v=1}.$$

Computing the  $n$ th coefficient of the second derivative amounts to expanding and combining binomial series.

Finally, after adding the expectation and subtracting the square of the expectation we find

**Theorem 2** *The variance of the expected number of ascents of size  $d$  or more in the compositions of  $n$  is*

$$\begin{aligned} \mathbb{V}(n) := & 2^{-d} \left\{ \frac{-2}{9} + \frac{n}{3} - \frac{d}{3} \right\} + 2^{-2d} \left\{ \frac{80}{81} + \frac{10d}{9} + \frac{d^2}{3} - \frac{13n}{27} - \frac{2nd}{9} \right\} \\ & + 2^{-3d} \left\{ -\frac{352}{441} + \frac{8n}{21} - \frac{8d}{7} \right\} + 2^{-n} \left\{ (-1)^n \left( \frac{5}{27} + \frac{2n}{27} - \frac{4d}{27} \right) - \frac{1}{3} + 2\alpha(n) \right\} \\ & + 2^{-n-d} (-1)^{n-d} \left\{ \frac{4d}{27} - \frac{4n}{27} + \frac{8}{81} \right\} - 2^{-2n} \frac{4}{81}, \end{aligned}$$

for  $n \geq 3d$ , where

$$\alpha(n) = \begin{cases} \frac{26}{147} & \text{if } n = 3m, \\ \frac{-4}{147} & \text{if } n = 3m - 2, \\ \frac{-22}{147} & \text{if } n = 3m - 1, \end{cases} \text{ for } m \in \mathbb{N}.$$

For fixed  $d$  we have

$$\mathbb{V}(n) \sim n \left\{ \frac{2^{-d}}{3} - 2^{-2d} \left( \frac{13}{27} + \frac{2d}{9} \right) + \frac{2^{-3d} 8}{21} \right\} \text{ as } n \rightarrow \infty.$$

## 4 Limiting distribution

We are interested in finding the limiting distribution of our random variable. We make use of Theorem IX.9 from Flajolet and Sedgewick (5). A short version is as follows:

Let  $F(z, u)$  be a bivariate function that is bivariate analytic at  $(z, u) = (0, 0)$  and has nonnegative coefficients there. Assume that  $F(z, 1)$  is meromorphic in  $z \leq r$  with only a simple pole at  $z = \rho$  for some positive  $\rho < r$ . Then, under further conditions stated in (5), the random variable with probability generating function

$$p_n(u) = \frac{[z^n]F(z, u)}{[z^n]F(z, 1)}$$

converges in distribution to a Gaussian variable with a speed of convergence that is  $O(n^{-1/2})$ .

Let us introduce the notation

$$c_{i,j} := \left. \frac{\partial^{i+j}}{\partial z^i \partial u^j} C(z, u) \right|_{(\rho, 1)}. \quad (4.1)$$

From Theorem IX.9 we need to show that

$$c_{0,1}c_{1,0} \neq 0. \quad (4.2)$$

In addition, we must show that

$$\rho c_{1,0}^2 c_{0,2} - \rho c_{1,0} c_{1,1} c_{0,1} + \rho c_{2,0} c_{0,1}^2 + c_{0,1}^2 c_{1,0} + c_{0,1} c_{1,0}^2 \rho \neq 0. \quad (4.3)$$

For our specific problem

$$F(z, v) = \frac{1}{1 - \tau(z, v)} \equiv \frac{B(z, v)}{C(z, v)},$$

so that

$$C(z, v) = 1 - \sum_{i \geq 1} \frac{(-1)^{i-1} z^i (1-v)^{i-1} z^{d \binom{i}{2}}}{(1-z)(1-z^2) \cdots (1-z^i)}.$$

We have  $\rho(1) = \rho = \frac{1}{2}$  and using (4.1),

$$c_{0,1} = -\frac{2^{1-d}}{3}, \quad c_{1,0} = -4, \quad c_{1,1} = -\frac{3d+11}{9} 2^{2-d}, \quad c_{0,2} = -\frac{2^{4-3d}}{21}, \quad c_{2,0} = -16.$$

We are now in a position to check the conditions listed in the theorem.

Equation (4.2) is satisfied since

$$c_{0,1} c_{1,0} = 8 \frac{2^{-d}}{3} \neq 0.$$

Equation (4.3) is equivalent to

$$2^{4-3d} \left( \frac{2^3}{21} + 2^d \frac{3d+11}{27} - \frac{2^{1+d}}{9} - \frac{2^{2+2d}}{9} \right) \neq 0$$

for non-negative integer values of  $d$ . Thus we deduce

**Theorem 3** *The distribution of the number of ascents of size  $d$  or more in compositions of  $n$  converges to a Gaussian distribution with a speed of convergence of  $O(n^{-1/2})$  with the mean  $\mu_n$  and the variance  $\sigma_n^2$  are as given in Theorems 1 and 2.*

**Remark** In Flajolet and Sedgewick (5) it is also shown that under the conditions of Theorem IX.9, the mean  $\mu_n$  and variance  $\sigma_n^2$  are of the form

$$\mu_n = \mathfrak{m} \left( \frac{\rho(1)}{\rho(u)} \right) n + O(1), \quad \sigma_n^2 = \mathfrak{v} \left( \frac{\rho(1)}{\rho(u)} \right) n + O(1),$$

where

$$\mathfrak{m}(f) = \frac{f'(1)}{f(1)} \quad \text{and} \quad \mathfrak{v}(f) = \frac{f''(1)}{f(1)} + \frac{f'(1)}{f(1)} - \left( \frac{f'(1)}{f(1)} \right)^2.$$

These asymptotic expressions are easily checked to be in agreement with the exact results for the mean and variance found previously in Theorems 1 and 2.

## 5 Size of the maximum ascent

Given a composition  $a_1 a_2 \dots a_k$  of  $n$  we shall study the size of the maximum ascent, that is, the parameter  $X$  where

$$X := \max\{a_{i+1} - a_i \mid 1 \leq i < k \text{ and } a_{i+1} \geq a_i\}.$$

We assign a value of 0 if no pair of consecutive integers satisfies this condition. The mean value of  $X$  is given by the expression

$$\sum_{d=0}^n \mathbb{P}(X > d) = \sum_{d=0}^n (1 - \mathbb{P}(X \leq d)).$$

Therefore for each fixed  $d$  we need to compute the probability that a composition of  $n$  has maximum ascent  $X \leq d$ .

We already know the generating function for compositions with no ascents of size  $d$  or more, which we will denote by  $F_d(z)$ . For this we use (2.3) with  $d + 1$  instead of  $d$  and  $v = 0$ , giving

$$F_{d+1}(z) := \frac{1}{1 - \tau(z, 0)} = \frac{1}{1 - \sum_{i \geq 1} \frac{(-1)^{i-1} z^{i+(d+1)\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)}}.$$

We want the generating function of compositions with  $X \leq d$ . This is equivalent to the generating function of the compositions with no ascent of size  $d + 1$  or more, which is  $F_{d+1}(z)$ . We now need to study the dominant poles,  $\rho_d$ , of  $F_{d+1}(z)$ , that is, the dominant zeros of

$$1 - \sum_{i \geq 1} \frac{(-1)^{i-1} z^{i+(d+1)\binom{i}{2}}}{(1-z)(1-z^2)\cdots(1-z^i)} = 0. \quad (5.1)$$

We follow the approach used by Gourdon and Prodinger in (6), which is also analogous to the one found in (10). Now for  $|z| \leq \frac{3}{5}$ , say, using the first two terms of the series above, the root  $\rho_d$  above can be approximated by the smallest positive root of

$$1 - \frac{z}{1-z} + \frac{z^{d+3}}{(1-z)(1-z^2)} + O(z^{3d}) = 0,$$

since the omitted terms in (5.1) are  $O(z^{3d})$ . This error will be majorized by subsequent  $O$  terms below. That is we want the root of

$$1 - 2z + \frac{z^{d+3}}{1-z^2} + O(z^{3d}) = 0.$$

The bootstrapping method gives a suitable approximation to  $\rho_d$ .

Let  $\rho_d := \frac{1}{2} + \varepsilon_d$ , then

$$1 - 2\left(\frac{1}{2} + \varepsilon_d\right) + \frac{4}{3} 2^{-d-3} = O\left(\left(\frac{3}{5}\right)^{3d}\right) + O\left(\frac{d}{2^{2d}}\right),$$

from which we find

$$\varepsilon_d = \frac{2^{-d}}{12} + O\left(\frac{d}{2^{2d}}\right) \text{ as } d \rightarrow \infty.$$

As  $F_{d+1}(z)$  has a simple pole at  $\rho_d$ , and by means of Rouché's Theorem, see IX.6.2 in (5), by comparing  $|\frac{1}{F_{d+1}(z)}|$  with  $|1 - \frac{z}{1-z}|$  on the circle  $|z| = \frac{3}{4}$ , we see that  $\frac{1}{F_{d+1}(z)}$  has no other zeros in  $|z| \leq 3/4$ .

It follows that

$$[z^n]F_{d+1}(z) = [z^n] \frac{A_d}{1 - z/\rho_d} + O\left(\left(\frac{4}{3}\right)^n\right) \text{ with } A_d = \frac{1}{\rho_d \frac{d\tau(z,0)}{dz} \Big|_{z=\rho_d}} = \frac{2 + O(2^{-d})}{\frac{d\tau(z,0)}{dz} \Big|_{z=\rho_d}},$$



where from (2.4) as  $d \rightarrow \infty$

$$\frac{d\tau(z, 0)}{dz} \Big|_{z=\rho} = \frac{\partial}{\partial z} \sum_{i \geq 1} \frac{(-1)^{i-1} z^{i+(d+1)\binom{i}{2}}}{(z; z)_i} \Big|_{z=\rho} = \left( \frac{z}{(1-z)^2} + O(z^d) \right) \Big|_{z=\rho} = 4 + O(2^{-d}).$$

Therefore as  $d \rightarrow \infty$ ,  $A_d = \frac{1}{2} + O(2^{-d})$ . Let us now restrict our attention to those  $d$  for which  $n^{-3} \leq 2^{-d} \leq \frac{\log n}{n}$ . The probability that  $X \leq d$  is then approximated as  $n \rightarrow \infty$  by

$$\begin{aligned} A_d \rho_d^{-n} &= A_d \left( \frac{1}{2} + \frac{2^{-d}}{12} + O\left(\frac{d}{2^{2d}}\right) \right)^{-n} = 2^{n-1} \exp\left(-\frac{2^{-d}n}{6}\right) (1 + O(2^{-d}) + O(nd2^{-2d})) \\ &= 2^{n-1} \exp\left(-\frac{2^{-d}n}{6}\right) \left(1 + O\left(\frac{\log^3 n}{n}\right)\right). \end{aligned}$$

So after dividing by  $2^{n-1}$  we have for  $n \rightarrow \infty$  and  $n^{-3} \leq 2^{-d} \leq \frac{\log n}{n}$ ,

$$\mathbb{P}(X \leq d) = \exp\left(-\frac{n}{2^d 6}\right) \left(1 + O\left(\frac{\log^3 n}{n}\right)\right). \quad (5.2)$$

Turning now to smaller values of  $d \geq 1$ , that is,  $d$  such that  $2^{-d} > \frac{\log n}{n}$ , a similar computation shows that (5.2) remains valid in this range, although now the probabilities  $\mathbb{P}(X \leq d)$  are small, since for such  $d$ ,  $\exp\left(-\frac{n}{2^d 6}\right) = O\left(\frac{1}{n}\right)$  as  $n \rightarrow \infty$ . Finally we must consider larger values of  $d \leq n$  that is,  $d$  for which  $n^{-3} > 2^{-d}$ , or equivalently,  $d \geq 3 \log_2 n$ . In this range we find that

$$\mathbb{P}(X \leq d) = 2^{n-1} \exp\left(-\frac{2^{-d}n}{6}\right) \left(1 + O\left(\frac{1}{n^2}\right)\right). \quad (5.3)$$

In view of the  $O$  estimates in (5.2) and (5.3) we may deduce that the mean value of  $X$  satisfies

$$\mathbb{E}_{max}(n) := \sum_{d=0}^n (1 - \mathbb{P}(X \leq d)) = \left( \sum_{d=0}^n \left(1 - \exp\left(-\frac{n}{2^d 6}\right)\right) \right) \left(1 + O\left(\frac{\log^4 n}{n}\right)\right) \text{ as } n \rightarrow \infty.$$

We now use the Mellin transform, (see (3)), to estimate the function

$$f(t) = \sum_{d \geq 0} \left(1 - \exp\left(-\frac{t}{2^d 6}\right)\right) \text{ as } t \rightarrow \infty.$$

The Mellin transform of  $f(t)$  is

$$f^*(s) = - \sum_{d \geq 0} (2^d 6)^s \Gamma(s) = - \frac{\Gamma(s) 6^s}{1 - 2^s} \text{ for } -1 < \Re s < 0.$$

Next we apply the Mellin inversion formula to recover  $f(t)$ :

$$f(t) = -\frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s) \frac{6^s}{1 - 2^s} t^{-s} ds.$$

We then shift the line of integration to the right and collect the (negative) residues at  $s = \frac{2k\pi i}{L}$  where  $k \in \mathbb{Z}$  and  $L = \log 2$ .

$$f(t) = \sum_{k \geq 0} \operatorname{Res} \Gamma(s) \frac{6^s}{1-2^s} t^{-s} \Big|_{s=\frac{2k\pi i}{L}} + \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s) \frac{6^s}{1-2^s} t^{-s} ds.$$

To evaluate the residue at the double pole  $s = 0$ , we need the expansion of the terms in the integrand to two terms as  $s \rightarrow 0$ , (here  $\gamma$  denotes Euler's constant),

$$\begin{aligned} \Gamma(s) &\sim \frac{1}{s} - \gamma, \\ 6^s &\sim 1 + s \log 6, \\ t^{-s} &\sim 1 - s \log t, \\ \frac{1}{1-2^s} &\sim \frac{1}{2} - \frac{1}{sL}. \end{aligned}$$

Hence the (negative) residue at  $s = 0$  is

$$\begin{aligned} [s^{-1}] \left\{ \left( \frac{1}{2} - \frac{1}{sL} \right) (1 + s \log 6) \left( \frac{1}{s} - \gamma \right) (1 - s \log t) \right\} \\ = \log_2 t - \frac{1}{2} - \log_2 3 + \frac{\gamma}{L}. \end{aligned}$$

For the residue at  $s = \frac{2k\pi i}{L}$  for  $k \neq 0$ , let  $\varepsilon = s - \frac{2k\pi i}{L}$  we have

$$[\varepsilon^{-1}] \Gamma \left( \frac{2k\pi i}{L} \right) \left( \frac{t}{6} \right)^{-\frac{2k\pi i}{L}} \frac{-1}{\varepsilon L} = -\frac{1}{L} \Gamma \left( \frac{2k\pi i}{L} \right) \left( \frac{t}{6} \right)^{-\frac{2k\pi i}{L}}.$$

The remainder integral  $\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}$  is of smaller order, so we have found

**Theorem 4** *The mean value of the size of the greatest ascent in the compositions of  $n$  satisfies as  $n \rightarrow \infty$ ,*

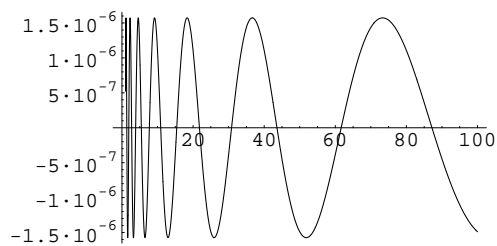
$$\mathbb{E}_{\max}(n) \sim \log_2 n - \frac{1}{2} - \log_2 3 + \frac{\gamma}{L} - \delta \left( \log_2 \frac{n}{6} \right)$$

where  $\delta(x)$  is a continuous periodic function of mean zero, period one and small amplitude with Fourier series

$$\delta(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma \left( -\frac{2k\pi i}{L} \right) e^{2k\pi i x}.$$

Computations show that  $\delta(x) < 1.7 \times 10^{-6}$ , as a result of the fast decrease of the gamma function with imaginary argument.

The diagram below shows the plot of  $\delta(\log_2 \frac{n}{6})$  for  $1 \leq n \leq 100$ .



**Fig. 1:**  $\delta(\log_2 \frac{n}{6})$  for  $1 \leq n \leq 100$

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