A short note about case distinctions in Tarski's geometry

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Abstract. In this paper we study some decidability properties in the context of Tarski's axiom system for geometry. We removed excluded middle from our assumptions and studied how our formal proof of the first thirteen chapters of [SST83] are impacted. We show that decidability of equality is equivalent to the decidability of congruence and betweenness. We prove that the decidability of the other predicates used in [SST83] can be derived from decidability of equality except for the predicate stating the existence of the intersection of two lines. All results have been proved formally using the Coq proof assistant.

1 Introduction

In this paper we study some decidability properties in the context of Tarski's axiom system for geometry. In previous work [Nar07,BN12,NBB14] we formalized, using the Coq proof assistant, the results about 2D geometry of the first thirteen chapters of [SST83] within classical logic. In this study, we take advantage of our formal proofs to study how classical logic is used in the proofs of Schwabhaüser, Szmielew and Tarski. We removed the excluded middle axiom ($\forall A.\ A \vee \neg A$) from our formal development and based on our formal proofs we studied which instances of the excluded middle axiom are used.

Studying these case distinctions has both a theoretical interest *per se* and also a practical interest in the context of automated deduction. Indeed, as noted[1] by Michael Beeson while reproducing proof of [SST83] using Otter:

> "These arguments by cases caused us a lot of trouble in finding Otter proofs."

The excluded middle axiom can be used at every step of the proof search process. This can generate a blow-up of the proof tree. Managing and guiding the automatic theorem prover for using the right case distinctions is essential.

[1] http://www.michaelbeeson.com/research/FormalTarski/index.php?include=archive11

1.1 Related work

In [DDS00], Christophe Dehlinger, Jean-François Dufourd and Pascal Schreck have carried a study similar to ours in the framework of Hilbert's axiom system [Hil71]. This first approach was realized in an intuitionist setting, and concluded that the decidability of point equality and collinearity is necessary to check Hilbert's proofs. In [MNS11], we proved that in the context of projective geometry decidability of incidence implies decidability of line and point equalities. Von Plato has proposed an axiom system for projective geometry based on the apartness predicate[vP95] which has been formalized in Coq by Gilles Khan [Kah95]. We have formalized the first thirteen chapters of [SST83] in Coq and shown that these results implies the axioms of Hilbert (except continuity) [Nar07,BN12]. Recently, Michael Beeson has written a note [Bee14c] which provides the connection between the proofs in [SST83] and Hilbert's axioms. This note is more readable than our Coq's proof. Michael Beeson and Larry Wos have reproduced the proof that Hilbert's axioms follows from Tarski's using Otter [BW14]. The work closest to ours is the one by Michael Beeson. He recently proposed a fully constructive variant of Tarski's axiom system [Bee14a,Bee14b] and study its properties.

Section 2 describes the Tarski's axiom system we are working with. Section 3 provides an overview of the occurrences of case distinction in our formal proofs. In Section 4 we show that all case distinctions can be reduced only to two axioms: decidability of equality and decidability of intersection. In Section 5, we study the role of decidability of intersection.

2 Tarski's axiom system

The axioms can be expressed using first order logic and two predicates.

betweenness The ternary *betweenness* predicate $\beta\ A\,B\,C$ informally states that B lies on the line AC between A and C. The relation holds also if $A = B$ or $B = C$.

congruence The quaternary *congruence* predicate $AB \equiv CD$ informally means that the distance from A to B is equal to the distance from C to D.

Note that in Tarski's geometry, only points are primitive objects. In particular, lines are *defined* by two distinct points whereas in Hilbert's axiom system lines and planes are *primitive objects*. Figure 1 provides the list of axioms that we used in our formalization.

2.1 Formalization in Coq

The formalization of this axiom system in Coq is straight-forward. We group our axioms in a structure. We use the type class mechanism developed by Matthieu Sozeau[SO08] to save us from the burden of providing the proper structure each

$$
\begin{array}{rl}
\text{Between Identity} & \beta\ A\,B\,A \Rightarrow A = B \\
\text{Pseudo-Transitivity} & AB \equiv CD \wedge AB \equiv EF \Rightarrow CD \equiv EF \\
\text{Symmetry} & AB \equiv BA \\
\text{Cong Identity} & AB \equiv CC \Rightarrow A = B \\
\text{Pasch} & \beta\ A\,P\,C \wedge \beta\ B\,Q\,C \Rightarrow \exists X, \beta\ P\,X\,B \wedge \beta\ Q\,X\,A \\
\text{Euclid} & \exists XY, \beta\ A\,D\,T \wedge \beta\ B\,D\,C \wedge A \neq D \Rightarrow \\
& \beta\ A\,B\,X \wedge \beta\ A\,C\,Y \wedge \beta\ X\,T\,Y \\
& AB \equiv A'B' \wedge BC \equiv B'C' \wedge \\
\text{5 segments} & AD \equiv A'D' \wedge BD \equiv B'D' \wedge \\
& \beta\ A\,B\,C \wedge \beta\ A'\,B'\,C' \wedge A \neq B \Rightarrow CD \equiv C'D' \\
\text{Construction} & \exists E, \beta\ A\,B\,E \wedge BE \equiv CD \\
\text{Lower Dimension} & \exists ABC, \neg\beta\ A\,B\,C \wedge \neg\beta\ B\,C\,A \wedge \neg\beta\ C\,A\,B \\
\text{Upper Dimension} & AP \equiv AQ \wedge BP \equiv BQ \wedge CP \equiv CQ \wedge P \neq Q \\
& \Rightarrow \beta\ A\,B\,C \vee \beta\ B\,C\,A \vee \beta\ C\,A\,B \\
\text{Continuity} & \forall XY, (\exists A, (\forall xy, x \in X \wedge y \in Y \Rightarrow \beta\ A\,x\,y)) \Rightarrow \\
& \exists B, (\forall xy, x \in X \Rightarrow y \in Y \Rightarrow \beta\ x\,B\,y).
\end{array}
$$

Fig. 1. Tarski's axiom system.

time we use a theorem (see Fig. 2). We slightly changed our formalization of the axiom system we described in [BN12] to separate the results which are valid in any dimension and in neutral geometry from the results which are valid only in 2D and also only in Euclidean geometry. Moreover, we define separate classes for decidability of equality (`EqDecidability`) and decidability of intersection (`InterDecidability`).

3 Case distinctions in Tarski's proofs

In our formalization of the first thirteen chapters of [SST83] there are more than 600 case distinctions. Note that our proof may perform more case distinctions than necessary. Case distinction is used only on atomic formulae and defined predicates. Table 1 lists the predicate with the number of occurrences of case distinction in our development. By far the decidability property which is used most often is decidability of equality of points. To obtain a self-contained paper, we recall here the definitions of the predicates which will be used in Sec. 4.

Definition 1 (Col). *To assert that three points A, B and C are collinear we note: $Col\ A\,B\,C$*

$$Col\ A\,B\,C := \beta\ A\,B\,C \vee \beta\ A\,C\,B \vee \beta\ B\,A\,C$$

Definition 2 (out).

$$Out\ P\,A\,B := A \neq P \wedge B \neq P \wedge (\beta\ P\,A\,B \vee \beta\ P\,B\,A)$$

Definition 3 (is_midpoint).

$$is\_midpoint\ M\,A\,B := \beta\ A\,M\,B \wedge AM \equiv BM$$

```
Class Tarski_neutral_dimensionless := {
 Tpoint : Type;
 Bet : Tpoint -> Tpoint -> Tpoint -> Prop;
 Cong : Tpoint -> Tpoint -> Tpoint -> Tpoint -> Prop;
 between_identity : forall A B, Bet A B A -> A=B;
 cong_pseudo_reflexivity : forall A B : Tpoint, Cong A B B A;
 cong_identity : forall A B C : Tpoint, Cong A B C C -> A = B;
 cong_inner_transitivity : forall A B C D E F : Tpoint,
   Cong A B C D -> Cong A B E F -> Cong C D E F;
 inner_pasch : forall A B C P Q : Tpoint,
      Bet A P C -> Bet B Q C ->
      exists x, Bet P x B /\ Bet Q x A;
 five_segments : forall A A' B B' C C' D D' : Tpoint,
    Cong A B A' B' ->
    Cong B C B' C' ->
    Cong A D A' D' ->
    Cong B D B' D' ->
    Bet A B C -> Bet A' B' C' -> A <> B -> Cong C D C' D';
 segment_construction : forall A B C D : Tpoint,
    exists E : Tpoint, Bet A B E /\ Cong B E C D;
 lower_dim : exists A, exists B, exists C,
    ~ (Bet A B C \/ Bet B C A \/ Bet C A B)
}

Class Tarski_2D `(Tn: Tarski_neutral_dimensionless) := {
  upper_dim : forall A B C P Q : Tpoint,
    P <> Q -> Cong A P A Q -> Cong B P B Q -> Cong C P C Q ->
    (Bet A B C \/ Bet B C A \/ Bet C A B)
}.

Class Tarski_2D_euclidean `(T2D:Tarski_2D) := {
  euclid : forall A B C D T : Tpoint,
    Bet A D T -> Bet B D C -> A<>D ->
    exists x, exists y,
    Bet A B x /\ Bet A C y /\ Bet x T y
}.

Class EqDecidability U := {
 eq_dec_points : forall A B : U, A=B \/ ~ A=B
}.

Class InterDecidability  U (Col : U -> U -> U -> Prop) := {
 inter_dec : forall A B C D,
   (exists I, Col I A B /\ Col I C D) \/
 ~ (exists I, Col I A B /\ Col I C D)
}.
```

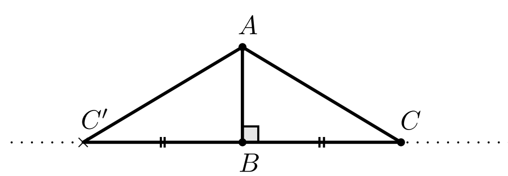
Fig. 2. Formalization of Tarski's axioms as a type class in Coq

Predicate	Meaning	Number of occ
`A = B`	points A and B are equal.	524
`Col A B C`	points A, B and C are collinear	84
`Par A B C D`	$AB \parallel CD$ (line AB is parallel to line CD)	16
`Bet A B C`	points A, B and C are collinear and B is between A and C, it can be the case the $A = B$ or $B = C$.	15
`Cong A B C D`	the segments AB and CD are congruent	13
`two_sides A B P Q`	P and Q are on opposite sides of the line AB	8
`out A B C`	C belongs to the half line $[AB($	7
`Per A B C`	the angle ABC is right angle	6
`is_null_anga A B C`	the angle A B C is null	5
`is_midpoint I A B`	I is the midpoint of segment AB	3
`inter A B C D`	exists I the intersection of line AB and CD	3
`Conga A B C D E F`	the angles $\angle ABC$ and $\angle DEF$ are congruent	2
`Perp_in X A B C D`	$AB \perp CD$ and X is the intersection of AB and CD	1
`is_image_spec P' P A B`	P' is the symmetric of P wrt line AB	1
`lg_null A B`	the length AB is null	1
`same_dir A B C D`	AB and CD are in the same direction	1

Table 1. Statistics about number of case distinctions.

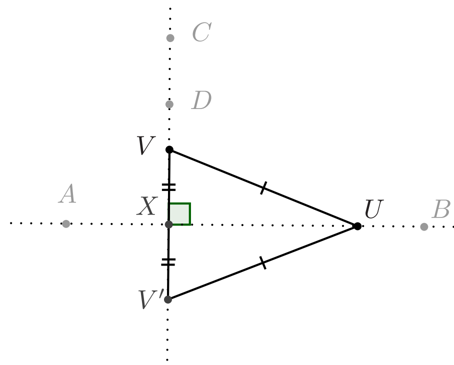
Definition 4 (Per).

$$Per\, A\, B\, C := \exists C', is\_midpoint\, B\, C\, C' \wedge AC \equiv AC'$$



Definition 5 (Perp_in).

$$\begin{aligned} Perp\_in\, X\, A\, B\, C\, D := A \neq B \;\wedge\; C \neq D \;\wedge\; Col\; X\, A\, B \;\wedge\; Col\; X\, C\, D \;\wedge \\ (\forall U\, V, Col\; U\, A\, B \Rightarrow Col\; V\, C\, D \Rightarrow Per\, U\, X\, V) \end{aligned}$$

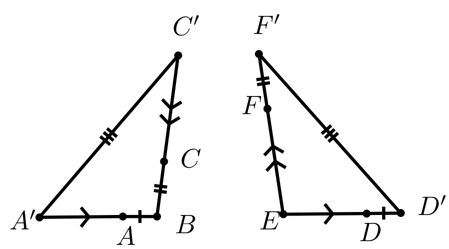


Definition 6 (is_image_spec).

$$is\_image\_spec\, P'\, P\, A\, B := (\exists X, is\_midpoint\, X\, P\, P' \;\wedge\; Col\, A\, B\, X) \wedge$$
$$(Perp\, A\, B\, P\, P' \;\vee\; P = P')$$

Definition 7 (conga).

$$\measuredangle ABC \cong \measuredangle DEF := A \neq B \;\Rightarrow B \neq C \;\Rightarrow D \neq E \;\Rightarrow F \neq F \;\Rightarrow$$
$$\exists A', \exists C', \exists D', \exists F' \quad \begin{cases} \beta\ B\, A\, A' \;\wedge AA' \equiv ED \;\wedge \\ \beta\ B\, C\, C' \;\wedge CC' \equiv EF \;\wedge \\ \beta\ E\, D\, D' \;\wedge DD' \equiv BA \;\wedge \\ \beta\ E\, F\, F' \;\wedge FF' \equiv BC \;\wedge \\ \qquad A'C' \equiv D'F' \end{cases}$$

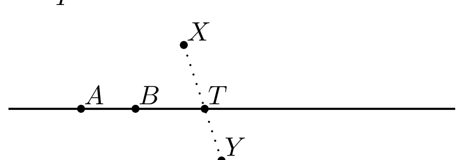


Definition 8 (Perp).

$$AB \perp CD := \exists X, Perp\_in\, X\, A\, B\, C\, D$$

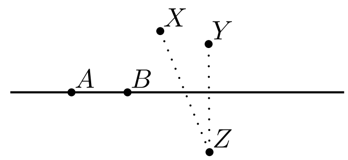
Definition 9 (opposite sides). *Given a line l defined by two distinct points A and B, two points X and Y not on l, are on opposite sides of l is written:* $A \frac{X}{Y} B$

$$A \frac{X}{Y} B := \exists T, Col\ A\, B\, T \wedge \beta\ X\, T\, Y$$



Definition 10 (same side). *Let l be a line defined by two distinct points A and B. Two points X and Y not on l, are on the same side of l is written:* $A \frac{}{X\,Y} B$

$$A \frac{}{X\,Y} B := \exists Z, A \frac{X}{Z} B \wedge A \frac{Y}{Z} B$$



Definition 11 (intersection).

$$inter\, A\, B\, C\, D := \exists I, Col\ I\, A\, B \wedge Col\ I\, C\, D$$

Definition 12 (parallelism).

$$AB \parallel CD := A \neq B\ \wedge C \neq D\ \wedge \neg\exists X, Col\ X\, A\, B \wedge Col\ X\, C\, D$$

Definition 13 (Parallelogram_strict).

$$Parallelogram\_strict\, A\, B\, A'\, B' := A \frac{}{B\, B'} A'\ \wedge\ AB \parallel A'B'\ \wedge\ AB \equiv A'B'$$

Definition 14 (Parallelogram_flat).

$$\begin{aligned} Parallelogram\_flat\, A\, B\, A'\, B' := {} & Col\ A\, B\, A'\ \wedge\ Col\ A\, B\, B'\ \wedge \\ & AB \equiv A'B'\ \wedge\ AB' \equiv A'B\ \wedge\ (A \neq A'\ \vee\ B \neq B') \end{aligned}$$

Definition 15 (Parallelogram).

$$\begin{aligned} Parallelogram\, A\, B\, A'\, B' :=\ & Parallelogram\_strict\, A\, B\, A'\, B' \vee \\ & Parallelogram\_flat\, A\, B\, A'\, B' \end{aligned}$$

Definition 16 (eqV).

$$eqV\ A\, B\, C\, D := Parallelogram\, A\, B\, D\, C\ \vee\ A = B\ \wedge\ C = D$$

Definition 17 (same_dir).

$$same\_dir\, A\, B\, C\, D := A = B\ \wedge\ C = D\ \vee\ \exists D', out\, C\, D\, D'\ \wedge\ eqV\ A\, B\, C\, D$$

Definition 18 (lg).

$$lg\, l := \exists A, \exists B, \forall XY, AB \equiv XY \Leftrightarrow l\, X\, Y$$

Definition 19 (lg_null).

$$lg\_null\, l := lg\, l\ \wedge\ \exists A, l\, A\, A$$

Definition 20 (in angle).

$$P \,\mathrm{in}\, \measuredangle ABC := A \neq B \,\wedge\, C \neq B \,\wedge\, P \neq B \,\wedge \\ \exists X, \beta\ A\,X\,C \,\wedge\, (X = B \,\vee\, out\, B\,X\,P)$$

Definition 21 (lea).

$$\measuredangle ABC \leq \measuredangle DEF := \exists P, P \,\mathrm{in}\, \measuredangle DEF \wedge \measuredangle ABC \cong \measuredangle DEP$$

Definition 22 (lta).

$$\measuredangle ABC < \measuredangle DEF := \measuredangle ABC \leq \measuredangle DEF \wedge \neg \measuredangle ABC \cong \measuredangle DEF$$

Definition 23 (acute).

$$acute\, A\,B\,C := \exists A', \exists B', \exists C', Per\, A'\,B'\,C' \,\wedge\, \measuredangle ABC < \measuredangle A'B'C'$$

Definition 24 (anga).

$$anga\, a := \exists A\,B\,C, acute\, A\,B\,C \,\wedge\, \forall X\,Y\,Z, \measuredangle ABC \cong \measuredangle XYZ \Leftrightarrow a\,X\,Y\,Z$$

Definition 25 (is_anga_null).

$$is\_null\_anga\, a := anga\, a \,\wedge\, \forall ABC, a\,A\,B\,C \Rightarrow out\,B\,A\,C$$

4 Decidability of equality and intersection are sufficient

In this section, we prove that decidability of equality implies decidability of all other predicates except the intersection predicate. Moreover, we prove that we could equivalently assume decidability of any of these three predicates (betweenness, congruence, equality).

First, we give a formal definition of the decidability property and a few lemmas necessary for the proofs.

Definition 26. *We say that a predicate P of arity n is decidable iff:*

$$\forall A_1, \ldots, A_n, P(A_1, \ldots, A_n) \vee \neg P(A_1, \ldots, A_n)$$

Lemma 3.1 of [SST83] states that:

Lemma 1 (between trivial). $\forall AB, \beta\ A\,B\,B$

Using lemma 4.6 of [SST83] we have:

Lemma 2. $\forall ABC,\ \beta\ A\,C\,B \wedge AC \equiv AB \Rightarrow C = B$

Using Lemma 2 and Lemma 5.2 of [SST83] we have:

Lemma 3. $\forall ABDE,\ A \neq B \wedge \beta\ A\,B\,D \wedge \beta\ A\,B\,E \wedge BD \equiv BE \Rightarrow D = E$

Using the segment construction axiom and Lemma 5.2 of [SST83], we can prove that we can construct a point on an half-line at a given distance:

Lemma 4. $\forall AQBC, A \neq Q \Rightarrow \exists X, (\beta\ Q\ A\ X \vee \beta\ Q\ X\ A) \wedge QX \equiv BC$

Theorem 1 (Decidability of basic relations). *In Tarski's geometry, the following properties are equivalent:*

1. *decidability of equality*
2. *decidability of congruence*
3. *decidability of betweenness*

Proof.

$1 \Rightarrow 2$ Assume decidability of equality, we prove decidability of congruence:

$$\forall AB, A = B \vee A \neq B \Rightarrow \forall ABCD, AB \equiv CD \vee \neg AB \equiv CD.$$

Let A, B, C and D be four points.

1. Case $A = B$.
   (a) Case $C = D$. We have $AB \equiv CD$.
   (b) Case $C \neq D$.
   Using axiom cong_identity we can conclude that $\neg AB \equiv CD$.
2. Case $A \neq B$.
   (a) Case $C = D$.
   Using axiom cong_identity we can conclude that $\neg AB \equiv CD$.
   (b) Case $C \neq D$. Using Lemma 4 we construct D' such that $\beta\ A\ B\ D' \vee bTAD'B$ and $AD' \equiv CD$. If $B = D'$ we have that $AB \equiv CD$. Otherwise $B \neq D'$. Assume that $AB \equiv CD$, then by transitivity $AB \equiv AD'$. By case distinction on $\beta\ A\ B\ D' \vee \beta\ A\ D'\ B$ we can show in both cases that $B = D'$ using Lemma 2, hence $\neg AB \equiv CD$.

$2 \Rightarrow 1$ Let us assume decidability of congruence, we prove decidability of equality. Let A and B be two points. By decidability of congruence we have that $AB \equiv AA \vee \neg AB \equiv AA$. If $AB \equiv AA$, by axiom cong_identity we have $A = B$. Otherwise $\neg AB \equiv AA$. Assuming $A = B$ we have $\neg AA \equiv AA$ this contradicts axiom cong symmetry hence $A \neq B$.

$1 \Rightarrow 3$ Assume decidability of equality, we prove decidability of betweenness. Construct C' a point such that $\beta\ A\ B\ C'$ and $BC \equiv BC'$. If $C = C'$ then $\beta\ A\ B\ C$. Otherwise $C \neq C'$. If $A = B$ then $\beta\ A\ B\ C$ by Lemma 1. Otherwise $A \neq B$. Assume $\beta\ A\ B\ C$ using Lemma 3 we obtain that $C = C'$, hence $\neg\beta\ A\ B\ C$.

$3 \Rightarrow 1$ Let us assume decidability of betweenness, we prove decidability of equality. Let A, B be two points. By decidability of betweenness we have that $\beta\ A\ B\ A \vee \neg\beta\ A\ B\ A$. If $\beta\ A\ B\ A$ then by between identity axiom we have $A = B$. If $\neg\beta\ A\ B\ A$, assume $A = B$ then by Lemma 1 we have $\beta\ A\ A\ A$, hence $A \neq B$.

□

For the predicates whose definition does not contain quantifiers and involves only predicates which have already been shown to be decidable, the decidability is trivial. This is the case for the predicates: Col, out, is_midpoint.

Lemma 5. *Col, out, is_midpoint are decidable.*

Proof. These predicates are conjunctions or disjunctions of decidable predicates. They are therefore decidable too. □

Lemma 6. *Per is decidable.*

Proof. Recall that by definition, $Per A B C \equiv \exists C', is\_midpoint B C C' \wedge AC \equiv AC'$. We construct C' the symmetric of C wrt B. If $AC \equiv AC'$ then $Per A B C$. Otherwise we can show that $\neg Per A B C$ using uniqueness of the symmetric point. □

Lemma 7. *Perp_in is decidable.*

Proof. To prove that *Perp_in* is decidable (let us name the points X, A, B, C and D) we first eliminate the trivial cases where A is equal to B, C is equal to D, X, A and B are not collinear and X, C and D are not collinear (in all cases *Perp_in* is trivially false). We then need to know if X is equal to B and/or D to be able to use decidability of *Per* since when the first two or the last two points of this predicate are equal we do not get any information about the angle formed by the lines AB and CD. So we are left with the 4 cases to handle (since X cannot be equal to both A and B (or C and D) as this would contradict the fact that A and B are different). Finally for each case we use decidability of *Per* with the correct points (we choose 3 different points : X, one on AB and one on CD) to complete the proof. □

Lemma 8. *is_image_spec is decidable.*

Proof. To obtain a proof that *is_image_spec* is decidable (let us name the points A, B, C and D) we first eliminate the trivial case where A is equal to B and C equal to D (*is_image_spec* is trivially true) and the one where A is not equal to B and C equal to D (*is_image_spec* is trivially false). Then we use existence of the symmetric of a point wrt a line to construct the point B' such that $is\_image\_spec\, A\, B'\, C\, D$. Finally we use the uniqueness of the symmetric of a point wrt a line and the decidability of equality of the points B and B' to complete this proof. □

Lemma 9. *Conga is decidable.*

Proof. To prove the decidability of the predicate *Conga* (let us name the points A, B, C, D, E and F) we first eliminate the trivial cases where A is equal to B, where B is equal to C, where D is equal to E and where E is equal to F (in all cases *Conga* is trivially false). We then construct the points A', C', D' and F' of the definition using the axiom of construction. Finally we use decidability of congruence with $A'C'$ and $D'F'$. If the congruence is true then $Conga\, A\, B\, C\, D\, E\, F$

is true by definition. If it is false we prove that $Conga\,A\,B\,C\,D\,E\,F$ is false by contradiction by proving that the congruence is as well true which is proved using the construction uniqueness. □

Lemma 10. *Perp is decidable.*

Proof. In order to prove decidability of the predicate *Perp* (let us name the points A, B, C and D) we first use decidability of collinearity with the points A, B and C. If A, B and C are collinear then $Perp\,A\,B\,C\,D$ is equivalent to $Perp\_in\,C\,A\,B\,C\,D$ and we are done as we already proved that *Perp_in* is decidable. If they are not collinear then we construct P the orthogonal projection of C on the line AB. Now if C is equal to D we can derive a contradiction from $\neg\ Perp\,A\,B\,D\,D$ since $Perp\,A\,B\,D\,D$ implies that D is different from D which is false. Finally when C and D are different we see that $Perp\,A\,B\,C\,D$ is equivalent to $Col\ P\,C\,D$ which is decidable. □

Lemma 11. *two_sides is decidable.*

Proof. Let us name the points A, B, C and D. We start by handling the trivial cases of $Col\ C\,A\,B$, $Col\ D\,A\,B$ and $A = B$ for which *two_sides* is trivial false. Now if C is equal to D *two_sides* is false as the only possible intersection point is D and we know that D, A and B are not collinear. From now on C and D are then different. Using the decidability of the intersection of two lines we see that in the case of no intersection *two_sides* (as there is no intersection between AB and CD it is obviously not between C and D) and in the case of an intersection we use decidability of betweenness to complete the proof. When the intersection is between C and D *two_sides* is true by definition and when it isn't we use uniqueness of intersection to derive a contradiction and prove that *two_sides* is false. □

Lemma 12. *one_side is decidable.*

Proof. Let us name the points A, B, C and D. We start by handling the trivial cases of $Col\ C\,A\,B$, and $Col\ D\,A\,B$ for which *two_sides* is trivial false. Then as neither C nor D is collinear to A and B we can prove that we have either $one\_side\,A\,B\,C\,D$ or $two\_side\,A\,B\,C\,D$. In the first case it is trivial. In the second we use the fact $two\_side\,A\,B\,C\,D$ implies $\neg one\_side\,A\,B\,C\,D$. □

Lemma 13. *Par is decidable.*

Proof. To prove that *Par* is decidable (let us name the points A, B, C and D) we first eliminate the trivial cases where A is equal to B, C is equal to D (in all cases *Par* is trivially false). We can then construct the parallel CD' to AB passing through C. Finally decidability of *Par* is equivalent in this context to decidability of *Col* which we already proved. □

Lemma 14. *same_dir is decidable.*

Proof. Let us name the points A, B, C and D. We start by handling the trivial cases of $A \neq B$ and $C \neq D$, $A \neq B$ and $C = D$ and finally $A = B$ and $C \neq D$ for which *same_dir* is trivial false. Now when $A \neq B$ and $C \neq D$ we construct the point E such that $ABEC$ is a parallelogram. Finally using the uniqueness of the parallelogram construction decidability of *same_dir* is equivalent to decidability of *out* applied to C, D and E which is already proven. $\square$

Lemma 15. *lg_null is decidable.*

Proof. In order to prove decidability of the predicate *lg_null* (let us name the length l) we first name the points representing the length l : A and B. Decidability of *lg_null* is then equivalent to decidability of equality of points applied with A and B which we already proved so the proof is complete. $\square$

Lemma 16. *is_null_anga is decidable.*

Proof. To obtain a proof that *is_null_anga* is decidable (let us name the angle a) we first name the points representing the angle a : A, B and C. Decidability of *is_null_anga* is then equivalent to decidability of *out* applied with B, A and C which we already proved so the proof is complete. $\square$

5 About the decidability of intersection

In this section we study the decidability of intersection. We could not manage to derive the decidability of intersection from decidability of equality. We can remark that in Tarski's axioms system minus Euclid axiom and the continuity axiom, the axioms containing an existential quantifier (Pasch, Lower dimension and Construction) allow to construct points which are not arbitrary far from the given points. Each application of an existential axiom can at most double the maximum distance between already constructed points. Using decidability of intersection we can construct points which are arbitrary far. Hence, we can not prove decidability of intersection from these other axioms. Michael Beeson has recently turned this proof sketch into a formal argument using Herbrand's theorem to obtain a new syntactic proof that Euclid's axiom is independent of the other axioms [BBN14].

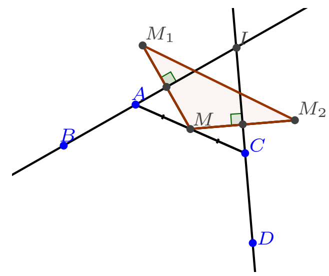
We can show also that if instead of the Euclid axiom given in [SST83] we assume the existence of the center of the circumscribed circle then we can derive decidability of equality.

Definition 27. *Circumscribed circle axiom*

$$\forall ABC, \neg\, Col\ A\, B\, C \quad \Rightarrow \quad \exists C',\ AC' \equiv BC' \wedge AC' \equiv CC'$$

We can now derive a lemma inspired from lemma 7.2 of [Bee14a]. We have a slightly different axiom system and we are interested in decidability rather than stability so we provide the proof in our context:

Theorem 2. *In the context of Tarski's Euclidean geometry in 2D, the decidability of intersection is equivalent to the circumscribed center principle.*



Proof. $\Leftarrow$ Let A, B, C and D be four points. We want to decide if line AB and CD have an intersection. Let M be the midpoint of AC. We can assume that M does not belong to line AB nor CD otherwise it is easy to see that the two lines intersect. We construct M_1 and M_2 the symmetric points of M wrt line AB and CD. If M, M_1 and M_2 are collinear then AB and CD are perpendicular to the same line. Hence, by a consequence of Euclid axiom they are parallel. Otherwise M, M_1 and M_2 are not collinear, by circumscribe circle axiom we can then construct I such that $IM_1 \equiv IM$ and $IM_2 \equiv IM$. By the definition of perpendicular lines, we can derive that I belongs to both AB and CD.

$\Rightarrow$ Using decidability of intersection we can prove all the results in the first 12 chapters of [SST83], this includes the existence of the center of the circumscribed circle. $\square$

6 Conclusion

We have shown in this paper that decidability of equality and decidability of intersection are sufficient properties to prove other decidability properties in the context of the geometry of Tarski. It is interesting to note that, in our experience, even if in theory it is possible, we *could not* have carried out this study without the use of a proof assistant. First, because it would have been very difficult to detect case distinction in an informal proof as often degenerated cases are omitted in the proofs as noted in our previous results [Nar07,BN12]. Second, because it is easy to enter in a circular argument proving a decidability property by using some other lemmas which have been proved using this decidability property or an indirect consequence of it. During the proving process, we modified some lemmas to remove unnecessary case distinctions, reordered many lemmas to obtain results when we need them to prove other results. Before the discovery of non-euclidean geometries, many incorrect proof of the axiom of Euclid have been proposed, often the error was due to circular arguments:

> Legendre's investigations of the provability of the parallel postulate extended over 40 years and appeared mostly in appendixes of successive editions of the *Eléments de géométrie*. All his attempts to derive the postulate from the other Euclidean axioms were deficient in that each one rested on some hypothesis that was logically equivalent to the desired statement.

Burton, D. (1999). The History of Mathematics - An Introduction, 6th ed., p572.

In the future, it would be interesting to have automatic tools to study case distinction in geometric proofs as for a human it is difficult to tell which case distinctions are necessary and in which order. Compared to Michael Beeson's constructive version of Tarski's axiom system [Bee14a] our study can be seen as less constructive as decidability of equality is a strong assumption. Michael Beeson only assumes stability of equality, betweenness and congruence. Still, we think our study is useful because we obtain results similar to his results about the connection between decidability of intersection and the existence of the circumscribed center [Bee14b]. Moreover in some models such as ruler-and-compass geometry the equality is decidable. A natural extension of this work consists in verifying using Coq the results of Michael Beeson in order to obtain a fully constructive formal development about Tarski's geometry. It would be interesting also to prove the equivalence with the axiom system of Lombard and Vesley [LV98].

Availability

The full Coq development is available here: `http://dpt-info.u-strasbg.fr/~narboux/tarski.html`

References

BBN14. Michael Beeson, Pierre Boutry, and Julien Narboux. Herbrand's theorem and non-Euclidean geometry. 10 pages, 2014.

Bee14a. Michael Beeson. A Constructive Version of Tarski's Geometry. 60 pages, `http://www.michaelbeeson.com/research/papers/AxiomatizingConstructiveGeometry.pdf`, 2014.

Bee14b. Michael Beeson. Constructive Geometry and the Parallel Postulate. 88 pages, `http://www.michaelbeeson.com/research/papers/ConstructiveGeometryAndTheParallelPostulate.pdf`, 2014.

Bee14c. Michel Beeson. Proving Hilbert's axioms in Tarski geometry. `http://www.michaelbeeson.com/research/papers/TarskiProvesHilbert.pdf`, 2014.

BN12. Gabriel Braun and Julien Narboux. From Tarski to Hilbert. In Tetsuo Ida and Jacques Fleuriot, editors, *Automated Deduction in Geometry 2012*, volume 7993 of *LNCS*, pages 89–109, Edinburgh, United Kingdom, September 2012. Jacques Fleuriot, Springer.

BW14. Michael Beeson and Larry Wos. OTTER proofs of theorems in Tarskian geometry. In *Proceedings of IJCAR*, LNAI. Springer, 2014.

DDS00. Christophe Dehlinger, Jean-François Dufourd, and Pascal Schreck. Higher-Order Intuitionistic Formalization and Proofs in Hilbert's Elementary Geometry. In *Automated Deduction in Geometry*, pages 306–324, 2000.

Hil71. David Hilbert. *Les fondements de la géométrie*. Dunod, Paris, Jacques Gabay edition, 1971. Edition critique avec introduction et compléments préparée par Paul Rossier.

Kah95. Gilles Kahn. Constructive Geometry according to Jan von Plato. Coq contribution, 1995. Coq V5.10.

LV98. Melinda Lombard and Richard Vesley. A common axiom set for classical and intuitionistic plane geometry. *Annals of Pure and Applied Logic*, 95(1–3):229–255, 1998.

MNS11. Nicolas Magaud, Julien Narboux, and Pascal Schreck. Formalizing Projective Plane Geometry in Coq. In Thomas Sturm, editor, *Post-proceedings of Automated Deduction in Geometry (ADG) 2008*, volume 6301 of *LNAI*, pages 141–162, Shanghai, China, 2011. Thomas Sturm, Springer.

Nar07. Julien Narboux. Mechanical Theorem Proving in Tarski's geometry. In Francisco Botana Eugenio Roanes Lozano, editor, *Automated Deduction in Geometry 2006*, volume 4869 of *LNCS*, pages 139–156, Pontevedra, Spain, 2007. Francisco Botana, Springer.

NBB14. Julien Narboux, Gabriel Braun, and Pierre Boutry. Tarski's Geometry Formalized in Coq, 2006-2014. http://dpt-info.u-strasbg.fr/ narboux/tarski.html.

SO08. Matthieu Sozeau and Nicolas Oury. First-Class Type Classes. In Otmane Aït Mohamed, César A. Muñoz, and Sofiène Tahar, editors, *TPHOLs*, volume 5170 of *Lecture Notes in Computer Science*, pages 278–293. Springer, 2008.

SST83. Wolfram Schwabhäuser, Wanda Szmielew, and Alfred Tarski. *Metamathematische Methoden in der Geometrie*. Springer-Verlag, Berlin, 1983.

vP95. Jan von Plato. The axioms of constructive geometry. In *Annals of Pure and Applied Logic*, volume 76, pages 169–200, 1995.