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# Clique-transversal sets and weak 2-colorings in graphs of small maximum degree<sup>†</sup>

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A clique-transversal set in a graph is a subset of the vertices that meets all maximal complete subgraphs on at least two vertices. We prove that every connected graph of order  $n$  and maximum degree three has a clique-transversal set of size  $\lfloor 19n/30 + 2/15 \rfloor$ . This bound is tight, since  $19n/30 - 1/15$  is a lower bound for infinitely many values of  $n$ . We also prove that the vertex set of any connected claw-free graph of maximum degree at most four, other than an odd cycle longer than three, can be partitioned into two clique-transversal sets. The proofs of both results yield polynomial-time algorithms that find corresponding solutions.

**2000 Mathematics Subject Classification:** 05C15, 05C69, 05C85

**Keywords:** clique-transversal set, weak coloring, clique coloring, cubic graph, claw-free graph, polynomial-time algorithm

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## 1 Introduction

We consider finite, simple undirected graphs  $G = (V, E)$ , with vertex set  $V$  and edge set  $E$ . The number of vertices will be denoted by  $n$ .

There are slight differences in the usage of the term ‘clique’ in graph theory. Throughout this paper, we use *clique* with the following restricted meaning: *inclusion-wise maximal complete subgraph with at least two vertices*. Hence, isolated vertices will *not* be called cliques, and maximality under inclusion will be required.

A *clique-transversal set* is a set  $S \subseteq V$  that meets all cliques of  $G$ . The smallest cardinality of a clique-transversal set in  $G$ , called *clique-transversal number*, is denoted by  $\tau_C(G)$ . A *weak 2-coloring* of  $G$  is a mapping  $\phi : V \rightarrow \{r, g\}$  (say red, green) such that both  $\phi^{-1}(r)$  and  $\phi^{-1}(g)$  are clique-transversal sets. If such  $\phi$  exists, we say that  $G$  is *weakly 2-colorable*.

This notion can be extended to *weak  $k$ -coloring*, also called  *$k$ -clique-coloring* in the literature, which assigns one of  $k$  colors to each vertex in such a way that no clique is monochromatic. Equivalently, graph  $G$  is *weakly  $k$ -colorable* if there exists a partition  $V_1 \cup \dots \cup V_k = V$  such that no  $V_i$  contains any cliques of  $G$ . The smallest nonnegative integer  $k$  admitting a weak  $k$ -coloring will be denoted by  $\chi_C(G)$ .

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## 1.1 Some standard terminology

For  $d = 3, 4$  we denote by  $\mathcal{G}_d$  the class of *connected* graphs of *maximum degree* at most  $d$ . The members of  $\mathcal{G}_3$  are the connected *subcubic* graphs, and those 3-regular ones are called *cubic*. The degree of vertex  $v$  will be denoted by  $d(v)$ . A vertex is *simplicial* if its neighbors are mutually adjacent.

Given a “forbidden” graph  $F$ , graph  $G$  is called *F-free* if no *induced* subgraph of  $G$  is isomorphic to  $F$ . In the cases of  $F = K_3$ ,  $F = K_{1,3}$ , and  $F = K_4 - e$  (one edge removed from  $K_4$ ), we use the standard terms *triangle-free*, *claw-free*, and *diamond-free*, respectively. A *hole* is a chordless cycle of length at least four.

## 1.2 Results and history

The graph invariant  $\tau_C$  was introduced by Gallai [13] and then studied by various authors. The earliest published results deal with chordal graphs [22], relating  $\tau_C$  with the number of vertices under the assumption that each edge is contained in some clique of given order. The case of  $\tau_C \leq n/2$  has been analyzed for line graphs and their complements [4], and some general bounds on  $\tau_C$  appeared in [11].

It should be noted that if  $G$  is triangle-free, then a set is a clique-transversal set of  $G$  if and only if it meets all edges—i.e., it is a vertex cover—therefore  $\tau_C(G)$  is equal to the number of vertices minus the independence number of  $G$  in this case. This also means that the determination of  $\tau_C$  is algorithmically hard, on many restricted classes of graphs. Structured hard classes with respect to  $\tau_C$  can be found in [9, 14, 8], whereas polynomial-time algorithms for other classes are given in [7, 9, 16].

Liang *et al.* [17] proved the following estimates on the clique-transversal number.

1. Every connected cubic graph  $G$  of order  $n > 4$  has  $5n/14 \leq \tau_C(G) \leq 2n/3$ .
2. There exist infinitely many cubic connected graphs with  $\tau_C(G) = 3n/5$ .

The estimates of 1 were proved also in [21], and the graphs attaining the lower bound were characterized in both papers. On the other hand, it remained an open problem to determine a tight upper bound on  $\tau_C(G)$  as a function of  $n$ . It has been explicitly raised in [17, page 114] whether  $\lceil 3n/5 \rceil$  is a valid upper bound. Here we disprove this guess and prove the following estimates.

**Theorem 1** *Consider the class  $\mathcal{G}_3$  of connected subcubic graphs.*

1. *If  $G \in \mathcal{G}_3$  of order  $n$  is not cubic, or contains a triangle, then  $\tau_C(G) \leq 19n/30 + 1/30$ .*
2. *If  $G \in \mathcal{G}_3$  of order  $n$  is cubic and triangle-free, then  $\tau_C(G) \leq 19n/30 + 2/15$ .*

*These bounds are tight in the sense that there exist infinitely many  $G \in \mathcal{G}_3$ , say of order  $n$ , such that*

3.  *$G$  is cubic and  $\tau_C(G) = 19n/30 - 1/15$ .*
4.  *$G$  is not cubic and  $\tau_C(G) = 19n/30 - 3/10$ .*

*Moreover, clique-transversal sets of sizes guaranteed in Parts 1 and 2 can be found in polynomial time for any  $G \in \mathcal{G}_3$ .*

Determining  $\chi_C(G)$  is hard: to decide  $\chi_C = 2$  is NP-complete on 3-chromatic *perfect* graphs [15], and can be even harder: it is  $\Sigma_2^P$ -complete on unrestricted input graphs [19]. On the positive side, all planar graphs have  $\chi_C \leq 3$  [20] and  $\chi_C = 2$  can be tested in polynomial time if the input is restricted to planar instances [15], hence  $\chi_C$  on planar graphs can be determined efficiently.

A necessary and sufficient condition for  $\chi_C \leq k$  on line graphs was given in [4]. Moreover, claw-free perfect graphs are weakly 2-colorable [6]. It was erroneously stated in [17, page 114] that the upper bound  $\tau_C \leq n/2$  implies  $\chi_C \leq 2$  for claw-free cubic graphs; later, however, in an unpublished manuscript the authors of [17] gave a proof for weak 2-colorability. Here we extend this latter result by dropping the condition of regularity and also weakening the condition on vertex degrees.

**Theorem 2** *Every connected claw-free graph of maximum degree at most four, other than an odd hole, is weakly 2-colorable. Moreover, a weak 2-coloring can be found in polynomial time.*

These results are proved in Sections 2 and 3, respectively. Some related problems are mentioned in the concluding section.

## 2 Transversal sets

In this section we prove Theorem 1. Let us begin with the proof of tightness, and then proceed with the upper bounds.

**Proof of Parts 3 and 4.** Locke [18] constructed an infinite family of connected cubic triangle-free graphs with  $n := 30k + 22$  vertices and independence number  $11k + 8$ . Thus, in every such graph  $G$  we have

$$\tau_C(G) = 19k + 14 = 19(n - 22)/30 + 14 = 19n/30 - 1/15.$$

If a non-regular connected graph is needed, we omit just one non-cutting vertex. Denoting  $n := 30k + 21$  we obtain

$$\tau_C(G) = 19k + 13 = 19(n - 21)/30 + 13 = 19n/30 - 3/10.$$

**Proof of Parts 1 and 2.** Let  $G = (V, E)$  be a subcubic connected graph of order  $n$ . Suppose first that  $G$  is *triangle-free*. If  $G$  is *not* 3-regular, we first run the  $O(n^4)$  algorithm of Fraughnaugh and Locke [12], which finds an independent set  $W$  of size at least  $11n/30 - 1/30$  in  $G$ . Then the set

$$S := V \setminus W, \quad |S| \leq 19n/30 + 1/30$$

meets all edges of  $G$  and hence is a clique-transversal set of required size, found in polynomial time. If  $G$  is triangle-free and *cubic*, then the algorithm in [12] guarantees a slightly weaker lower bound  $|W| \geq 11n/30 - 2/15$  on the size of independent set  $W$ , and we obtain  $|S| \leq 19n/30 + 2/15$  in this case.

Suppose from now on that  $G$  contains a triangle, say  $T$  with vertex set  $\{x_1, x_2, x_3\}$ . Each  $x_i \in T$  ( $i = 1, 2, 3$ ) has at most one neighbor outside  $T$ . We assume  $d(x_1) \geq d(x_2) \geq d(x_3)$ , and if  $d(x_i) = 3$  then denote the neighbor of  $x_i$  outside  $T$  by  $y_i$ .

If  $d(x_1) = 2$ , then  $G \simeq K_3$ ; and if  $d(x_3) = 3$  and  $y_1 = y_2 = y_3$ , then  $G \simeq K_4$ . In either case,  $\tau_C(G) = 1 \leq n/3$  holds, and we have nothing to prove. Similarly, it is easy to check that  $\tau_C(G) \leq n/2$  is valid if  $n \leq 4$ . Hence, we assume  $d(x_1) = 3$  and  $n > 4$ .

We shall apply induction on  $n$ , assuming that the upper bound  $\tau_C(G') \leq 19n'/30 + 1/30$  is valid for all non-cubic  $G' \in \mathcal{G}_3$  of order  $n' < n$ . For disconnected subcubic graphs with  $K$  components, none of which is cubic, this equivalently means  $\tau_C(G') \leq 19n'/30 + K/30$ . Note that no proper subgraph of  $G$  can have cubic components, because  $G$  is connected. The following simple fact will also be useful.

**Remark 1** Removing any set  $U$  of vertices, the number of components in the remaining graph cannot be larger than the edges connecting  $U$  with  $V \setminus U$ .

We now proceed with the inductive step for the upper bound on  $\tau_C$ . If  $d(x_2) = 2$ , then  $G - T$  is connected and it has a clique-transversal set  $S'$  of size at most  $19(n-3)/30 + 1/30$  by the induction hypothesis. Since  $S := S' \cup \{x_1\}$  is a clique-transversal set in  $G$ , the upper bound  $\tau_C(G) \leq 19n/30 - 13/15$  follows.

Suppose  $d(x_2) = 3$  and  $y_1 \neq y_2$ . If  $d(x_3) = 2$ , or  $d(x_3) = 3$  but  $y_3 = y_1$  (or  $y_3 = y_2$ ), we consider the graph  $G - T - y_1$  (or  $G - T - y_2$ ). Since it has at most three connected components by Remark 1, it contains a clique-transversal set  $S'$  of size at most  $19(n-4)/30 + 3/30$ , and then  $S := S' \cup \{y_1, x_2\}$  meets all cliques of  $G$ . Thus,  $\tau_C(G) \leq 19n/30 + (3 - 76 + 60)/30 = 19n/30 - 13/30$ .

Finally, suppose  $d(x_3) = 3$  and  $y_1 \neq y_2 \neq y_3 \neq y_1$ . We now consider  $G - T - y_1 - y_2$ . By Remark 1 it has at most five connected components. Hence, by the induction hypothesis, it has a clique-transversal set  $S'$  of size at most  $19(n-5)/30 + 5/30$ , and  $S := S' \cup \{y_1, y_2, x_3\}$  is a clique-transversal set in  $G$ . Thus,  $\tau_C(G) \leq 19n/30 + (5 - 95 + 90)/30 = 19n/30$ .

*Time analysis.* Let us choose a polynomial  $P(x)$  satisfying the following properties:  $P(x)$  is monotone increasing for  $x > 0$ ,  $P(n)$  is an upper bound for all  $n$  on the running time of the  $O(n^4)$  algorithm in [12] for triangle-free subcubic graphs, moreover

$$P(x') + P(x'') \leq P(x' + x'') \quad \text{and} \quad P(x-3) + cx \leq P(x)$$

for all  $x', x'' \geq 1$ , all  $x \geq 4$ , and for some constant  $c$  to be fixed later. For instance, if  $\sum_{i=0}^4 a_i x^i$  is a valid bound for [12], then  $P(x) := \sum_{i=0}^4 |a_i| x^i + cx^2$  will do; and any faster algorithm for triangle-free graphs would yield a stronger estimate for the general case, too.

If  $G$  is triangle-free, then the algorithm terminates in at most  $P(n)$  steps by assumption. Otherwise, triangle  $T$  can be found in  $c_1 n$  steps for some constant  $c_1$ , e.g. applying breadth-first search and checking at each vertex whether its two descendants (or possibly three for the root vertex) are adjacent or not.

The removal of 3, 4, or 5 vertices takes constant time. Assuming that the remaining graph has connected components of orders  $n_1, \dots, n_k$ , we need at most  $c_2(n_1 + \dots + n_k)$  steps to determine its components and at most  $P(n_1) + \dots + P(n_k) \leq P(n_1 + \dots + n_k) \leq P(n-3)$  steps to find the partial clique-transversal set  $S'$ . In this way, choosing  $c = c_1 + c_2$  we obtain that  $P(n)$  is an upper bound on the total running time.  $\square$

### 3 Weak 2-coloring

In this section we prove Theorem 2. Since even cycles are trivial to 2-color, we assume that  $G$  is not a cycle. It will turn out that diamond-free graphs admit a more elegant approach than general ones, therefore we treat them first; and afterwards the idea will be to identify a diamond  $D$ , find a weak 2-coloring of  $G - D$ , and prove that it can be extended to a weak 2-coloring of  $G$ .

So, assume first that  $G$  is connected, claw-free and also diamond-free, has maximum degree at most four, and is not a chordless cycle of length greater than three. Under these conditions we say that  $G$  is a *safe graph*. Moreover, Let us call a vertex  $x$  *safe* if it satisfies the following requirements:

1.  $G - x$  is connected,
2.  $G - x$  is not a cycle longer than three,

3.  $x$  is either a pendant vertex or contained in a  $K_3 \subseteq G$ .

For a safe vertex  $x$  we define its *critical neighbor*  $y$ —whose choice is not always unique—as follows.

- If  $d(x) = 1$ , then  $y$  is the unique neighbor of  $x$ .
- If  $x$  is in some triangle  $T_x$ , let  $K_x$  be the (unique) clique containing  $T_x$ .
  - If  $x$  has neighbor(s) outside  $K_x$ , let  $y \notin K_x$  be any such neighbor.
  - Otherwise, let  $y \in K_x$  be any neighbor of  $x$ .

Note that  $K_x$  is well-defined because each edge (and hence also each triangle) of  $G$  lies in a unique clique, otherwise  $G$  would not be diamond-free. For the same reason,  $x$  cannot occur in two triangle cliques which share a further vertex. And  $x$  cannot be involved in two cliques of size two either, because they would induce a claw with a vertex of  $T_x$ . On the other hand, it can happen that  $x$  is incident with two edge-disjoint triangles, in this case  $T_x = K_x$  can be chosen as any one of them.

We proceed with some properties concerning safe vertices in safe graphs.

**Lemma 1** *If  $x$  is a safe vertex in a safe graph  $G$ , and  $x$  is contained in a triangle  $T_x$ , then also  $K_x - x$  is a clique in  $G - x$  for the unique clique  $K_x$  containing  $T_x$  in  $G$ .*

**Proof:** Otherwise, there is a vertex  $z$  adjacent to all vertices of  $K_x$ . In this case,  $xy$  must be a non-edge, by the maximality of  $K_x$ . But then  $T_x \cup \{z\}$  induces a diamond, a contradiction.  $\square$

**Lemma 2** *Every safe graph of order greater than one has a safe vertex.*

**Proof:** Let  $G$  be a safe graph. Suppose first that  $G$  has a leaf  $x$ . The only safe-vertex-defining condition which could be violated is 2, but then we would find a claw in  $G$ . Thus,  $x$  is safe.

Assume next that  $G$  has no pendant vertices. Then  $G$  is not a tree, and it contains a chordless cycle. If this cycle can be chosen with length at least four, then we denote it by  $C$ . Since  $G$  is not a cycle, there exists some vertex  $u$  adjacent to  $C$ . Claw-freeness implies that there is an edge  $e = xy$  in  $C$  such that  $xyu$  is a triangle. If  $G - x$  is disconnected, then the two neighbors of  $x$  on  $C$  and a third neighbor in another component of  $G - x$  form a claw with center  $x$ . Hence,  $G - x$  has to be connected, and again it suffices to check whether Condition 2 is valid.

Suppose on the contrary that the graph  $G - x$  is a chordless cycle. Let  $z \neq y$  be the other neighbor of  $x$  on  $C$ . In this case,  $G - x$  consists of two paths, namely  $P := C - x$  from  $y$  to  $z$  and a  $z-u$  path  $Q$ , completed to a chordless cycle with edge  $uy$ . The neighbors of  $x$  are  $y, u, z$ , and the neighbor of  $z$  on  $Q$ . This is the only situation where  $x$  violates Condition 2. But then both  $y$  and  $z$  are safe in  $G$ .

Finally, if  $G$  has no chordless cycles of length at least four, then  $G$  is chordal, by definition. It is a well-known fact that a chordal graph has a simplicial vertex  $x$ , which clearly is safe.  $\square$

**Lemma 3** *Let  $x$  be a safe vertex in a safe graph  $G = (V, E)$ , with critical neighbor  $y$ . If  $\phi : V \setminus \{x\} \rightarrow \{r, g\}$  is a weak 2-coloring of  $G - x$ , then  $\phi(x) := \{r, g\} \setminus \{\phi(y)\}$  extends it to a weak 2-coloring of  $G$ .*

**Proof:** Suppose on the contrary that some monochromatic clique  $R$  occurs in  $G$ , say completely red. Of course,  $x \in R$  and  $|R| \geq 2$ . Let  $W$  be the complete subgraph  $R - x$ . This  $W$  is not maximal in  $G - x$

since  $\phi$  is a weak 2-coloring of  $G - x$ . Hence, By Lemma 1 we have  $|W| = 1$ , say  $W = \{w\}$ . Note that  $w \neq y$  because  $\phi(w) = \phi(x) \neq \phi(y)$ .

Vertex  $x$  is not pendant, therefore its  $K_x$  is well-defined. Since the edge  $wx$  is a clique in  $G$  and so it cannot be contained in any triangle, we see that  $wy \notin E$ , moreover  $w$  is not in  $K_x - x$ .

By Lemma 1,  $K_x - x$  is a clique in  $G - x$ , consequently both  $y$  and  $w$  have some non-neighbors in  $K_x - x$ ; denote one non-neighbor by  $y'$  and  $w'$ , respectively. Then  $yy'$  is an edge, otherwise  $\{x, y, w, w'\}$  would induce a claw. But now  $yy' \notin E$  implies  $y' \neq w'$  and that  $\{x, y, y', w'\}$  induces a diamond, a contradiction.  $\square$

Based on these lemmas, we design Algorithm 1 as a subroutine for the general algorithm to find a weak 2-coloring.

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**Algorithm 1** SAFECOL( $G$ ) — Weak 2-coloring of safe graphs

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**Require:** Safe graph  $G = (V, E)$ .

**Ensure:** Weak 2-coloring  $\phi : V \rightarrow \{r, g\}$ .

- 1: **if**  $|V| = 1$  **then** {assume  $V = \{v\}$ }
  - 2:    $\phi(v) := g$
  - 3: **else**
  - 4: Find safe vertex  $x$  and its critical neighbor  $y$
  - 5: SAFECOL( $G - x$ )
  - 6:  $\phi(x) := \{r, g\} \setminus \{\phi(y)\}$
- 

*Time analysis for Algorithm 1.* Apart from the recursive call in Step 5, the only time-consuming instruction is to identify a safe vertex in Step 4. Efficient implementation is ensured by the following claim.

**Lemma 4** *A safe vertex in a safe graph can be found in linear time.*

**Proof:** The non-cutting vertices  $x$  of  $G$  can be enumerated in  $O(n)$  steps, and since  $G$  has bounded maximum degree (and also because it is claw-free), for each  $x$  it can be tested in constant time whether or not  $x$  is incident with a triangle. Finally,  $G - x$  can be a cycle for at most one choice of  $x$ .  $\square$

Hence, storing the eliminated vertices in a stack, the recursive call of Step 5 (which yields iterated executions of Steps 4 and 6) can be implemented efficiently. As a consequence, Algorithm 1 requires not more than  $O(n^2)$  steps.

The following side-product of our method appears to be of interest on its own right, too.

**Remark 2** *Since every subgraph of any safe  $G \not\cong K_1$  contains a safe vertex, a “safe elimination order” can be determined.*

From now on we suppose that  $G$  contains a diamond  $D \simeq K_4 - e$ . Some cliques of  $G$  have vertices in both  $D$  and  $G - D$ ; we call them *crossing cliques*. If a crossing clique  $Q$  has just one vertex in  $D$ , we say that  $Q$  is a *strong crossing clique*; and otherwise we say that  $Q$  is *weak*.

As for notation, we assume that the diamond  $D$  found in  $G$  has vertex set  $\{c_1, c_2, d_1, d_2\}$ , where the only non-edge is  $\{c_1, c_2\}$ . By the degree assumption, there can occur at most one edge from  $d_i$  to  $M := G - D$ , and at most two edges from  $c_i$  to  $M$  ( $i = 1, 2$ ). Due to these degree constraints and the assumption that  $G$  is claw-free, combinations of the following crossing cliques may occur:

- strong edge:  $c_i a_i$  (at most one for each  $i \in \{1, 2\}$ )
- strong triangle:  $c_i b'_i b''_i$  (at most one for each  $i \in \{1, 2\}$ )
- weak triangle:  $c_i d_j w_{i,j}$  (at most one for each pair  $(i, j)$ )
- weak 4-clique:  $c_i d_1 d_2 z_i$  (at most one for each  $i \in \{1, 2\}$ )

Degree bounds on  $d_1, d_2$  imply that if both  $w_{1,j}, w_{2,j}$  exist, then  $w_{1,j} = w_{2,j}$ ; and similarly, if both  $z_1, z_2$  exist, then  $z_1 = z_2$ . Moreover, weak triangles of type  $d_1 d_2 v$  would create a claw, hence are excluded.

The procedure can now be formalized as described in Algorithm 2. The heart of the proof is expressed in the following assertion.

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**Algorithm 2** CLQCOL( $G$ ) — Determination of weak 2-coloring

---

**Require:** Claw-free connected graph  $G = (V, E)$  of maximum degree at most 4, not a hole.

**Ensure:** Weak 2-coloring  $\phi : V \rightarrow \{r, g\}$ .

- 1: **if**  $G$  is diamond-free **then**  $\{G$  is safe $\}$
  - 2: SAFECOL( $G$ )
  - 3: **else**
  - 4: Find diamond  $D$ , label its vertices  $c_1, c_2, d_1, d_2$  such that  $c_1 c_2 \notin E$
  - 5: **for all** components  $H$  of  $G - D$  **do**
  - 6:   **if**  $H$  not a cycle longer than 3 **then**
  - 7:     CLQCOL( $H$ )
  - 8:   **else**  $\{$ assume  $H \simeq C_\ell, \ell \geq 4$ , vertices labeled  $x_1, \dots, x_\ell$  sequentially along  $H$  $\}$
  - 9:     **if**  $\ell$  is even **then**
  - 10:        $\phi(x_i) := g$  for  $i$  odd ( $i = 1, 3, \dots, \ell - 1$ ),  $\phi(x_i) := r$  for  $i$  even ( $i = 2, 4, \dots, \ell$ )
  - 11:     **if**  $\ell$  is odd **then**
  - 12:       Find edge  $e \in E(H)$  contained in a crossing clique  $Q$   $\{$ assume  $e = x_1 x_\ell$  $\}$
  - 13:        $\phi(x_i) := g$  for  $i$  odd ( $i = 1, 3, \dots, \ell$ ),  $\phi(x_i) := r$  for  $i$  even ( $i = 2, 4, \dots, \ell - 1$ )
  - 14: Find  $\phi : \{c_1, c_2, d_1, d_2\} \rightarrow \{r, g\}$  with  $\phi(d_1) \neq \phi(d_2)$ , s.t. no monochromatic crossing clique occurs  $\{$ such  $\phi$  exists; see text $\}$
- 

**Lemma 5** *Let  $G \in \mathcal{G}_4$  be claw-free, and  $D$  a diamond in  $G$ . If no component of  $G - D$  is an odd hole, then every weak 2-coloring of  $G - D$  can be extended to a weak 2-coloring of  $G$  in such a way that the two vertices of degree three inside  $D$  get distinct colors.*

**Proof:** Suppose that a weak 2-coloring  $\phi$  of  $G - D$  has been fixed. We wish to extend it to the entire  $G$  without changing any color in  $G - D$ ; the extension will also be denoted by  $\phi$ .

Once we decide that  $\phi(d_1) \neq \phi(d_2)$  holds, all cliques of  $G$  with three vertices in  $D$  are 2-colored. This includes the triangles of  $D$  and the weak 4-cliques, too, if there are any. Therefore, we only have to show that the crossing cliques of orders two and three—strong edge, strong triangle, weak triangle—are 2-colorable under this condition.

A strong crossing clique may determine the color of  $c_i$ . Namely,  $\phi(c_i) = \{r, g\} \setminus \{\phi(a_i)\}$  must hold in a strong edge, and likewise,  $\phi(b'_i) = \phi(b''_i)$  in a strong triangle forces  $\phi(c_i) = \{r, g\} \setminus \{\phi(b'_i)\}$ . Since each  $c_i$  is incident with at most one strong clique, two contradictory conditions of this kind cannot occur at  $c_i$ . Moreover, apart from these situations, we have no *a priori* restriction on the colors of  $c_1$  and  $c_2$ .



Suppose first that  $c_1a_1$  is a strong edge. Then  $c_1$  cannot be incident with any crossing triangles: a strong one is impossible by the degree condition, and a weak triangle  $c_1d_1w_{1,1}$  would create a claw on  $\{c_1, d_2, a_1, w_{1,1}\}$  because  $c_1a_1$  is a clique and hence  $a_1$  cannot be adjacent to any neighbor of  $c_1$ . Consequently,  $\phi(c_1) := \{r, g\} \setminus \{\phi(a_1)\}$  yields a 2-coloring for all crossing cliques incident with  $c_1$ . The same argument applies if there is a strong edge  $c_2a_2$ .

The situation is similar and only slightly more complicated if there is a strong triangle, say  $c_1b'_1b''_1$ . In this case further edges  $b'_1d_1$  and/or  $b''_1d_2$  may be present, creating one or two weak triangles (or weak 4-cliques). If  $\phi(b'_1) = \phi(b''_1)$ , the choice  $\phi(c_i) := \{r, g\} \setminus \{\phi(b'_1)\}$  2-colors those weak triangles as well, and the proof is done. On the other hand, if  $\phi(b'_1) \neq \phi(b''_1)$ , then we may disregard the strong triangle because it is already 2-colored, independently of the actual color of  $c_2$ .

From now on we may assume that  $c_1$  and  $c_2$  are contained in weak triangles only. We select one  $c_1d_iw_{1,i}$  and one  $c_2d_jw_{2,j}$ , and define  $\phi(c_1) := \{r, g\} \setminus \{\phi(w_{1,i})\}$ ,  $\phi(c_2) := \{r, g\} \setminus \{\phi(w_{2,j})\}$ . This leaves at most one monochromatic weak triangle on each of  $c_1$  and  $c_2$ . If such a triangle remains on one of  $c_1$  and  $c_2$  only, then some of  $(\phi(d_1), \phi(d_2)) := (g, r)$  and  $(\phi(d_1), \phi(d_2)) := (r, g)$  surely makes it 2-colored. In the other case both  $d_1$  and  $d_2$  occur in two weak triangles; but each  $d_i$  has only one neighbor in  $G - D$ , therefore we must have  $w_{1,1} = w_{2,1} \neq w_{1,2} = w_{2,2}$ . Here  $w_{2,1} \neq w_{1,2}$  holds because otherwise two weak 4-cliques would occur instead of four weak triangles.

If  $\phi(w_{2,1}) \neq \phi(w_{1,2})$ , a simple completion of the coloring is to put  $\phi(d_1) := \phi(w_{1,2})$  and  $\phi(d_2) := \phi(w_{2,1})$ ; and if  $\phi(w_{2,1}) = \phi(w_{1,2})$ , then all the four weak triangles have a vertex of opposite color at  $c_1$  or  $c_2$ , and we obtain a weak 2-coloring by assigning  $(\phi(d_1), \phi(d_2)) := (g, r)$ .  $\square$

Based on Lemma 5, the soundness of Algorithm 2 can be verified easily, although it needs a little case distinction because odd hole components in  $G - D$  are not weakly 2-colorable. If a component  $H \not\cong K_3$  of  $G - D$  is an odd cycle longer than three, however, then any edge connecting  $H$  with  $D$  has to be extendable to a triangle with two vertices in  $H$ , for otherwise a claw would occur. Hence, edge  $e$  in Step 10 is well-defined, and it induces a strong triangle with  $c_1$  or  $c_2$ . That is, the situation is the same as if the strong triangle occurred from a non-cycle component, and the argument given in the proof of Lemma 5 verifies that all crossing cliques are 2-colored.

*Time analysis for Algorithm 2.* As it has been shown, Algorithm 1 called in Step 2 runs within  $cn_i^2$  time on any graph of order  $n_i$ , for some absolute constant  $c$ . Observe further that, no matter how many times it is performed during the recursive calls of Step 7, the safe subgraphs occurring in the procedure are mutually vertex-disjoint. Consequently, the overall running time of this part of Algorithm 2 does not exceed  $cn^2$ .

Even better, cycles in Steps 8–13 need time proportional to  $\ell$ , and also those cycles are mutually vertex-disjoint. Hence, they require  $O(n)$  time altogether. Also, Step 14 requires constant time for  $D$ , because only few crossing cliques can occur and they can be enumerated in constant time. These constants sum up to  $O(n)$  through all iterations.

Since the vertex degrees are bounded, we need at most  $c'n$  time to determine diamond  $D$  in Step 4. Also, we can enumerate the components of  $G - D$  in Step 5 and check the condition in Step 6 in linear time. Hence, reduction to a smaller problem instance takes linear time. Thus, the overall running time of the algorithm is  $O(n^2)$ .  $\square$

## 4 Concluding remarks

Here we put a couple of simple observations and mention some problems, which would be of interest for future research.

**NP-completeness.** From the well-known fact that the independence number is NP-complete to determine on cubic graphs, in connection with Theorem 1 we can derive that the complexity of finding  $\tau_C$  is NP-complete on triangle-free cubic graphs. The proof can be done in two steps:

- Given a cubic graph  $G = (V, E)$ , replace each edge  $e = xy \in E$  by a path  $xv_e w_e y$  of length three. This operation yields a subcubic triangle-free graph  $H$ , and increases the independence number by exactly  $|E|$ .
- Take two copies  $H', H''$  of  $H$  and insert the edges  $v'_e w''_e$  and  $v''_e w'_e$  for all  $e \in E$ . This results in a cubic triangle-free graph whose independence number is the double of that in  $H$ .

**Optimum running time.** Although our algorithms run in polynomial time, we expect that the orders of those polynomials are not optimal. For this reason, it is natural to ask:

**Problem 1** Determine the best asymptotic running time of an algorithm for

1. finding clique-transversal sets of size at most  $19n/30 + O(1)$  in connected subcubic graphs,
2. finding weak 2-colorings in claw-free graphs of maximum degree four.

**Clique-transversal number vs. clique size.** The flavor of results in [22] is that if every edge of a ‘nicely structured’ graph lies in a ‘large’ clique, then  $\tau_C$  is ‘small’. This direction has been pursued in [3] and recently in [5]. We think that there are many further classes of graphs for which such kind of results would be of interest to study.

**Line graphs.** The line graph of  $K_6$  is 8-regular and is not weakly 2-colorable. This fact, together with our Theorem 2, leads to the following problem.

**Problem 2** Find the largest integer  $d$  such that every claw-free graph of maximum degree  $d$  is weakly 2-colorable.

**Perfect graphs.** A long-standing open problem of Duffus *et al.* [10] asks whether  $\chi_C$  is bounded above by a constant on the class of perfect graphs. In fact, no examples of perfect graphs  $G$  with  $\chi_C(G) > 3$  are known. The upper bound  $\chi_C \leq 3$  has been proved for some classes of perfect graphs in [6]. Moreover, it is immediate by definition that every strongly perfect graph is weakly 2-colorable.

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