Monotone numerical schemes and feedback construction for hybrid control systems

Roberto Ferretti, Hasnaa Zidani

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Abstract

Hybrid systems are a general framework which can model a large class of control systems arising whenever a set of continuous and discrete dynamics are mixed in a single system. In this paper, we study the convergence of monotone numerical approximations of value functions associated to control problems governed by hybrid systems. We discuss also the feedback reconstruction and derive a convergence result for the approximate feedback control law. Some numerical examples are given to show the robustness of the monotone approximation schemes.

keywords. Hybrid systems; Approximation of the value function; Optimal feedback law

AMS subject classification. 34K34, 34K35, 49L20, 65N12

1 Introduction

In recent times, the notion of hybrid system has provided a very general framework to treat heterogeneous control systems, in which a collection of continuous and discrete dynamics are integrated in a single model. In this setting, a control may consist either in applying a conventional input (as in "classical" control models), or in performing discontinuous transitions in the state space, or even by switching among different dynamics. It is well-known, for example, that a restocking causes a jump in the state in economic models, and other classical examples may be provided for impulsive jumps. Switches in the dynamics occur, for instance, in hybrid and multi-gear vehicles in which the different dynamics correspond respectively to different engines and different mechanical transmission ratios. For example, a hybrid engine is obliged to switch to internal combustion whenever the battery is discharged, but may switch back to the electric power as soon as travel conditions ar! e favourable (two such examples will be examined in the section on numerical tests).

In this setting, the discrete dynamics may impose switching between two continuous dynamics, jumps in the system trajectory, or both. Moreover, as in the example above, we will discriminate in general between autonomous (mandatory) and controlled (optional) transitions, and, in defining an optimal control problem, we will also associate to such transitions suitable switching/jumping costs. General, recent reviews on the control theory for hybrid systems can be found in [1, 2].

Here, we will focus on optimal control problems. While necessary optimality conditions in the form of a Maximum Principle have been given by various authors (see, e.g., [3]), we will rather work in the framework

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of Dynamic Programming techniques. As a general analytical background, the reader is referred to [4, 5, 6] for a study of the characterization and properties of the value function.
The contribution of the present paper is on the numerical side. Our first point of interest is to approximate the value function of the problem – in this part, construction of a convergent approximation will result from an adaptation of monotone schemes to the hybrid case, which involves a dynamic programming equation in the form of a quasi-variational inequality. Moreover, in order to give a more constructive result, we also study the synthesis of an approximate optimal feedback control. We will define a procedure to compute a piecewise constant control given the state and the numerical value function, and prove that (under suitable assumptions) this construction provides an asymptotically optimal solution. Note that, in general, the value function of most relevant hybrid control problems is not expected to be continuous unless a certain number of technical assumptions are satisfied. In what follows, we will keep ourselves within this more restrictive framework, but, as numerical examples will show, the approximation strategies under consideration are robust enough to provide good results even in more general situations.

The outline of the paper is the following. In Section 2 we will set the basic assumptions on the control problem and review the characterization of the value function in terms of a suitable Dynamic Programming equation. In Section 3 we will study the numerical approximation via monotone schemes and discuss convergence and solvability of the numerical scheme. Section 4 will treat the construction of the approximate optimal feedback, while finally Section 5 will present some numerical examples of approximation of the value function and construction of the optimal control.

2 Setting of the Problem. Preliminaries

We start by introducing some notations used in the article. By $|\cdot|$ we mean the standard Euclidean norm. The notation $C_b(\mathbb{R}^d)$ will denote the space of continuous and bounded functions from $\mathbb{R}^d$ to $\mathbb{R}$. Let us also recall that in the inductive limit topology on $\mathbb{R}^d \times \mathbb{I}$, the concept of converging sequence is stated as follows:

> $(x_n, q_n) \in \mathbb{R}^d \times \mathbb{I}$ converges to $(x, q) \in \mathbb{R}^d \times \mathbb{I}$ if, for any $\varepsilon > 0$, there exists N_ε such that $q_n = q$, and $|x_n - x| < \varepsilon$ for any $n \geq N_\varepsilon$.

Among the various mathematical formulations of optimal control problems for hybrid systems, we will adopt here the one given in [4, 6, 7]. Let therefore $\mathbb{I}$ be a finite set, and consider the controlled system (X, Q) satisfying:

$$\begin{cases} \dot{X}(t) = f(X(t), Q(t), u(t)), \\ X(0) = x, \ Q(0^+) = q, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^d$, and $q \in \mathbb{I}$. Here, X and Q represent respectively the continuous and the discrete component of the state. Note that throughout the paper we will term *switch* a transition in the state which involves only a change in the $Q(t)$ component, whereas *jump* will denote a transition which might also involve a discontinuous change in $X(t)$.
The function $f : \mathbb{R}^d \times \mathbb{I} \times U \to \mathbb{R}^d$ is the continuous dynamics and the continuous control set is:

$$\mathscr{U} = \{u :]0, \infty[\to U \mid u \text{ measurable, } U \text{ compact metric space}\}.$$

The trajectory undergoes discrete transitions when it enters two predefined sets A (the *autonomous* jump set) and C (the *controlled* jump set), both of them subsets of $\mathbb{R}^d \times \mathbb{I}$. More precisely:

- On hitting A, the trajectory jumps to a predefined destination set D, possibly with a different discrete state $q' \in \mathbb{I}$. This jump is driven by a prescribed transition map $g : \mathbb{R}^d \times \mathbb{I} \times \mathscr{V} \to D$, where $\mathscr{V}$ is a discrete finite control set.
  We denote by τ_i an arrival time to A, and by $(X(\tau_i^-), Q(\tau_i^-))$ the position of the state before the jump. The arrival point after the jump and the new discrete state value will be denoted by $(X(\tau_i^+), Q(\tau_i^+)) = g(X(\tau_i^-), Q(\tau_i^-), w_i)$ and will depend on a discrete control action $w_i \in \mathscr{V}$.

- When the trajectory evolves in the set C, the controller can choose either to jump or not. If the controller chooses to jump, then the continuous trajectory is moved to a new point in D.
  By ξ_i we denote a (controlled) transition time. The state $(X(\xi_i^-), Q(\xi_i^-))$ is moved by the controlled jump the to the destination $(X(\xi_i^+), Q(\xi_i^+)) \in D$.

The trajectory starting from $x \in \mathbb{R}^d$ with discrete state $q \in \mathbb{I}$ is therefore composed of a continuous evolution given by ((1)) between two discrete jumps at the transition times. For example, assuming $\tau_i < \xi_k < \tau_{i+1}$, the evolution of the hybrid system would be given by:

$$
\begin{aligned}
&(X(\tau_i^+), Q(\tau_i^+)) = g(X(\tau_i^-), Q(\tau_i^-), w) \\
&\dot{X}(t) = f(X(t), Q(\tau_i^+), u(t)) \qquad \tau_i < t < \xi_k \\
&(X(\xi_k^+), Q(\xi_k^+)) \in D \qquad \text{(destination of the controlled jump at } \xi_k) \\
&\dot{X}(t) = f(X(t), Q(\xi_k^+), u(t)) \qquad \xi_k < t < \tau_{i+1}.
\end{aligned}
$$

2.1 Basic Assumptions

In the product space $\mathbb{R}^d \times \mathbb{I}$, we consider sets (and in particular the sets A, C and D) of the form

$$
S = \{(x,q) \in \mathbb{R}^d \times \mathbb{I} : x \in S_i, q = i\}, \tag{2}
$$

in which S_i represents the subset of S in which $q = i$. We make the following standing assumptions on the sets A, C, D and on the functions f and g:

(A1) For each $i \in \mathbb{I}$, A_i, C_i, and D_i are closed subsets of $\mathbb{R}^d$, and D_i is bounded. ∂A_i and ∂C_i are C^2.

(A2) The function f is Lipschitz continuous with Lipschitz constant L_f in the state variable x and uniformly continuous in the control variable u. Moreover, for all $(x,q) \in \mathbb{R}^d \times \mathbb{I}$ and $u \in U$,

$$
|f(x,q,u)| \le M_f.
$$

(A3) The map $g : A \times \mathscr{V} \to D$ is bounded and uniformly Lipschitz continuous with respect to x, with Lipschitz constant L_g.

(A4) ∂A is a compact set, and for some $\alpha > 0$, the following transversality condition:

$$
f(x,q,u) \cdot \eta_{x,q} \le -2\alpha
$$

holds for all $x \in \partial A_q$, and all $u \in U$, where $\eta_{x,q}$ denotes the unit outward normal to ∂A_q at x. We also assume similar transversality conditions on ∂C.

(A5) We assume that, for all $i \in \mathbb{I}$,

$$d(A_i, C_i) \geq \beta > 0$$
$$d(A_i, D_i) \geq \beta > 0$$

where d is the appropriate Euclidean distance.

In what follows, a control policy for the hybrid system consists of two parts: continuous input u and discrete inputs. A continuous control is a measurable function $u \in \mathscr{U}$ acting on the trajectory through the continuous dynamics (1). The discrete inputs take place at the transition times

$$0 \leq \tau_0 \leq \tau_1 \leq \cdots \tau_i \leq \tau_{i+1} \cdots$$
$$0 \leq \xi_0 \leq \xi_1 \leq \cdots \xi_k \leq \xi_{k+1} \cdots$$

in which at time τ_i (which *cannot* be selected by the controller) the trajectory undergoes a discrete transition under the action of the discrete control $w_i \in \mathscr{V}$, while at time ξ_k (which *can* be selected by the controller) the trajectory moves to a new position $(x'_k, q'_k) \in D \times \mathbb{I}$. The discrete inputs are therefore of two forms $\{w_i\}_{i \geq 0}$ and $\{(\xi_k, x'_k, q'_k)\}_{k \geq 0}$. To shorten the notation, we will denote by

$$\theta := (u(\cdot), \{w_i\}, \{(\xi_k, x'_k, q'_k)\})$$

a hybrid control strategy, and by Θ the set of all admissible strategies.

Concerning the sets of control values, we assume that

(A6) The control set U is a compact metric space, and $\mathscr{V}$ is a finite discrete set.

Remark 2.1 *From the definition of admissible controls, it clearly takes no time to have a discrete transition on the state. However, assumption (A5), along with suitable assumptions on the cost functional, prevents from any pathological executions of multiple discrete transitions at one single time or of infinite number of discrete transitions in any finite period of time. These kind of transitions are known in the literature as "Zeno executions" (see also [5, 8, 9] and the references therein for other kind of sufficient assumptions that allow to avoid Zeno executions).*

Now, for every control strategy $\theta \in \Theta$, we associate the cost defined by:

$$\begin{aligned} J(x,q;\theta) \quad := \quad & \int_0^{+\infty} \ell(X(t),Q(t),u(t))e^{-\lambda t}dt + \sum_{i=0}^{\infty} C_a(X(\tau_i^-),Q(\tau_i^-),w_i)e^{-\lambda \tau_i} \\ & + \sum_{k=0}^{\infty} C_c(X(\xi_k^-),Q(\xi_k^-),X(\xi_k^+),Q(\xi_k^+))e^{-\lambda \xi_k}, \end{aligned} \tag{3}$$

where $\lambda > 0$ is the discount factor, $\ell : \mathbb{R}^d \times \mathbb{I} \times U \to \mathbb{R}_+$ is the running cost, the functions $C_a : A \times \mathscr{V} \to \mathbb{R}_+$ and $C_c : C \times D \to \mathbb{R}_+$ are the costs for respectively autonomous and controlled transitions. The value function V is then defined as:

$$V(x,q) := \inf_{\theta \in \Theta} J(x,q;\theta). \tag{4}$$

The framework of infinite horizon control problem is quite general and the ideas developed in this paper are still valid in the context of finite horizon problems. We assume the following conditions on the cost functional:

(A7) $\ell : \mathbb{R}^d \times \mathbb{I} \times U$ is a bounded and non-negative function, Lipschitz continuous with respect to the x variable, and uniformly continuous w.r.t. the u variable.

(A8) $C_a(x,q,w)$ and $C_c(x,q,x',q')$ are uniformly Lipschitz continuous in the variables x and x', and bounded with a strictly positive infimum. Moreover, for any x and q, the function C_c satisfies the inequality

$$C_c(x,q,x',q') < C_c(x,q,\bar{x},\bar{q}) + C_c(\bar{x},\bar{q},x',q') - \Delta$$

for some $\Delta \geq 0$ (in Section 4, we will require Δ to be positive).

We briefly review the main theoretical facts about the value function (4) .

2.2 Characterization of the Value Function

It is quite straightforward to derive a Dynamic Programming Principle for the control problem (1)-(3) as follows (see [5]):

1. For any $(x,q) \in (\mathbb{R}^d \times \mathbb{I}) \setminus (A \cup C)$ there exists $s_0 > 0$ such that, for every $s \in]0, s_0[$, we have:

$$V(x,q) = \inf_{u \in \mathscr{U}} \left[\int_0^s \ell(X(t),q,u(t)) e^{-\lambda t}\, dt + e^{-\lambda s} V(X(s),q) \right]. \tag{5}$$

2. For $(x,q) \in A$, we have:

$$V(x,q) = \inf_{w \in \mathscr{V}} \left[V(g(x,q,w)) + C_a(x,q,w) \right]. \tag{6}$$

3. For $(x,q) \in C$, we have:

$$V(x,q) \leq \inf_{(x',q') \in D} \left[V(x',q') + C_c(x,q,x',q') \right]. \tag{7}$$

   If it happens that $V(x,q) < \inf_{(x',q') \in D} [V(x',q') + C_c(x,q,x',q')]$, then there exists $s_0 > 0$ such that for every $0 < s < s_0$, we have:

$$V(x,q) = \inf_{u \in \mathscr{U}} \left[\int_0^s \ell(X(t),q,u(t)) e^{-\lambda t}\, dt + e^{-\lambda s} V(X(s),q) \right]. \tag{8}$$

Moreover, it is not difficult to show uniform continuity of the value function V. More precisely, we have:

Lemma 2.1 *Under assumptions (A1)–(A8), the value function is Hölder continuous and bounded.*

From the Dynamic Programming Principle, it can be verified that the value function satisfies, in the viscosity sense, a quasi-variational inequality. To give a precise statement of this result, we first introduce the Hamiltonian $H : \mathbb{R}^d \times \mathbb{I} \times \mathbb{R}^d \to \mathbb{R}$ defined, for $x, p \in \mathbb{R}^d$ and $q \in \mathbb{I}$, by:

$$H(x,q,p) := \sup_{u \in U} \{ -\ell(x,q,u) - f(x,q,u) \cdot p \}. \tag{9}$$

We also define the transition operators $\mathscr{M}$ (respectively $\mathscr{N}$) mapping $C^0(\mathbb{R}^d \times \mathbb{I})$ into $C^0(A)$ (respectively $C^0(C)$) by:

$$\mathscr{M}\phi(x,q) := \inf_{w \in \mathscr{V}} \{ \phi(g(x,q,w)) + C_a(x,q,w) \} \qquad (x,q) \in A, \tag{10}$$

$$\text{(respectively} \quad \mathscr{N}\phi(x,q) := \inf_{(x',q') \in D} \{ \phi(x',q') + C_c(x,q,x',q') \} \quad (x,q) \in C). \tag{11}$$

From [10], the following properties hold for $\mathscr{M}$ and $\mathscr{N}$.

Proposition 2.1 *Let $\phi, \psi : \mathbb{R}^d \times \mathbb{I} \to \mathbb{R}$, and $\mathscr{M}$, $\mathscr{N}$ be defined by (10)–(11). Then:*

1. *If $\phi \geq \psi$, then $\mathscr{M}\phi \geq \mathscr{M}\psi$;*
2. *$\mathscr{M}(t\phi + (1-t)\psi) \geq t\mathscr{M}\phi + (1-t)\mathscr{M}\psi$ for $t \in [0,1]$;*
3. *$\mathscr{M}(\phi + c) = \mathscr{M}\phi + c$, for $c \in \mathbb{R}$;*
4. *$|\mathscr{M}\phi - \mathscr{M}\psi|_0 \leq |\phi - \psi|_0$.*

The same properties also hold for the operator $\mathscr{N}$.

Now, let us come back to the characterization of the value function V.

Proposition 2.2 *Assume (A1)–(A8). The function V is a bounded and Hölder continuous viscosity solution of:*

$$\lambda V(x,q) + H(x,q,D_x V(x,q)) = 0 \qquad (x,q) \in (\mathbb{R}^d \times \mathbb{I}) \setminus (A \cup C), \tag{12}$$

$$\max(V(x,q) - \mathscr{N}V(x,q), \lambda V(x,q) + H(x,q,D_x V(x,q)) = 0 \quad (x,q) \in C, \tag{13}$$

$$V(x,q) - \mathscr{M}V(x,q) = 0 \qquad (x,q) \in A. \tag{14}$$

The proof is given in [4, Theorem 3.5]. The same arguments of the proof of Theorem 5.1 in [4] can then be used to obtain a strong comparison principle (and hence, uniqueness of the solution) as follows:

Theorem 2.1 *Assume (A1)–(A8). Let u (respectively, v) be a bounded upper semi-continuous (respectively, lower semi-continuous) function on $\mathbb{R}^d$. Assume that u is a sub-solution (respectively, v is a super-solution) of (12)–(14) in the following sense:*

$$\lambda V(x,q) + H(x,q,D_x V(x,q)) \leq 0 \quad (\geq 0) \qquad (x,q) \in (\mathbb{R}^d \times \mathbb{I}) \setminus (A \cup C);$$

$$\max(V(x,q) - \mathscr{N}V(x,q), \lambda V(x,q) + H(x,q,D_x V(x,q)) \leq 0 \;\; (\geq 0) \quad (x,q) \in C;$$

$$V(x,q) - \mathscr{M}V(x,q) \leq 0 \quad (\geq 0) \qquad (x,q) \in A.$$

Then, $u \leq v$.

Recall that the viscosity framework turns out to be a convenient tool for the study of the theoretical properties of the value function and also for the analysis of the convergence of numerical schemes.

3 The Numerical Scheme

Consider monotone approximation schemes of (12)–(14), of the following form:

$$S^h(x,q,V^h(x,q),V^h) = 0 \qquad (x,q) \in (\mathbb{R}^d \times \mathbb{I}) \setminus (A \cup C), \tag{15}$$

$$\max\left\{S^h(x,q,V^h(x,q),V^h), V^h(x,q) - N^h V^h(x,q)\right\} = 0 \quad (x,q) \in C, \tag{16}$$

$$V^h(x,q) - M^h V^h(x,q) = 0 \qquad (x,q) \in A. \tag{17}$$

Here and in what follows, $V^h \in C_b(\mathbb{R}^d \times \mathbb{I})$ is the solution of (15)–(17), and we denote by $S^h : \mathbb{R}^d \times \mathbb{I} \times \mathbb{R} \times C_b(\mathbb{R}^d \times \mathbb{I}) \to \mathbb{R}$ a family of consistent, monotonic operators (indexed by h) which is considered to be an approximation of the HJB equation (12) (see assumptions (B1)–(B4) for the precise properties). Accordingly, M^h and N^h are consistent approximations of the transition operators $\mathscr{M}$ and $\mathscr{N}$. We will denote the discretization parameter by $h \in \mathbb{R}_+$, implicitly assuming that if multiple discretization parameters appear in the scheme, then they are related one another (e.g., by CFL-type conditions).

The abstract notations of the scheme was introduced by Barles and Souganidis [11] to display clearly the monotonicity of the scheme: $S^h(x,q,r,v)$ is nondecreasing in r and nonincreasing in v. Typical approximation schemes that can be put in this framework are classical Finite Differences [12, 13], Semi-Lagrangian schemes [14, 15, 16], and Markov Chain Approximations [13]. To rephrase Barles–Souganidis theory for the case under consideration, we shall make the following standing assumptions on the discrete scheme (15)–(17):

(B1) Stability: the solution of (15)–(17) is uniformly bounded in L^∞ for all $h \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $q \in \mathbb{I}$.

(B2) Monotonicity of S^h: $S^h(x,q,r,\phi) \le S^h(x,q,r,\psi)$
for all $h \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $q \in \mathbb{I}$, $r \in \mathbb{R}$, and $\phi, \psi \in C_b(\mathbb{R}^d \times \mathbb{I})$ such that $\phi \ge \psi$ in $\mathbb{R}^d \times \mathbb{I}$.

(B3) Boundedness of S^h: for all $h \in \mathbb{R}_+$ and $\phi \in C_b(\mathbb{R}^d \times \mathbb{I})$, $S^h(x,q,r,\phi)$ is locally bounded.

(B4) Consistency of S^h: S^h is consistent, that is, given a smooth function ϕ, then

$$S^h(x,q,\phi(x,q),\phi) \to \lambda\phi(x,q) + H(x,q,D_x\phi(x,q))$$

as $h \to 0$, for all $x \in \mathbb{R}^d$ and $q \in \mathbb{I}$.

(B5) Monotonicity of M^h and N^h: Both operators M^h and N^h are monotone, that is, given two functions $\phi, \psi \in C_b(\mathbb{R}^d \times \mathbb{I})$ such that $\phi \ge \psi$ in $\mathbb{R}^d \times \mathbb{I}$, then

$$M^h\phi(x,q) \ge M^h\psi(x,q)$$
$$N^h\phi(x,q) \ge N^h\psi(x,q)$$

for all $x \in \mathbb{R}^d$ and $q \in \mathbb{I}$.

(B6) Invariance w.r.t. constants: Both operators M^h and N^h are invariant with respect to the addition of constants, that is, given a function ϕ and a constant c, then

$$M^h(\phi + c)(x,q) = M^h\phi(x,q) + c$$
$$N^h(\phi + c)(x,q) = N^h\phi(x,q) + c$$

for all $x \in \mathbb{R}^d$ and $q \in \mathbb{I}$. Note that, by a well-known result [17], assumptions (B5) and (B6) imply that M^h and N^h are L^∞-nonexpansive.

(B7) Consistency: Both operators M^h and N^h are consistent, that is, given a smooth function ϕ, then

$$\phi(x,q) - N^h\phi(x,q) \to \phi(x,q) - \mathscr{N}\phi(x,q),$$
$$\phi(x,q) - M^h\phi(x,q) \to \phi(x,q) - \mathscr{M}\phi(x,q),$$

as $h \to 0$, for all $x \in \mathbb{R}^d$ and $q \in \mathbb{I}$.

Note that assumptions (B5)-(B7) are analogous to the properties satisfied by the continuous operators $\mathscr{M}$ and $\mathscr{N}$ (see Proposition 2.1).

3.1 Monotonicity and Convergence

Our plan is to obtain a monotone scheme, so that its theoretical analysis might be performed by the Barles–Souganidis Theorem (see [11]). By (B1), we are assuming that the numerical scheme yields uniformly bounded solutions V^h. This assumption will be discussed and checked for specific schemes in next section. Here, we show that the scheme (15)–(17), which for shortness will be rewritten as

$$F^h(x,q,V_h(x,q),V_h) = 0$$

is also monotone and consistent in the sense of [11].

Monotonicity Note that Barles–Souganidis Theorem requires the function F^h to be decreasing with respect to its last argument, that is, if $\phi(x,q) \geq \psi(x,q)$ on $\mathbb{R}^d \times \mathbb{I}$, then

$$F^h(x,q,t,\phi) \leq F^h(x,q,t,\psi). \tag{18}$$

By (B2), this is satisfied in (15). Moreover, by (B5) this property also holds for the terms $\phi - M^h\phi$ and $\phi - N^h\phi$. Therefore, monotonicity is satisfied for (17) and, taking into account that the max of two monotone operators is monotone, for (16) as well.

Consistency According to what has been done for the scheme, we rewrite the Quasi-Variational Inequality (12)–(14) as

$$\mathscr{F}(x,q,\phi(x,q),D_x\phi(x,q)) = 0. \tag{19}$$

Assumptions (B4) and (B7) immediately imply that the whole operator F^h is consistent. Indeed, the scheme results from consistent terms, or from the max of consistent terms, so that, given a smooth function ϕ, the limit

$$\lim_{h\to 0} F^h(x,q,\phi(x,q),\phi)$$

converges to the operator $\mathscr{F}$ defined in (19), computed on ϕ, at all points of continuity of $\mathscr{F}$. On the other hand, by convergence of each term, if the point $\bar{x}$ is at the interface $\partial A_q \cup \partial C_q$, then

$$\begin{aligned}
\mathscr{F}_*(\bar{x},q,\phi(\bar{x},q),D_x\phi(\bar{x},q)) \quad &\leq \quad \liminf_{\substack{h\to 0\\ x\to\bar{x}}} F^h(x,q,\phi(x,q),\phi) \\
&\leq \quad \limsup_{\substack{h\to 0\\ x\to\bar{x}}} F^h(x,q,\phi(x,q),\phi) \leq \mathscr{F}^*(\bar{x},q,\phi(\bar{x},q),D_x\phi(\bar{x},q)),
\end{aligned}$$

(where $\mathscr{F}_*$ and $\mathscr{F}^*$ denote respectively the lower and the upper semicontinuous envelopes of $\mathscr{F}$) which corresponds to the generalized consistency notion required in [11].

Convergence Last, combining consistency, monotonicity and L^∞ stability, it is a standard matter to recover convergence by means of Barles–Souganidis Theorem [11]:

Theorem 3.1 *Assume (B1)–(B7). Then, the unique solution V^h of (15)–(17) converges locally uniformly to the solution of (12)–(14).*

3.2 A Practical Construction of the Scheme

While the framework introduced so far is general and abstract enough to include a number of different numerical approaches, we give in this section an example of implementable scheme which satisfies the general assumptions made above.

Consider a family of space grids in the computational domain, indexed by the discretization parameter h. A numerical approximation typically computes the solution as a vector of node values, which still needs a numerical reconstruction to be extended to the whole of $\mathbb{R}^d \times \mathbb{I}$. We will denote by $\mathscr{I}[\phi]$ the operator which interpolates the node values of the function ϕ, so that $\mathscr{I}[\phi](x,q)$ will be the corresponding interpolated value computed at $x \in \mathbb{R}^d$ and $q \in \mathbb{I}$.

Note that, since q is a discrete variable, $\mathscr{I}$ needs not to be a $(d+1)$-dimensional interpolation, but rather a vector of d-dimensional interpolations. We make the standing assumption that this interpolation is monotone:

(C1) The interpolation operator $\mathscr{I}[\phi]$ is monotone with respect to ϕ, that is, given two functions ϕ, ψ such that $\phi \geq \psi$ for $(x,q) \in \mathbb{R}^d \times \mathbb{I}$, then

$$\mathscr{I}[\phi](x,q) \geq \mathscr{I}[\psi](x,q)$$

for all $x \in \mathbb{R}^d$ and $q \in \mathbb{I}$.

For example, $\mathbb{P}_1$ (piecewise linear) or $\mathbb{Q}_1$ (piecewise multi-linear) interpolations would satisfy (C1). Now, assume that (15) is in the form

$$S^h(x,q,V^h(x,q),V^h) = V^h(x,q) - \Sigma^h(x,q,V^h). \tag{20}$$

We consider schemes approximating (12)–(14) in fixed point form:

$$V^h(x,q) = T^h(x,q,V^h) := \begin{cases} M^h V^h(x,q) & \text{if } x \in A_q \\ \min\left\{N^h V^h(x,q), \Sigma^h(x,q,V^h)\right\} & \text{if } x \in C_q \\ \Sigma^h(x,q,V^h) & \text{else.} \end{cases} \tag{21}$$

Here, V^h is the numerical solution, and more precisely, its extension to all $x \in \mathbb{R}^d$ and $q \in \mathbb{I}$ by means of the interpolation $\mathscr{I}$.

The form (15)–(17), although not the most general possibility, has in fact some definite advantages:

- It may immediately be constructed from the time-marching formulation of a stationary problem (in this case, h is usually regarded as a *time step* and the fixed-point iteration on (15)–(17) is termed *value iteration*);
- It typically gives a contraction for the right-hand side T^h;
- It inherits monotonicity of the discretization from the time-dependent version.

Before entering the details of theoretical analysis, we start by giving some practical examples, and checking at the same time that assumptions (B1)–(B7) are satisfied.

Jump operators. The most natural way of discretizing the jump operators $\mathscr{M}$ and $\mathscr{N}$, is to compute the exact operators at the nodes, then extending the result by interpolation to all x and q. Assuming that the infimum can be computed as a minimum (which is the way it is computed in a practical scheme), the resulting discretization of the operators $\mathscr{M}$ and $\mathscr{N}$ is therefore

$$M^h\phi(x,q) := \mathscr{I}[\mathscr{M}\phi](x,q),$$
$$N^h\phi(x,q) := \mathscr{I}[\mathscr{N}\phi](x,q).$$

Monotonicity of discrete operators requires to use monotone interpolations, for which (C1) holds. This construction of M^h and N^h satisfies assumptions (B5)–(B7).

Continuous control. A monotone discretization of (15) in the form (20) can be constructed, as we remarked above, in time-marching form, for example with a Semi-Lagrangian scheme (see [16]):

$$v(x_j,q) = \min_{u\in U}\left\{h\,\ell(x_j,q,u) + (1-\lambda h)V^h(x_j + h\,f(x_j,q,u))\right\}. \tag{22}$$

Note that the left-hand side of (22) represents the value of V_h at a node x_j, and is extended to all $x \in \mathbb{R}^d$ via the interpolation $\mathscr{I}$. Note also that, for $h = \Delta t$ small enough, the right-hand side is a contraction with contraction coefficient $L = 1 - \lambda h$.

Using other difference schemes (e.g., Upwind, Lax–Friedrichs), the operator Σ^h turns out again to be a contraction with a Lipschitz constant $L = 1 - |O(h)|$. Therefore, it is not overly restrictive to assume in addition that:

(C2) The operator Σ^h is a contraction for h small enough, and more precisely,

$$\left\|\Sigma^h(x,q,\phi) - \Sigma^h(x,q,\psi)\right\|_\infty \le L\,\|\phi-\psi\|_\infty$$

with a Lipschitz constant $L = 1 - C_L h$, for any $h < h_0$.

We prove now that the scheme, as stated in (15)–(17), admits a unique solution which might be computed iteratively.

Theorem 3.2 *Assume (B5)–(B6) and (C2). Assume in addition that $h < h_0$. Then, the iteration*

$$V^h_{k+1} = T^h(V^h_k) \tag{23}$$

converges, for $k \to \infty$, to a unique fixed point V^h in $L^\infty(\mathbb{R}^d \times \mathbb{I})$ for any initial guess $V^h_0 \in L^\infty(\mathbb{R}^d \times \mathbb{I})$.

Proof. Our plan is to prove that there exists $0 < L < 1$ such that:

$$\|T^h(\phi) - T^h(\psi)\|_\infty \le \|\phi-\psi\|_\infty \tag{24}$$

$$\|(T^h\circ T^h)(\phi) - (T^h\circ T^h)(\psi)\|_\infty \le L\|\phi-\psi\|_\infty, \tag{25}$$

that is, even though the application of T^h does not necessarily contracts L^∞ norms, twice its application does. Then, by a standard argument, we infer that the iteration is equivalent to a contraction with a Lipschitz constant $\sqrt{L} < 1$, and is therefore convergent.
First, setting $E = (\mathbb{R}^d \times \mathbb{I}) \setminus (C \cup A)$, we rewrite the ∞-norm as

$$\|\cdot\|_\infty = \max(\|\cdot\|_{\infty,A}, \|\cdot\|_{\infty,C}, \|\cdot\|_{\infty,E}),$$

where $\|\cdot\|_{\infty,S}$ stands for the ∞-norm on the set S.
Taking into account that the destination set D is included in E, that T^h corresponds to M^h on the set A and that M^h itself is nonexpansive in L^∞, we have

$$\|T^h(\phi) - T^h(\psi)\|_{\infty,A} \le \|\phi - \psi\|_{\infty,E}. \tag{26}$$

On the other hand, by (C2) we also have that

$$\|T^h(\phi) - T^h(\psi)\|_{\infty,E} \le L\|\phi - \psi\|_{\infty,E} \le L\|\phi - \psi\|_{\infty}. \tag{27}$$

Consider now $x \in C_q$. By (15)–(17), we can single out four cases:

1. First case:

$$\min\{N^h\phi(x,q), \Sigma^h(x,q,\phi)\} = N^h\phi(x,q),$$
$$\min\{N^h\psi(x,q), \Sigma^h(x,q,\psi)\} = N^h\psi(x,q).$$

   In this case, by nonexpansivity of N^h, we obtain

$$|T^h(x,q,\phi) - T^h(x,q,\psi)| \le \|\phi - \psi\|_{\infty,E}. \tag{28}$$

2. Second case:

$$\min\{N^h\phi(x,q), \Sigma^h(x,q,\phi)\} = \Sigma^h(x,q,\phi),$$
$$\min\{N^h\psi(x,q), \Sigma^h(x,q,\psi)\} = \Sigma^h(x,q,\psi).$$

   If this is the case, by contractivity of Σ^h, we get

$$|T^h(x,q,\phi) - T^h(x,q,\psi)| \le L\|\phi - \psi\|_{\infty,E}. \tag{29}$$

3. Third case:

$$\min\{N^h\phi(x,q), \Sigma^h(x,q,\phi)\} = N^h\phi(x,q),$$
$$\min\{N^h\psi(x,q,), \Sigma^h(x,q,\psi)\} = \Sigma^h(x,q,\psi).$$

   Here, assume first that $N^h\phi(x,q) \ge \Sigma^h(x,q,\psi)$. If this happens, it is possible to bound from above $|T^h(x,q,\phi) - T^h(x,q,\psi)|$ by replacing $N^h\phi(x,q)$ with a suboptimal choice, so that

$$\begin{aligned} |T^h(x,q,\phi) - T^h(x,q,\psi)| &\le \Sigma^h(x,q,\phi) - \Sigma^h(x,q,\psi) \\ &\le L\|\phi - \psi\|_{\infty,E}. \end{aligned} \tag{30}$$

   If, on the contrary, $N^h\phi(x,q) \le \Sigma^h(x,q,\psi)$, then $\Sigma^h(x,q,\psi)$ is replaced with a suboptimal choice, and we obtain

$$\begin{aligned} |T^h(x,q,\phi) - T^h(x,q,\psi)| &\le N^h\psi(x,q) - N^h\phi(x,q) \\ &\le \|\phi - \psi\|_{\infty,E}. \end{aligned} \tag{31}$$

4. Fourth case:

$$\min\{N^h\phi(x,q),\Sigma^h(x,q,\phi)\} = \Sigma^h(x,q,\phi),$$
$$\min\{N^h\psi(x,q),\Sigma^h(x,q,\psi)\} = N^h\psi(x,q).$$

This case completely parallels the third one and is therefore omitted.

To sum up, (28)–(31) imply the bound

$$\|T^h(\phi) - T^h(\psi)\|_{\infty,C} \le \|\phi - \psi\|_{\infty,E}, \tag{32}$$

so that, using also (26) and (27), we get

$$\|T^h(\phi) - T^h(\psi)\|_{\infty} \le \|\phi - \psi\|_{\infty,E}, \tag{33}$$

which entails (24). Finally, applying a second time T^h, we get

$$\|(T^h \circ T^h)(\phi) - (T^h \circ T^h)(\psi)\|_\infty \le L\|T^h(\phi) - T^h(\psi)\|_{\infty,E},$$

and hence, by (27), (25) follows.

□

Note that, for a scheme in time-marching form, the consistency condition reads

$$\lim_{h\to 0} \frac{\phi(x,q) - \Sigma^h(x,q,\phi)}{h} = \lambda\phi(x,q) + H(x,q,D\phi(x,q)).$$

This condition implies that, if for example the iteration (23) is started fron the initial condition

$$V_0^h \equiv 0,$$

then $\|\Sigma^h(\cdot,\cdot,V_0^h)\|_\infty = O(h)$ and therefore, as it is easy to check, $\|V_1^h\|_\infty = O(h)$. This entails that the norm of the limit solution,

$$\begin{aligned} \left\|V^h\right\|_\infty &\le \frac{\left\|V_1^h - V_0^h\right\|_\infty}{1-\sqrt{1-C_L h}} \\ &\le \frac{2\left\|V_1^h\right\|_\infty}{C_L h}, \end{aligned}$$

is uniformly bounded with respect to h, i.e., that the scheme is stable in L^∞, and therefore (B1) is also satisfied.

4 Construction of the Approximate Feedback

We have focused so far on the approximation of the value function V. However, from the control point of view, the interest is rather on the construction of the (approximate) optimal control given the information on the value function. We consider therefore an approximate strategy which leads to construct a quasi-optimal feedback solution starting from the approximate value function V^h, at least in any set on which V^h converges uniformly to V. To this end, we need more strict assumptions, in particular to ensure the existence of an optimal control. In addition to the basic assumptions, we assume therefore (C1), plus the following set of hypotheses:

(D1) The set of admissible controls U is convex;

(D2) The running cost $\ell(x,q,u)$ is convex w.r.t. the control u;

(D3) The dynamics $f(x,q,u)$ is linear w.r.t. the control u, that is,

$$f(x,q,u) = f_1(x,q) + f_2(x,q)u.$$

(D4) The value function is Hölder continuous, so that

$$|V(x,q) - V(y,q)| \leq L_V |x-y|^\gamma,$$

and $\gamma > 1/2$.

Concerning (D4), we recall that an explicit bound on the Hölder exponent γ is proved in [4]. In order to unify the two cases of transition, the jump operator $\mathscr{M}$ will be put in the equivalent form

$$\mathscr{M}\phi(x,q) = \min_{(x',q')\in D_a(x,q)} \{\phi(x',q') + C_a(x,q,x',q')\},$$

in which $D_a(x,q) \subset D$ stands for the image of the function g at the point (x,q), and with a small abuse of notation C_a denotes now the equivalent cost function for the transition.

Once set a time grid $t_0, t_1, \ldots, t_k, \ldots$ with $t_k = k\delta$ for some "small" parameter δ (to be suitably chosen in the sequel depending on h), the general idea is to sample the state at the time t_k and define a constant control on the interval $]t_k, t_{k+1}]$.

The approximate feedback is defined on the basis of the approximate value function and of the point $(x,q) = (X(t_k), Q(t_k))$, with a discrete dynamic programming procedure. More precisely:

1. If $(x,q) \in A$, then define

$$(x_*^\delta, q_*^\delta) := \operatorname*{argmin}_{(x',q')\in D_a(x,q)} \left\{ V^h(x',q') + C_a(x,q,x',q') \right\}, \tag{34}$$

   and in the time interval $]t_k, t_{k+1}[$ perform an autonomous transition to (x_*^δ, q_*^δ);

2. If $(x,q) \in (\mathbb{R}^d \times \mathbb{I}) \backslash (A \cup C)$, then define

$$u_*^\delta := \operatorname*{argmin}_{u\in U} \left\{ \delta\ell(x,q,u) + e^{-\lambda\delta} V^h(x + \delta f(x,q,u), q) \right\}, \tag{35}$$

   and apply the control $u(t) \equiv u_*^\delta$ for $t \in]t_k, t_{k+1}]$;

3. If $(x,q) \in C$, set:

$$c_1 = \min_{u\in U} \left\{ \delta\ell(x,q,u) + e^{-\lambda\delta} V^h(x + \delta f(x,q,u), q) \right\}$$
$$c_2 = \min_{(x',q')\in D} \left\{ V^h(x',q') + C_c(x,q,x',q') \right\}.$$

   Two subcases are possible:

(a) If $c_1 \geq c_2$, then define

$$(x_*^\delta, q_*^\delta) := \underset{(x',q')\in D}{\operatorname{argmin}} \left\{ V^h(x',q') + C_c(x,q,x',q') \right\}$$

and in the time interval $]t_k, t_{k+1}[$ perform a jump to the state (x_*^δ, q_*^δ);

(b) If $c1 < c_2$, then define a constant control via (35) and apply it for $t \in]t_k, t_{k+1}]$.

Note that in the time interval $]0,\delta]$ this approximate feedback either applies a constant control, or performs a jump, but does not blend the two forms of control. Note also that, since $\delta \to 0$, asymptotically no more than a single jump can occur in the interval for the exact optimal strategy, see Remark 2.1. To avoid that this could occur for the approximate optimal control, we will assume in addition that (A8) holds with $\Delta > 0$.

Here and in what follows, we use the superscript δ to refer to the discretization operated by the controller, while the superscript h refers to the numerical scheme.

Consider a given initial condition (x,q). Let θ_* denote an optimal strategy (continuous control, autonomous transition control, controlled transition times and controlled transition states) for the hybrid system restricted to the time interval $]0,\delta]$, and θ_*^δ denote the strategy obtained by the approximate feedback outlined above on the same time interval. For any $\theta \in \Theta$, we denote by $\Gamma(\delta,\theta)$ the cost accumulated on the interval $]0,\delta]$ starting from the state (x,q) with the strategy θ, that is,

$$\begin{aligned}
\Gamma(\delta,\theta) \quad &= \quad \int_0^\delta \ell(X(t),Q(t),u(t))e^{-\lambda t}dt \\
&\quad + \sum_{\tau_k\in[0,\delta)} C_a(X(\tau_k^-),Q(\tau_k^-),X(\tau_k^+),Q(\tau_k^+))e^{-\lambda \tau_k} \\
&\quad + \sum_{\xi_i\in[0,\delta)} C_c(X(\xi_i^-),Q(\xi_i^-),X(\xi_i^+),Q(\xi_i^+))e^{-\lambda \xi_i}.
\end{aligned}$$

We point out that, in the case of the optimal strategy θ_*, the two summations at the right-hand side contain at most one term if δ is small enough. Moreover, given an approximate control strategy θ^δ in the form above, set

$$\left(X^\delta(\delta,\theta^\delta), Q^\delta(\delta,\theta^\delta)\right) = \begin{cases} \left(x+\delta f(x,u^\delta),q\right) & \text{if } \theta^\delta \text{ applies the control } u^\delta, \\ (x^\delta,q^\delta) & \text{if } \theta^\delta \text{ jumps to the state } (x^\delta,q^\delta). \end{cases}$$

An approximate cost is then defined accordingly, as

$$\Gamma^\delta(\delta,\theta^\delta) = \begin{cases} \delta\ell(x,q,u^\delta) & \text{if } \theta^\delta \text{ applies the control } u^\delta, \\ C_a(x,q,x^\delta,q^\delta) & \text{if } \theta^\delta \text{ jumps to the state } (x^\delta,q^\delta) \text{ and } x\in A, \\ C_c(x,q,x^\delta,q^\delta) & \text{if } \theta^\delta \text{ jumps to the state } (x^\delta,q^\delta) \text{ and } x\in C. \end{cases}$$

Once written on an interval $[0,\delta[$, the Dynamic Programming Principle reads

$$\begin{aligned}
V(x,q) \quad &= \quad \min_{\theta\in\Theta}\left\{\Gamma(\delta,\theta) + e^{-\lambda\delta}V(X(\delta,\theta),Q(\delta,\theta))\right\} \\
&= \quad \Gamma(\delta,\theta_*) + e^{-\lambda\delta}V(X(\delta,\theta_*),Q(\delta,\theta_*)),
\end{aligned}$$

while, on the basis of the approximate feedback defined above, we also define

$$\begin{aligned}
W(x,q) \quad &:= \quad \min_{\theta^\delta}\left\{\Gamma^\delta(\delta,\theta^\delta) + e^{-\lambda\delta}V^h(X^\delta(\delta,\theta^\delta),Q^\delta(\delta,\theta^\delta))\right\} \\
&= \quad \Gamma^\delta(\delta,\theta_*^\delta) + e^{-\lambda\delta}V^h(X^\delta(\delta,\theta_*^\delta),Q^\delta(\delta,\theta_*^\delta)).
\end{aligned}$$

In what follows, we will need a unilateral estimate on $W-V$. Replacing in W the actual minimizer θ_*^δ with a different strategy $\bar\theta_*$, we have:

$$\begin{aligned}
W(x,q)-V(x,q) &\le \Gamma^\delta(\delta,\bar\theta_*)-\Gamma(\delta,\theta_*)\\
&\quad +e^{-\lambda\delta}\left[V^h(X^\delta(\delta,\bar\theta_*),Q^\delta(\delta,\bar\theta_*))-V(X(\delta,\theta_*),Q(\delta,\theta_*))\right]\\
&= \Gamma^\delta(\delta,\bar\theta_*)-\Gamma(\delta,\theta_*)+e^{-\lambda\delta}\left[(V^h-V)(X^\delta(\delta,\bar\theta_*),Q^\delta(\delta,\bar\theta_*))\right.\\
&\quad \left.+V(X^\delta(\delta,\bar\theta_*),Q^\delta(\delta,\bar\theta_*))-V(X(\delta,\theta_*),Q(\delta,\theta_*))\right],
\end{aligned}$$

in which L_V is the Hölder constant in (D4). We obtain therefore:

$$\begin{aligned}
W(x,q)-V(x,q) &\le \Gamma^\delta(\delta,\bar\theta_*)-\Gamma(\delta,\theta_*)\\
&\quad +e^{-\lambda\delta}\left[\left\|V^h-V\right\|_\infty+L_V\left|X^\delta(\delta,\bar\theta_*)-X(\delta,\theta_*)\right|^\gamma\right].
\end{aligned} \tag{36}$$

Note that (36) holds for any strategy $\bar\theta_*$, but in its derivation we have assumed that $\bar\theta_*$ is defined in such a way to have

$$Q^\delta(\delta,\bar\theta_*)=Q(\delta,\theta_*)).$$

We will denote the rightmost term of (36) as $\varepsilon(h,\delta)$, and will give later a more explicit expression of it, depending on the kind of control involved.

In order to evaluate the increase in the cost introduced by the suboptimal strategy θ_*^δ, consider an evolution in which this strategy is only applied in the first time step, while at successive times the exact optimal control θ_* is used. The (non-negative) optimality gap of this control strategy is given by

$$\Delta_{opt}:=\Gamma(\delta,\theta_*^\delta)+e^{-\lambda\delta}V(X(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta))-V(x,q), \tag{37}$$

so that, if we prove that, for $h,\delta\to 0$, it satisfies one of the conditions

$$\Delta_{opt}=\begin{cases} o(\delta) & \\ o(\Gamma(\delta,\theta_*)) & \\ o(1) & \text{if no accumulation occurs}\end{cases} \tag{38}$$

then we obtain that the global cost associated to the control strategy θ_*^δ converges to the optimal cost, i.e., that θ_*^δ is an asymptotically optimal strategy. The third condition in (38) requires some more words – it means that an optimality gap $o(1)$ on a time interval $[t_k,t_{k+1}[$ is still acceptable if intervals in which this situation occurs are uniformly apart one another.

We start by giving a general upper bound on (37). Using the function W defined above, we can write:

$$\begin{aligned}
\Delta_{opt} &= \Gamma(\delta,\theta_*^\delta)+e^{-\lambda\delta}V(X(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta))+(W(x,q)-V(x,q))-W(x,q)\\
&\le \Gamma(\delta,\theta_*^\delta)-\Gamma^\delta(\delta,\theta_*^\delta)+\varepsilon(h,\delta)\\
&\quad +e^{-\lambda\delta}\left[V(X(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta))-V^h(X^\delta(\delta,\theta_*^\delta),Q^\delta(\delta,\theta_*^\delta))\right].
\end{aligned}$$

Then, by further manipulation, we obtain

$$\begin{aligned}
\Delta_{opt} &\le \Gamma(\delta,\theta_*^\delta)-\Gamma^\delta(\delta,\theta_*^\delta)+\varepsilon(h,\delta)+e^{-\lambda\delta}\left[(V-V^h)(X^\delta(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta))\right.\\
&\quad \left.+V(X(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta))-V(X^\delta(\delta,\theta_*^\delta),Q^\delta(\delta,\theta_*^\delta))\right]\\
&\le \Gamma(\delta,\theta_*^\delta)-\Gamma^\delta(\delta,\theta_*^\delta)+\varepsilon(h,\delta)\\
&\quad +e^{-\lambda\delta}\left[\left\|V-V^h\right\|_\infty+L_V\left|X(\delta,\theta_*^\delta)-X^\delta(\delta,\theta_*^\delta)\right|^\gamma\right].
\end{aligned} \tag{39}$$

In order to make this estimate explicit, we must consider the various kinds of control strategy, namely autonomous jumps, controlled jumps and continuous control, and apply (39) (with $\varepsilon(h,\delta)$ given by (36)) with a specific estimation of all terms.

We start by examining the two main situation, namely discontinuous transitions and continuous control.

Jumps in the state (cases 1 and 3a above)

In this case, considering either transition operator, we have that the discrete approximation only performs a jump in the time interval considered, while the exact system is also subject to some control u before and/or after the jump is performed. The control strategy is therefore given by the arrival point in the discrete system, by the arrival point plus the control u before/after the jump in the exact system. We set

$$\min_{x',q'}\left\{V(x',q')+C(x,q,x',q')\right\}=V(x_*,q_*)+C(x,q,x_*,q_*),$$

$$\min_{x',q'}\left\{V^h(x',q')+C(x,q,x',q')\right\}=V(x_*^\delta,q_*^\delta)+C(x,q,x_*^\delta,q_*^\delta),$$

(with $C=C_c$ and $(x',q')\in D$, or $C=C_a$ and $(x',q')\in D_a(x,q)$ depending on the transition operator) and define $\bar{\theta}_*$ to be the jump to (x_*,q_*), whereas the application of the discrete strategy θ_*^δ to the continuous system would consist in a jump to (x_*^δ,q_*^δ) at 0^+, plus a continuous control u on $[0^+,\delta]$. We avoid to specify this control, whose definition results in a $O(\delta)$ term in both the cost and the displacement, and turns out to be irrelevant (for example, this could consist in applying the discrete optimal control computed at the arrival point (x_*^δ,q_*^δ)).

According to this setting, we have for the discrete system

$$(X^\delta(\delta,\theta_*^\delta),Q^\delta(\delta,\theta_*^\delta))=(x_*^\delta,q_*^\delta),\quad \Gamma^\delta(\delta,\theta_*^\delta)=C(x,q,x_*^\delta,q_*^\delta), \tag{40}$$

$$(X^\delta(\delta,\bar{\theta}_*),Q^\delta(\delta,\bar{\theta}_*))=(x_*,q_*),\quad \Gamma^\delta(\delta,\bar{\theta}_*)=C(x,q,x_*,q_*), \tag{41}$$

while, for the continuous system,

$$(X(\delta,\theta_*),Q(\delta,\theta_*))=(x_*+O(\delta),q_*),\quad \Gamma(\delta,\theta_*)=C(x,q,x_*,q_*)+O(\delta), \tag{42}$$

$$(X(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta))=(x_*^\delta+O(\delta),q_*^\delta),\quad \Gamma(\delta,\theta_*^\delta)=C(x,q,x_*^\delta,q_*^\delta)+O(\delta). \tag{43}$$

Continuous control (cases 2 and 3b above)

In this setting, the approximate controller produces a piecewise constant control. Note that, under a constant control on the time interval $[0,\delta[$, we have

$$X(\delta,\theta^\delta)=x+\delta f(x,q,u^\delta)+O(\delta^2), \tag{44}$$

$$\Gamma(\delta,\theta^\delta)=\delta\ell(x,q,u^\delta)+O(\delta^2). \tag{45}$$

We have now, for the discrete system:

$$(X^\delta(\delta,\theta_*^\delta),Q^\delta(\delta,\theta_*^\delta))=(x+\delta f(x,q,u_*^\delta),q),\quad \Gamma^\delta(\delta,\theta_*^\delta)=\delta\ell(x,q,u_*^\delta), \tag{46}$$

and, once defined $\bar{\theta}_*$ as the integral mean value $\bar{u}_*$ of the control $u_*(t)$ over $[0,\delta)$,

$$(X^\delta(\delta,\bar{\theta}_*),Q^\delta(\delta,\bar{\theta}_*))=(x+\delta f(x,q,\bar{u}_*),q),\quad \Gamma^\delta(\delta,\bar{\theta}_*)=\delta\ell(x,q,\bar{u}_*). \tag{47}$$

On the other hand, we have for the continuous system:

$$(X(\delta,\theta_*),Q(\delta,\theta_*)) = (X_*(\delta),q), \quad \Gamma(\delta,\theta_*) = \int_0^\delta \ell(X_*(t),q,u_*(t))dt, \tag{48}$$

and according to (44)–(45),

$$(X(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta)) = (x+\delta f(x,q,u_*^\delta)+O(\delta^2),q), \quad \Gamma(\delta,\theta_*^\delta) = \delta\ell(x,q,u_*^\delta)+O(\delta^2). \tag{49}$$

Depending on the kind of control performed by both θ_* and θ_*^δ, we consider now four possible situations.

Case (i). Both θ_* and θ_*^δ perform a jump.

We use in this case (41) and (42), obtaining:

$$\varepsilon(h,\delta) \le \|V-V^h\|_\infty + O(\delta^\gamma)$$

so that, using also (40) and (43), we obtain from (39)

$$\Delta_{opt} \le 2\|V-V^h\|_\infty + O(\delta^\gamma),$$

and since $\Gamma(\delta,\theta_*)$ is bounded from below, we also obtain that $\Delta_{opt} = o(\Gamma(\delta,\theta_*))$ (which corresponds to the second condition in (38)) since $\|V-V^h\|_\infty = o(1)$.

Case (ii). Both θ_* and θ_*^δ perform a continuous control.

In this case, we recall that the discrete constant control $\bar{u}_*$ is defined as the integral mean of $u_*(t)$. Then, taking into account the Lipschitz continuity of f and ℓ, ε is estimated by means of (47)–(48), which give for the evolution of the state:

$$\begin{aligned}
X(\delta,\theta_*) &= x+\int_0^\delta f_1(X_*(t),q)dt + \int_0^\delta f_2(X_*(t),q)u_*(t)dt \\
&= x+\int_0^\delta f_1(x+O(\delta),q)dt + \int_0^\delta f_2(x+O(\delta),q)u_*(t)dt \\
&= x+\delta f_1(x,q)dt + \delta f_2(x,q)\left(\frac{1}{\delta}\int_0^\delta u_*(t)dt\right) + O(\delta^2) \\
&= X^\delta(\delta,\bar{\theta}_*) + O(\delta^2)
\end{aligned}$$

and, for the cost term,

$$\begin{aligned}
\Gamma(\delta,\theta_*) &= \int_0^\delta \ell(X_*(t),q,u_*(t))dt \\
&= \int_0^\delta \ell(x+O(\delta),q,u_*(t))dt \\
&= \int_0^\delta \ell(x,q,u_*(t))dt + O(\delta^2) \\
&\ge \delta\ell(x,q,\bar{u}_*) + O(\delta^2) \\
&= \Gamma^\delta(\delta,\bar{\theta}_*) + O(\delta^2),
\end{aligned}$$

where we have applied Jensen's inequality by convexity of ℓ. This results in estimating ε as

$$\varepsilon(h,\delta) \leq \|V - V^h\|_\infty + O(\delta^{2\gamma}).$$

Plugging this estimate in (39) and using (46) and (49), we finally obtain

$$\Delta_{opt} \leq 2\|V - V^h\|_\infty + O(\delta^{2\gamma}).$$

Therefore, if $\|V - V^h\|_\infty = o(\delta)$, then by (D4) the first condition in (38) is satisfied.

Case (iii). θ_* performs a jump, θ_*^δ performs a continuous control.

In this situation, ε is estimated via (41)–(42), which give

$$\varepsilon(h,\delta) \leq \|V - V^h\|_\infty + O(\delta^{\gamma}).$$

Using in (39) this bound, along with (46) and (49), we obtain

$$\Delta_{opt} \leq 2\|V - V^h\|_\infty + O(\delta^{\gamma}),$$

and since, as in case (i), $\Gamma(\delta,\theta_*)$ is bounded from below, we have $\Delta_{opt} = o(\Gamma(\delta,\theta_*))$ (i.e., the second condition in (38) is satisfied) since $\|V - V^h\|_\infty = o(1)$.

Case (iv). θ_* performs a continuous control, θ_*^δ performs a jump.

Using (47)–(48) as in case (ii), we obtain

$$\varepsilon(h,\delta) \leq \|V - V^h\|_\infty + O(\delta^{2\gamma}).$$

Plugging this expression, (40) and (43) in (39), we get

$$\Delta_{opt} \leq 2\|V - V^h\|_\infty + O(\delta^{\gamma}).$$

Note that $\Gamma(\delta,\theta_*) = O(\delta)$ so that only the third condition in (38) can be satisfied. It remains then to check that the situation of an unbounded number of transitions in a finite time cannot occur, even if the approximate strategy θ_*^δ is used. This is clearly true in autonomous transitions, for which the destination set is well separated from the jump set A, so we only need to prove this fact for controlled transitions.

First, note first that $W(x,q)$ converges to $V(x,q)$, as it can be seen by using (36) and the opposite inequality,

$$\begin{aligned} V(x,q) - W(x,q) \quad \leq \quad & \Gamma(\delta,\theta_*^\delta) - \Gamma^\delta(\delta,\theta_*^\delta) \\ & + e^{-\lambda\delta}\left[V(X(\delta,\theta_*^\delta),Q(\delta,\theta_*^\delta)) - V^h(X^\delta(\delta,\theta_*^\delta),Q^\delta(\delta,\theta_*^\delta))\right], \end{aligned}$$

which is obtained replacing θ_* with θ_*^δ, and can be tackled by similar ideas. Thus, the three functions $V(x,q)$, $V^h(x,q)$ and $W(x,q)$ converge to the same value.

Now, arguing by contradiction, we prove that two controlled transitions for the approximate strategy θ_*^δ must be uniformly apart one another. Assume therefore that, for h and δ small enough, at some point (x_0,q_0) the strategy θ_*^δ causes first a jump to (x_1,q_1), then a continuous control to (x_2,q_1), then a new jump to (x_3,q_3).

Note that, in order to obtain a contradiction, we can assume that (x_1,q_1) and (x_2,q_1) are arbitrarily close one another. Note also that the cost associated to the discrete transitions are respectively $C_c(x_0,q_0,x_1,q_1)$

and $C_c(x_2,q_1,x_3,q_3)$. Therefore, in view of the Lipschitz continuity of C_c and of the convergence of V^h and W to V, we can set

$$\begin{aligned}V^h(x_1,q_1) &= W(x_2,q_1)+\varepsilon_1,\\ C_c(x_2,q_1,x_3,q_3) &= C_c(x_1,q_1,x_3,q_3)+\varepsilon_2,\\ W(x_0,q_0) &= V(x_0,q_0)-\varepsilon_3,\\ V^h(x_3,q_3) &= V(x_3,q_3)+\varepsilon_4,\end{aligned}$$

for some $\varepsilon_1,\dots,\varepsilon_4$ arbitrarily small. Therefore, we have

$$\begin{aligned}W(x_0,q_0) &= C_c(x_0,q_0,x_1,q_1)+V^h(x_1,q_1)\\ &= C_c(x_0,q_0,x_1,q_1)+W(x_2,q_1)+\varepsilon_1\\ &= C_c(x_0,q_0,x_1,q_1)+C_c(x_2,q_1,x_3,q_3)+V^h(x_3,q_3)+\varepsilon_1\\ &= C_c(x_0,q_0,x_1,q_1)+C_c(x_1,q_1,x_3,q_3)+V^h(x_3,q_3)+\varepsilon_1+\varepsilon_2,\end{aligned}$$

and finally,

$$V(x_0,q_0)=C_c(x_0,q_0,x_1,q_1)+C_c(x_1,q_1,x_3,q_3)+V(x_3,q_3)+\sum_{i=1}^{4}\varepsilon_i.$$

This would indicate that the two jumps (x_0,q_0) to (x_1,q_1) to (x_3,q_3) provide an optimal strategy up to the quantity $\sum_i \varepsilon_i$. Having assumed that $\Delta>0$, this is in contradiction with assumption (A8), which gives a strictly lower cost by jumping directly from (x_0,q_0) to (x_3,q_3).

We can summarize this analysis in the following result. Note that it will be stated *assuming* that V^h converges uniformly to V. This will allow us to avoid any reference to the specific class of schemes used to construct the approximate value function.

Theorem 4.1 *Assume (A1)–(A8) and (D1)–(D4). Assume moreover that*

$$\|V^h-V\|_\infty \to 0 \quad (h\to 0)$$

and that δ is chosen in such a way to have $\|V^h-V\|_\infty=o(\delta)$. Then, the control θ_*^δ defined by the points 1–3 above satisfies

$$J(x,q;\theta_*^\delta)\to V(x,q)$$

as $h,\delta\to 0$.

Remark 4.1 *In numerical practice, it seems that the condition $\|V^h-V\|_\infty=o(\delta)$ is overly restrictive. In fact, this condition is introduced because the above proof only uses L^∞ convergence of solutions, while usual numerical schemes also show some form of convergence for gradients. For example, in Semi-Lagrangian schemes a practical choice (although not justified by the theory) could be to set $\delta=\Delta t$ and approximate the optimal feedback with the argmin used by the scheme.*

5 Numerical Tests

We give in this section some numerical examples, starting with the simpler case of one dimension. The numerical solution has been computed with the monotone Semi-Lagrangian scheme (22) under Courant numbers $\Delta t/\Delta x \in [0.5, 2.5]$ (note that this scheme is known to be unconditionally stable with respect to the choice of Δt and Δx, see [16]). The scheme has been implemented with a piecewise (multi)linear space reconstruction, while the Euler method has been used both in the scheme and in the simulation of optimal trajectories. The plots show the value functions and, in the one-dimensional case, a sample of (approximate) optimal solutions, in which the optimal feedback control is obtained by the argmin used at each node by the scheme, as discussed in Remark 4.1. In the two-dimensional example, the optimal feedback is plotted instead.

The set of assumptions introduced in Section 2.1, and needed for the theoretical characterization of the value function and for the convergence of numerical schemes, are quite restrictive in many examples. These basic assumptions may fail in the examples presented here – in a way, this shows the robustness of the procedure.

Example 5.1 *We start with an example somewhat typical in the framework of hybrid control, which has the following physical setting. A vehicle is driven by a hybrid system, which can switch between an electric and a thermic motor, the latter being used also for charging the battery of the former. Switching is mandatory in two cases: when the battery is fully charged the electric motor is activated, and when the battery is fully discharged the thermic motor is activated. At any intermediate level of charge, the switching is driven by an optimal cost strategy, in which:*

- *The cost of switching from electric to thermic propulsion is constant;*
- *The cost of switching from thermic to electric propulsion is zero;*
- *The cost per time unit of thermic propulsion is proportional to some convex function of the speed $u(t)$, e.g., to the squared speed u^2;*
- *The cost per time unit of electric propulsion is zero.*

Here, the state variable $x \in [0,1]$ represents the level of charge of the battery, and the speed is represented by a control $u(t)$, which is restricted to a bounded interval, e.g., $u(t) \in [0,1]$. When the electric motor is active, the level of charge of the battery decreases at a rate which becomes faster during the discharge. On the other hand, the level of charge increases proportionally to $u(t)$ when the thermic motor is active. This leads to define the dynamics of respectively the electric and the thermic part as

$$f(X(t),1,u(t)) := -(c_1 - c_2 X(t))u(t),$$
$$f(X(t),2,u(t)) := c_3 u(t).$$

In the numerical test, we have chosen $c_1 = 1$, $c_2 = 0.5$ and $c_3 = 0.5$. Moreover, the electric to thermic transition cost is $c_4 = 0.2$, the cost per unit time of the thermic propulsion is $c_5 u^2$, with $c_5 = 1$, and the discount factor has been chosen as $\lambda = 1$. The optimization criterion is a combination of maximum speed and minimum fuel consumption, the running cost ℓ being defined as

$$\ell(X(t),1,u(t)) := -u(t),$$
$$\ell(X(t),2,u(t)) := c_5 u(t)^2 - u(t).$$

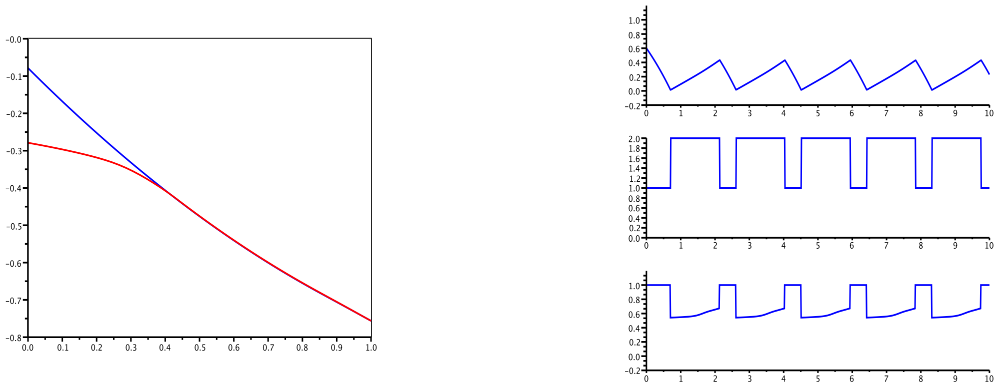


Figure 1: Example 5.1. Left, value functions. Right, an optimal trajectory: $X(t)$ (upper), $Q(t)$ (middle) and $u(t)$ (lower).

In this test, boundary conditions are to be assigned at $(x,q) = (0,1)$ (electric propulsion, discharged battery) and at $(x,q) = (1,2)$ (thermic propulsion, fully charged battery), and are both of autonomous switching. In practice, at each iteration they are implemented as Dirichlet conditions, in which the assigned value depends on the solution at the previous iteration. More precisely, in computing the $(k+1)$th iteration, we set

$$V^h_{k+1}(0,1) = V^h_k(0,2) + c_4, \quad V^h_{k+1}(1,2) = V^h_k(1,1).$$

Figure 1 shows both components of the value function (computed with 50 nodes) in the left plot and an optimal solution in the right plots, which display respectively $X(t)$, $Q(t)$ and $u(t)$. In the optimal trajectory, the switching between thermic and electric motion is apparent. Note that there exists a contact set $[\bar{x}, 1]$ in which the optimal cost of both dynamics coincides, this meaning that the optimal strategy in this interval is to switch to electric propulsion. On the other hand, at the level of fully discharged battery ($x = 0$), the two components of the value function differ for a constant which is precisely c_4, because of the mandatory transition to thermic propulsion.

If the system is under thermic propulsion and the level of charge is below $\bar{x}$, it does not switch until it reaches $\bar{x}$, then continues by evolving from the level $\bar{x}$ to zero (by electric propulsion) and back (by thermic propulsion). If the initial level of charge is above $\bar{x}$ (as in the sample trajectory), then the optimal solution starts by using electric power, then eventually it reaches zero charge level, and starts switching as in the case above.The continuous control takes always the maximum value $u(t) \equiv 1$ under electric power, while it varies according to a nonconstant optimal feedback law under thermic power.

Example 5.2 *In this second example, we try to simulate a more complicate behaviour. We consider a problem of stabilization, posed on the set $\Omega = [-1,1]$, with a control in the set $U = [-1,1]$. Here, an unstable system might be controlled (and brought to the origin) by means of two different dynamics:*

- *The two controllers are described by*

$$f(X(t),1,u(t)) := X(t) + c_1 u(t)$$
$$f(X(t),2,u(t)) := X(t) + c_2 u(t),$$

with $c_2 \gg c_1$, this meaning that the second dynamics is "stronger";

- *Accordingly, the associated running costs are*

$$\ell(X(t),1,u(t)) := X(t)^2 + c_5 u(t)^2,$$
$$\ell(X(t),2,u(t)) := X(t)^2 + c_6 u(t)^2,$$

with $c_6 \gg c_5$, this meaning that the second dynamics is "more expensive";

- *The first dynamics cannot stabilize the system for any $x \in [-1,1]$ if $c_1 < 1$, so it must switch to the second on hitting the boundary of Ω, whereas switching between the two dynamics in internal points is optional. The cost for a switching is c_3 if from the first to the second dynamics, and c_4 if vice versa.*

Here, the boundary at $(x,q) = (\pm 1,2)$ is treated as a state constraint, while a mandatory switching is performed at the boundary $(x,q) = (\pm 1,1)$ (transition from the "weak" to the "strong" controller), which is treated by setting

$$V^h_{k+1}(\pm 1,1) = V^h_k(\pm 1,2) + c_3.$$

The tests have been carried with $\lambda = 1$, $c_1 = 0.5$, $c_2 = 2$, $c_3 = 0.2$, $c_5 = 0.05$ and $c_6 = 1$. Figure 2 shows (again on a 50 nodes grid) two different situations, obtained by modifying the switching cost c_4. In the first row, we have set $c_4 = 0.1$, and the optimal strategy is basically to use the "strong" controller when starting far from the origin, and the "weak" controller otherwise. The second row, obtained with $c_4 = 0$, illustrates a different situation, in which the "far" points activate the "strong" dynamics, whereas as soon as the state returns close to the origin, the optimal strategy requires to switch back to the "weak" controller. The situation is illustrated by the two sample trajectories (right in Fig. 2). In the first one, the trajectory is unstable at the start, then a switch to the "strong" dynamics occurs, whereas the second trajectory switches one more time to the "weak" (and cheaper) dynamics as the state approaches the origin.

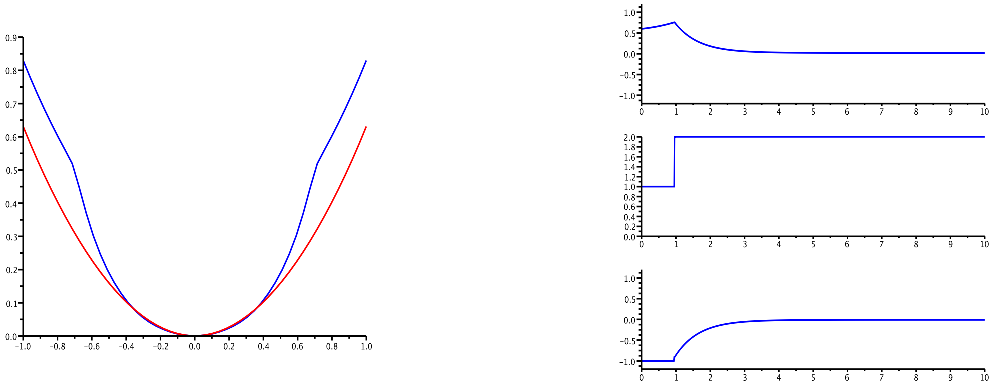
For this test, we also include a convergence study. In lack of an explicit solution, the numerical errors are related to a reference numerical solution computed with 1025 nodes and a time step $\Delta t = 5 \cdot 10^{-3}$. Table 1 shows the L_∞ errors obtained using a linear refinement law, and a linear increase of the number of iterations, which amounts, in a time-marching scheme, to computing the solutions at the same final time (in our case, at the final time $T = 10$ instead of $T = +\infty$). In order to remove the effect of the iterative solver, the same time horizon has been used to compute the reference solution.

The experimental convergence rate is slightly above the unity, which makes it reasonable to apply the construction of feedback discussed in Remark 4.1. Note that, in principle, a first-order convergence is expected, since the consistency error of each building block of the scheme is of order $O(\Delta t + \Delta x^2/\Delta t)$ and the refinement is performed at constant Courant number (see [16]).

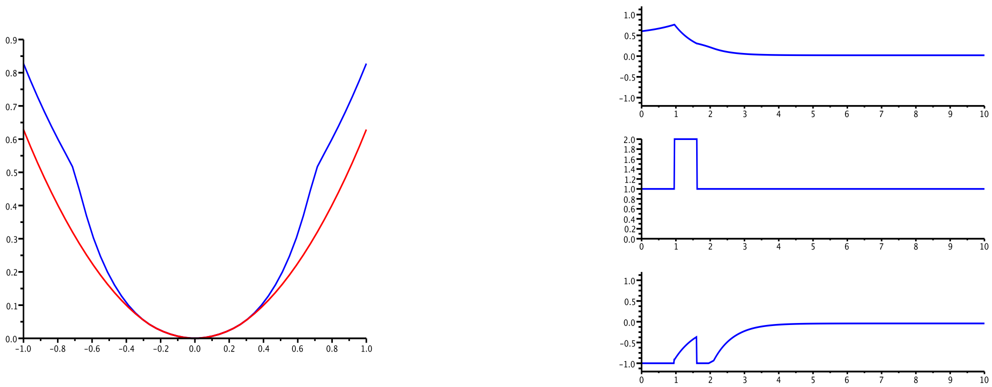
Table 1: Errors for Example 5.2b, SL method with linear interpolation.

N	Δt	Iterations	L_∞ error
33	0.15	67	$1.42 \cdot 10^{-1}$
65	0.075	134	$6.76 \cdot 10^{-2}$
129	0.0375	268	$3.11 \cdot 10^{-2}$

Example 5.3 *We come back to the first example, but this time we consider a 2-dimensional problem where the state $x = (x_1, x_2)$ represents the state of charge of the battery and the quantity of fuel available in the vehicle.*



(a)



(b)

Figure 2: Examples 5.2*a* (a) and 5.2*b* (b). Left, value functions. Right, an optimal trajectory: $X(t)$ (upper), $Q(t)$ (middle) and $u(t)$ (lower).

Here, the state variable $x \in [0,1]^2$ has two components, which represent respectively the state of charge of the battery, and the level of the fuel reservoir. As in Example 5.1, the control $u(t)$ represents the speed and is assumed to take its values in $[0,1]$. We assume here that when the electric motor is active, no fuel is consumed, and the level of charge of the battery decreases at a rate which becomes faster during the discharge. On the other hand, when the thermic engine is on, the level of charge in the battery increases while the level of the fuel decreases proportionally to $u(t)$. This leads to define the dynamics of respectively the electric and the thermic part as

$$f(X,1,u) := \begin{pmatrix} -(c_1 - c_2 X_1)u \\ 0 \end{pmatrix} \qquad\qquad f(X,2,u) := \begin{pmatrix} c_3 u \\ -c_4 u \end{pmatrix}.$$

In the numerical test, we have chosen $c_1 = 1$, $c_2 = c_3 = c_4 = 0.5$. We consider that the switches are mandatory in three cases: when the battery is fully charged the electric motor is activated, when the battery is fully discharged the thermic motor is activated, when the reservoir of fuel is empty it should be refilled. At any intermediate level of charge, the switching is driven by an optimal cost strategy, in which:

- *The cost of switching from electric to thermic propulsion is constant: $c_5 = 0.2$;*

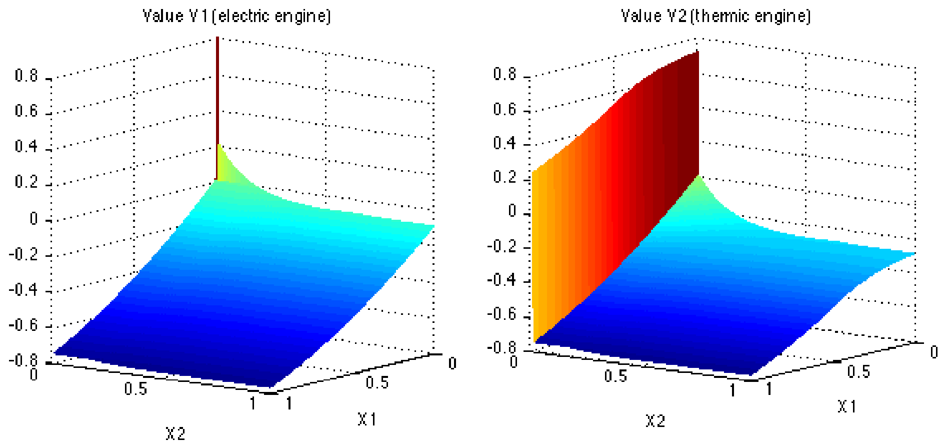


Figure 3: Example 5.3: value functions for electric engine ($q = 1$, left) and thermic engine ($q = 2$, right).

- *The cost of switching from thermic to electric propulsion is zero;*
- *The cost to fill the fuel reservoir is constant and takes the value $c_6 = 1$;*
- *The cost per time unit of thermic propulsion depends on the speed $u(t)$ and is given by the following form: $c_7 u^2 - u$, with $c_7 = 1$ when the thermic engine is on while $c_7 = 0$ if the thermic engine is off.*

The discount factor has been chosen as $\lambda = 1$. The approximation of this example is performed with a discretization using 250×250 space nodes in $[0,1] \times [0,1]$, and about 1500 time steps with $\Delta t = 2.3 \cdot 10^{-3}$.

Figure 3 shows both components of the value functions, while the corresponding optimal feedback laws and optimal switching are given in Figure 4. In this example, the autonomous transition for filling the fuel reservoir creates an apparent discontinuity of the value functions (in the axis $x_2 = 0$ for the value function related to the thermic engine and on the nodes $x_1 = x_2 = 0$ for the value function related to the electric engine). In fact, if the reservoir is empty ($x_2 = 0$), it is mandatory to refill it, whereas, at small but positive reservoir levels, the vehicle may still use electric power. Note that, similar to the one-dimensional case, the optimal feedback gives $u \equiv 1$ under electric motion, while it provides some slight modulation of the control under thermic motion. Moreover, the thermic to electric switch (i.e., the discontinuity in the second component of the optimal switch) occurs at a roughly constant state of charge of! the battery, whereas the reverse switch (occurring in the first component of the optimal switch) takes place only when the battery is fully discharged.

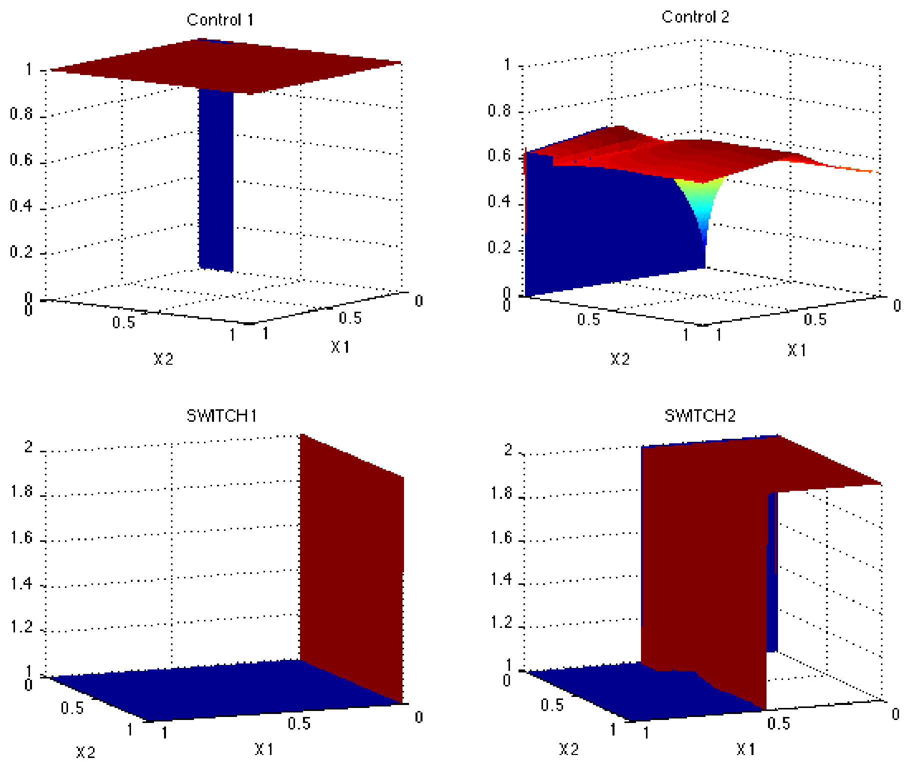


Figure 4: Example 5.3: control variables for electric engine ($q = 1$, left) and thermic engine ($q = 2$, right). Optimal feedback (upper) and optimal switching (lower).

6 Conclusions

Hybrid systems provide a general framework for complex problems where the dynamics involve continuous and discrete inputs. We prove in this paper that, under suitable assumptions on the problem, the Hamilton–Jacobi approach allows for an effective way (at least for a low dimension of the state space) to compute the value function and to approximate the optimal control law along with the corresponding optimal trajectories. The robustness of the method is shown on simple numerical examples.

It would be interesting now to further the analysis of the numerical method and provide an error estimate for the approximation of the value function. Also, forthcoming work will include the implementation of a policy iteration method in place of the slower value iteration approach used in this paper.

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