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Symmetry and dependence properties within a semiparametric family of bivariate copulas

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Abstract

In this paper, we study a semiparametric family of bivariate copulas. The family is generated by an univariate function, determining the symmetry (radial symmetry, joint symmetry) and dependence property (quadrant dependence, total positivity, ...) of the copulas. We provide bounds on different measures of association (such as Kendall's Tau, Spearman's Rho) for this family and several choices of generating functions allowing to reach these bounds.

Keywords: Copulas, semiparametric family, measures of association, positive dependence, dependence orderings.

AMS Subject classifications: Primary 62H05, secondary 62H20.

1 Introduction

The theory of copulas provides a relevant tool to build multivariate probability laws, from fixed margins and required degree of dependence. From Sklar's Theorem [16], the dependence properties of a continuous multivariate distribution H can be entirely summarized, independently of its margins, by a copula, uniquely associated with H. Several families of copulas, such as Archimedean copulas [6] or copulas with polynomial sections [14, 12] have been proposed. In [12], the authors point out that the copulas with quadratic section proposed in [14] can only model a weak dependence for a vector of random variables; weak in the sense that for such copulas, the Sperman's Rho and Kendall's Tau are respectively lower than 1/2 and 1/3. Then, they introduce copulas with cubic sections and conclude that copulas with higher order polynomial sections would increase the degree of dependence but simultaneously the complexity of the model. We propose to give up the polynomial form to work with a semiparametric family of copulas. The induced parametric families of copulas [1] are generated as simply as Archimedean copulas, that is by an univariate function. Furthermore, dependence properties of copulas with polynomial sections are preserved and the degree of dependence can be increased without increasing significantly the complexity of the model. Note that, in [5], a class of symmetric bivariate copulas with a wide range of correlation coefficients is introduced, but these copulas are less convenient to perform classical calculations on probability laws than the copulas studied here. In Section 2, a specific semiparametric family is defined and its basic properties are derived. Symmetry properties are established in Section 3. Properties of three measures of association are investigated in Section 4. Section 5 is devoted to the dependence structure and the dependence ordering of the family.

2 Definition and basic properties

Throughout this paper, we note $I = [0, 1]$. A bivariate copula defined on the unit square I^2 is a bivariate cumulative distribution function with univariate uniform I margins. Equivalently, C must satisfy the following properties :

(P1) $C(u, 0) = C(0, v) = 0$, $\forall (u, v) \in I^2$,

(P2) $C(u, 1) = u$ and $C(1, v) = v$, $\forall (u, v) \in I^2$,

(P3) $\Delta(u_1, u_2, v_1, v_2) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$, $\forall (u_1, u_2, v_1, v_2) \in I^4$, such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

Let us recall that, from Sklar's Theorem, any bivariate distribution with cumulative distribution function H and marginal cumulative distribution functions F and G can be written $H(x, y) = C(F(x), G(y))$, where C is a copula. This result justifies the use of copulas for building bivariate distributions.
We consider the semiparametric family of functions defined on I^2 by :

$$C_\theta(u, v) = uv + \theta\phi(u)\phi(v), \quad \theta \in [-1, 1], \tag{2.1}$$

where ϕ is a function on I. This family is a particular case of Farlie's family introduced in [4]. Let us note first that, the independent copula $C_0(u, v) = uv$ belongs to any parametric family $\{C_\theta\}$ generated by a function ϕ. Second, the functions ϕ and $(-\phi)$ clearly define the same function C_θ.
The following theorem, which is very similar to Theorem 2.2 in [14], concerning bivariate copulas with quadratic sections, gives necessary and sufficient conditions on ϕ to ensure that C_θ is a copula.

Theorem 1 *ϕ generates a parametric family of copulas $\{C_\theta,\ \theta \in [-1, 1]\}$ if and only if it satisfies the following conditions :*

(a) $\phi(0) = \phi(1) = 0$,

(b) *ϕ satisfies the Lipschitz condition :* $|\phi(x) - \phi(y)| \leq |x - y|, \ \ \forall (x, y) \in I^2$.

Furthermore, C_θ is absolutely continuous.

Proof: It is clear that **(P1)**$\Leftrightarrow$ $(\phi(0) = 0)$ and **(P2)**$\Leftrightarrow$ $(\phi(1) = 0)$. In the case of the C_θ function, Δ can be rewritten as

$$\Delta(u_1, u_2, v_1, v_2) = (u_2 - u_1)(v_2 - v_1) + \theta(\phi(u_2) - \phi(u_1))(\phi(v_2) - \phi(v_1)),$$

and then **(b)**$\Rightarrow$ **(P3)**. Conversely, if there are $x < y$ such that $|\phi(x) - \phi(y)| > |x - y|$, then with the choice $\theta = -1$, $u_1 = v_1 = x$ and $u_2 = v_2 = y$, we obtain:

$$\Delta(x, y, x, y) = (x - y)^2 - (\phi(x) - \phi(y))^2 < 0.$$

Therefore **(P3)** $\Rightarrow$ **(b)**. Lastly, C_θ is absolutely continuous since ϕ satisfies the Lipschitz condition. □

The following corollary provides a new characterization of the functions ϕ generating parametric families of copulas. It will turn out to be useful in the sequel.

Corollary 1 *ϕ generates a parametric family of copulas $\{C_\theta,\ \theta \in [-1, 1]\}$ if and only if it satisfies the following conditions :*

(i) *ϕ est absolutely continuous,*

(ii) *$|\phi'(x)| \leq 1$ almost everywhere in I,*

(iii) *$|\phi(x)| \leq \min(x, 1 - x)$, $\forall x \in I$.*

In such a case, C_θ is absolutely continuous.

Proof: From a classical analysis result ([17], Lemma 2.1), **(b)** is satisfied if and only if **(i)** and **(ii)** both hold. Now, assume that conditions **(i)**–**(iii)** are satisfied. Taking successively $x = 0$ and $x = 1$ in **(iii)** yields **(a)**.
Conversely, assume that **(b)** holds. Taking successively $y = 0$ and $y = 1$ in the Lipschitz condition gives **(iii)**. □

The function ϕ plays a role similar to the generating function in Archimedean copulas [6]. Each copula C_θ is entirely described by the univariate function ϕ and the parameter θ, which tunes the dependence between the margins (see Section 4). Symmetry and dependence properties of the copula C_θ will have a geometrical interpretation on the graph of ϕ. Furthermore, the choice of ϕ determines the vertical and horizontal sections of the copula up to a multiplicative factor and an additive linear function.
We now give some examples of functions ϕ generating parametric families of copulas. These examples will be used to illustrate the symmetry and dependence properties introduced in the following sections.

Example 1 *The following functions ϕ generate parametric families of copulas.*

1. *$\phi^{[1]}(x) = \min(x, 1 - x)$ is the upper bound of Corollary 1***(iii)***.*
2. *$\phi^{[2]}(x) = x(1 - x)$ generates the Farlie-Gumbel-Morgenstern (FGM) family of copulas [9], which contains all copulas with both horizontal and vertical quadratic sections [14].*
3. *$\phi^{[3]}(x) = x(1 - x)(1 - 2x)$ defines the parametric family of symmetric copulas with cubic sections proposed in [12], equation (4.4).*
4. *$\phi^{[4]}(x) = \frac{1}{\pi}\sin(\pi x)$ induces a family of copulas able to model strong dependence.*

In the sequel, we note $\{C_\theta^{[i]}\}$ the parametric family of copulas associated to the function $\phi^{[i]}$, $i \in \{1, \dots, 4\}$. Graphs of the functions $\phi^{[i]}$ are plotted in Figure 1. As a consequence of the condition **(iii)** of Corollary 1, graphs of $\phi^{[1]}$ and $(-\phi^{[1]})$ are the edges of a square K, within which lies the graph of ϕ.
In the following, we review several concepts of symmetry and dependence. Throughout the rest of this paper, ϕ denotes a function satisfying the conditions given in Theorem 1, and $\{C_\theta\}$ represents the parametric family of copulas generated by the function ϕ.

3 Symmetry properties

Let $(a, b) \in \mathbb{R}^2$ and (X, Y) a random vector. We say that X is symmetric about a if the cumulative distribution function of $(X - a)$ and $(a - X)$ are identical. The following definitions generalize this symmetry concept to the bivariate case:

- X and Y are exchangeable if (X, Y) and (Y, X) are identically distributed.
- (X, Y) is marginally symmetric about (a, b) if X and Y are symmetric about a and b respectively.
- (X, Y) is radially symmetric about (a, b) if $(X - a, Y - b)$ and $(a - X, b - Y)$ follow the same joint cumulative distribution function.
- (X, Y) is jointly symmetric about (a, b) if the vectors of random variables $(X - a, Y - b)$, $(a - X, b - Y)$, $(X - a, b - Y)$ and $(a - X, Y - b)$ have a common joint cumulative distribution function.

The following theorem provides conditions on ϕ to ensure that the couple (X, Y) with associated copula C_θ is radially (or jointly) symmetric.

Theorem 2

(i) *If X and Y are identically distributed then X and Y are exchangeable.*

Besides, if (X, Y) is marginally symmetric about (a, b) then:

(ii) *(X, Y) is radially symmetric about (a, b) if and only if*

$$\text{either } \forall u \in I,\ \phi(u) = \phi(1 - u) \text{ or } \forall u \in I,\ \phi(u) = -\phi(1 - u). \tag{3.1}$$

(iii) *(X, Y) is jointly symmetric about (a, b) if and only if $\forall u \in I,\ \phi(u) = -\phi(1 - u)$.*

Proof:

(i) When X and Y are identically distributed, exchangeability is equivalent to the symmetry of the copula. In other words, C_θ must verify $C_\theta(u, v) = C_\theta(v, u)$, $\forall (u, v) \in I^2$, which is the case by definition, see (2.1).

(ii) Assume (X,Y) is marginally symmetric about (a,b). Then, from Theorem 2.7.3 in [13], (X,Y) is radially symmetric about (a,b) if and only if

$$\forall (u,v) \in I^2, \;\; \delta_\theta(u,v) = 0, \tag{3.2}$$

where

$$\delta_\theta(u,v) = C_\theta(u,v) - C_\theta(1-u,1-v) - u - v + 1 = \theta[\phi(u)\phi(v) - \phi(1-u)\phi(1-v)].$$

Obviously (3.1) implies (3.2). Conversely, suppose that (3.2) holds. Then, $\delta_1(u,u) = \phi^2(u) - \phi^2(1-u) = 0$ for all $u \in I$ and consequently, either $\phi(u) = \phi(1-u)$ or $\phi(u) = -\phi(1-u)$. If there exist (u_1,u_2) such that $\phi(u_1) = \phi(1-u_1) \neq 0$ and $\phi(u_2) = -\phi(1-u_2) \neq 0$, then $\delta_1(u_1,u_2) = 2\phi(u_1)\phi(u_2) \neq 0$. As a conclusion (3.2) implies (3.1).

(iii) Assume (X,Y) is marginally symmetric about (a,b). Then (X,Y) is jointly symmetric about (a,b) if and only if

$$\forall (u,v) \in I^2, \;\; d_\theta(u,v) = 0, \tag{3.3}$$

where

$$d_\theta(u,v) = C_\theta(u,v) + C_\theta(u,1-v) - u = \theta\phi(u)[\phi(v) + \phi(1-v)].$$

It immediatly follows that (3.3) is equivalent to $\phi(u) = -\phi(1-u)$, $\forall u \in I$. □

As an example, any marginally symmetric random vector (X,Y) associated to a copula of the families $\{C_\theta^{[i]}\}$, $1 \leq i \leq 4$, is radially symmetric. Moreover, in the case $i = 3$, (X,Y) is jointly symmetric.
We now focus on the dependence properties of the semiparametric family of copulas.

4 Measures of association

In the next two sections, we note (X,Y) a random vector with distribution H, assumed to have a density h, copula C and margins F and G. We consider three standard measures of association between the components of the random vector (X,Y):

- the normalized volume between the graphs of H and FG [15],

$$\sigma = 12 \int_0^1 \int_0^1 |C(u,v) - uv| dudv,$$

- Kendall's Tau [2,3], defined as the probability of concordance minus the probability of discordance of two vectors (X_1,Y_1) and (X_2,Y_2) described by the same joint bivariate law H,

$$\tau = 4 \int_0^1 \int_0^1 C(u,v) dC(u,v) - 1, \tag{4.1}$$

- Spearman's Rho [2,3], which is the probability of concordance minus the probability of discordance of two vectors (X_1,Y_1) and (X_2,Y_2) with respective joint cumulative law H and FG,

$$\rho = 12 \int_0^1 \int_0^1 C(u,v) dudv - 3.$$

In the case of a copula generated by (2.1), these measures can be rewritten only in terms of the function ϕ.

Proposition 1 *Let (X,Y) be a random vector with copula C_θ given by (2.1). The coefficients of association are:*

$$\sigma_\theta = 12|\theta| \left(\int_0^1 |\phi(u)|du\right)^2, \ \ \tau_\theta = 8\theta \left(\int_0^1 \phi(u)du\right)^2, \ \ \rho_\theta = 12\theta \left(\int_0^1 \phi(u)du\right)^2 = \frac{3}{2}\tau_\theta.$$

Proof: The proofs are very similar for the three coefficients. Let us consider the example of Kendall's Tau. The density $c_\theta(u,v)$ of C_θ is

$$c_\theta(u,v) = 1 + \theta\phi'(u)\phi'(v). \tag{4.2}$$

Replacing in (4.1), it yields

$$\begin{aligned} \tau_\theta &= 4\theta \left[\int_0^1 \phi(u)du\right]^2 + 4\theta \left[\int_0^1 u\phi'(u)du\right]^2 + 4\theta^2 \left[\int_0^1 \phi(u)\phi'(u)du\right]^2, \\ &= 4\theta \left[\int_0^1 \phi(u)du\right]^2 + 4\theta \left[\phi(1) - \int_0^1 \phi(u)du\right]^2 + 2\theta^2[\phi^2(1) - \phi^2(0)]. \end{aligned}$$

Theorem 1 entails $\phi(1) = \phi(0) = 0$ and the result follows. □

The measures τ_θ and ρ_θ linearly increase with θ, which appears as an association parameter. They are also proportionnals to the square of the surface lying between the graph of ϕ and the x-axis. Bounds for each of these measures of association are deduced from Corollary 1 **(iii)**.

Proposition 2 $\forall\theta \in [-1,1]$, $0 \le \sigma_\theta \le 3|\theta|/4$, $|\tau_\theta| \le |\theta|/2$ *and* $|\rho_\theta| \le 3|\theta|/4$.

Therefore the range of each coefficient of association is $0 \le \sigma_\theta \le 3/4$, $-1/2 \le \tau_\theta \le 1/2$ and $-3/4 \le \rho_\theta \le 3/4$. The semiparametric family of copulas defined by (2.1) seems to be a good tool to build low complexity models with moderate dependences ($0 \le |\tau| \le 1/2$). This is illustrated in Table 1 where the values obtained for the copulas presented in Example 1 are reported. Results obtained for the cubic sections copulas illustrate the fact that τ and ρ can be null although X and Y are not independent. All the values associated to $\{C_\theta^{[4]}\}$ are larger than those obtained with the FGM family. The bounds of Proposition 2 are obtained via the family $\{C_\theta^{[1]}\}$. This hierarchy of results graphically appears in Figure 1. For each fixed θ, the closer to the edges of the square K is the graph of ϕ, the larger is the coefficient of association of the copula induced by ϕ. Non-differentiability of $\phi^{[1]}$ can be a drawback. In order to build more regular copulas, with measures of association closed from the upper bounds, we define the sequence of $\mathcal{C}^1$ functions $(\phi_n^{[5]})$ as following:

$$\forall n > 0, \ \phi_n^{[5]}(x) = \begin{cases} x & \text{if} \quad 0 \le x \le \dfrac{1}{2} - \dfrac{1}{n} \\ \dfrac{1}{2}\left(1 - \dfrac{1}{n}\right) - \dfrac{n}{2}\left(x - \dfrac{1}{2}\right)^2 & \text{if} \quad \dfrac{1}{2} - \dfrac{1}{n} < x < \dfrac{1}{2} + \dfrac{1}{n} \\ 1 - x & \text{if} \quad \dfrac{1}{2} + \dfrac{1}{n} \le x \le 1. \end{cases}$$

Moreover each copula is differentiable and the associated Kendall's Tau sequence is given by

$$\tau_{n,\theta}^{[5]} = 8\theta \left(\frac{1}{4} - \frac{1}{3n^2} \right)^2, \quad n > 0.$$

It converges to the upper Kendall's Tau, $\tau_{n,\theta}^{[5]} \to \theta/2$ as $n \to \infty$. It is also possible to build $\mathcal{C}^\infty$ copulas, such that the coefficients of association are closed from the upper bounds. Define the sequence $(\phi_n^{[6]})$ of $\mathcal{C}^\infty$ functions by:

$$\forall n \geq 2, \quad \phi_n^{[6]}(x) = 1 - (x^n + (1-x)^n)^{1/n}.$$

Similarly to the previous example, for each $n \geq 2$, $\phi_n^{[6]}$ induces a parametric family of copulas $\{C_{n,\theta}^{[6]}\}$. Moreover $\forall x \in I$, $\phi_n^{[6]}(x) \to \phi^{[1]}(x)$ as $n \to \infty$ and $|\phi_n^{[6]}| \leq 1$. So, by Lebesgue's dominated convergence theorem, $\tau_{n,\theta}^{[6]}$ converges to the upper bound as $n \to \infty$. Graphs of $\phi_2^{[6]}$, $\phi_4^{[6]}$, and $\phi_8^{[6]}$ are plotted in Figure 2. We clearly see that the surface lying between the $\phi_n^{[6]}$ graph and the x-axis increases with n. The sequences $(\tau_{n,1}^{[6]})$ and $(\tau_{n,1}^{[5]})$ are plotted in Figure 3. We can observe that for $n \geq 2$, $\tau_{n,1}^{[5]}$ and $\tau_{n,1}^{[6]}$ are both larger than the value obtained by the FGM copula $C_1^{[2]}$ and that they exceed the value reached by the copula $C_1^{[4]}$, whenever $n \geq 3$.

5 Concepts of dependence

The property of independence clearly corresponds to the subset all of whose distribution members have the copula $C_0(u,v) = uv$. In the same way, the dependence properties can be thought of as a subset of the set of all joint distrubutions. Many dependence properties can be described by identifying the copulas or simple properties of the copulas, which corresponds to the subset.

5.1 Positive dependence

In this subsection, we will use several concepts of positive dependence, which express that two variables are large (or small) simultaneously.

- PFD: X and Y are Positive Function Dependent if for all integrable real-valued function g, we have $\mathbb{E}_h[g(X)g(Y)] - \mathbb{E}_h[g(X)]\mathbb{E}_h[g(Y)] \geq 0$ where $\mathbb{E}_h$ is the expectation symbol relative to the density h.
- PQD: X and Y are Positively Quadrant Dependent if

$$\forall (x,y) \in \mathbb{R}^2, \quad P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y). \tag{5.1}$$

- LTD($Y|X$): Y is Left Tail Decreasing in X if $P(Y \leq y | X \leq x)$ is nonincreasing in x for all y. A similar definition can be given for $LTD(X|Y)$.
- RTI($Y|X$): Y is Right Tail Increasing in X if $P(Y > y | X > x)$ is nondecreasing in x for all y.
- SI($Y|X$): Y is Stochastically Increasing in X if $P(Y > y | X = x)$ is nondecreasing in x for all y.

- LCSD: X and Y are Left Corner Set Decreasing if $P(X \leq x, Y \leq y | X \leq x', Y \leq y')$ is nonincreasing in x' and y' for all x and y.
- RCSI: X and Y are Right Corner Set Increasing if $P(X > x, Y > y | X > x', Y > y')$ is nondecreasing in x' and y' for all x and y.
- TP2 density: (X, Y) have the TP2 density property if h is a totally positive function of order 2 *i.e.* $h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1) \geq 0$ for all $(x_1, x_2, y_1, y_2) \in I^4$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

When X and Y are exchangeable, Y is Stochastically Increasing in X if and only if X is Stochastically Increasing in Y. Will will denote this property SI. Similarly, we will denote LTD the equivalent properties LTD($Y|X$) and LTD($X|Y$), and RTI, RTI($Y|X$) or RTI($X|Y$). The links between these concepts [8, 10] are illustrated in Figure 4. The following theorem summarizes the properties of positive dependence of any vector (X, Y) associated with the copula C_θ defined by (2.1). Similar results can be established for the corresponding concepts of negative dependence.

Theorem 3 *Let $\theta > 0$ and (X, Y) a random vector with copula C_θ.*

(i) *X and Y are PFD.*

(ii) *X and Y are PQD if and only if either $\forall u \in I,\ \phi(u) \geq 0$ or $\forall u \in I,\ \phi(u) \leq 0$.*

(iii) *X and Y are LTD if and only if $\phi(u)/u$ is monotone.*

(iv) *X and Y are RTI if and only if $\phi(u)/(u-1)$ is monotone.*

(v) *X and Y are LCSD if and only if they are LTD.*

(vi) *X and Y are RCSI if and only if they are RTI.*

(vii) *X and Y are SI if and only if ϕ is either concave or convex.*

(viii) *X and Y have the TP2 density property if and only if they are SI.*

Proof:

(i) Let g be an integrable real-valued function on I. The density c_θ of C_θ is given by (4.2). Routine calculations yield

$$\mathbb{E}_{c_\theta}(g(X)g(Y)) - \mathbb{E}_{c_\theta}(g(X))\mathbb{E}_{c_\theta}(g(Y)) = \theta \left[\int_0^1 g(t)\phi'(t)dt \right]^2 \geq 0,$$

since $\theta \geq 0$.

(ii) The vector (X, Y) is PQD if and only if the uniform I-margins vector (U, V) with distribution C_θ is PQD. For (U, V), condition (5.1) can simply be rewritten as $\theta\phi(u)\phi(v) \geq 0$, $\forall (u, v) \in I^2$ and the conclusion follows.

(iii) Since C_θ is symmetric, the necessary and sufficient conditions given by Theorem 5.2.5 in [13], simply reduce to the unique condition: $C_\theta(u,v)/u = v + \theta\phi(v)\phi(u)/u$ is nonincreasing in u for all $v \in I$. Suppose for instance $\phi(u)/u$ is nonincreasing. Then, since $\phi(1) = 0$, we have $\phi(v) \geq 0$ for all v in I and thus $C_\theta(u,v)/u$ is nonincreasing in u for all $v \in I$. The case of a nondecreasing function $\phi(u)/u$ is similar.
Conversely, suppose $C_\theta(u,v)/u$ is nonincreasing in u for all $v \in I$. If there exist $v \in I$ such that $\phi(v) > 0$ then $\phi(u)/u$ is nonincreasing too. The case where there is $v \in I$ such that $\phi(v) < 0$ is similar. The case $\phi(v) = 0$ for all $v \in I$ is trivial. Hence we have shown that $C_\theta(u,v)/u$ is nonincreasing in u for all $v \in I$ if and only if $\phi(u)/u$ is monotone.

(iv) Is similar to **(iii)**.

(v) In view of Corollary 5.2.17 in [13], a necessary and sufficient condition for X and Y to be LCSD is: C_θ is a totally positive function of order 2. For all $u_1 \leq u_2$ and $v_1 \leq v_2$, the quantity

$$C_\theta(u_1,v_1)C_\theta(u_2,v_2) - C_\theta(u_1,v_2)C_\theta(u_2,v_1) = \theta\left[\frac{\phi(v_2)}{v_2} - \frac{\phi(v_1)}{v_1}\right]\left[\frac{\phi(u_2)}{u_2} - \frac{\phi(u_1)}{u_1}\right]u_1u_2v_1v_2$$

is nonnegative if and only if $\phi(u)/u$ is monotone. This is the necessary and sufficient condition for X and Y to be LTD given in **(iii)**.

(vi) Is similar to **(v)**.

(vii) In view of the C_θ symmetry, the geometric interpretation of stochastic monotonicity given by Corollary 5.2.11 in [13] provides the necessary and sufficient condition: $C_\theta(u,v)$ is a concave function of u. Suppose ϕ is a concave function. Then, taking into account that $\phi(0) = \phi(1) = 1$, we easily show that $\phi(v) \geq 0$ for all $v \in I$ and therefore $C_\theta(u,v)$ is a concave function of u. The case of a convex function ϕ is similar.
Conversely, suppose $C_\theta(u,v)$ is a concave function of u. If there exists a v such that $\phi(v) > 0$ then (2.1) shows that ϕ is concave. The case where there exists a $v \in I$ such that $\phi(v) < 0$ is similar. The case $\phi(v) = 0$ for all $v \in I$ is trivial. Hence we have shown the equivalence between $C_\theta(u,v)$ is a concave function of u and ϕ is concave or convex.

(viii) X and Y have the TP2 density property if and only if the density of the copula satisfies

$$\forall u_1 \leq u_2,\ v_1 \leq v_2,\ \ c_\theta(u_1,v_1)c_\theta(u_2,v_2) - c_\theta(u_1,v_2)c_\theta(u_2,v_1) \geq 0,$$

which can be rewritten as

$$\forall u_1 \leq u_2,\ v_1 \leq v_2,\ \ [\phi'(u_1) - \phi'(u_2)][\phi'(v_1) - \phi'(v_2)] \geq 0.$$

Equivalently, ϕ' is either nonincreasing or nondecreasing which means that ϕ is either concave or convex. This is the necessary and sufficient condition for X and Y to be SI as established in **(vii)**. □

The results of Theorem 3 are illustrated in Figure 5. For instance, random vectors with copula in the parametric families generated by $\phi^{[1]}$, $\phi^{[2]}$, $\phi^{[4]}$, $\phi_n^{[5]}$ or $\phi_n^{[6]}$ have the TP2 density property since these functions are concave. Consequently, these random vectors are also SI, LCSD, RCSI, LTD, RTI, PQD, and PFD. When the copula belongs to the family $\{C_\theta^{[3]}\}$, the associated random vector is PFD but not PQD.

5.2 Dependence orderings

In Theorem 3 positive dependence properties of the copula C_θ are established for fixed $\theta > 0$. In this subsection we investigate how measures of dependence change as θ increases. To this end, it is necessary to define a dependence ordering which can decide whether a multivariate cumulative distribution function is more dependent than another. Consider H and H^* two bivariate cumulative distribution functions with the same margins. We study two dependence orderings:

- H^* is more PQD (or more concordant) than H if $H(x_1, x_2) \leq H^*(x_1, x_2)$ for all (x_1, x_2). We denote $H \prec_C H^*$ this case.
- Note $H_{2|1}$ and ${H^*}_{2|1}$ the conditional distributions of the second random variable given the first one. H^* is more SI than H if ${H^*}_{2|1}^{-1}(H_{2|1}(x_2|x_1)|x_1)$ is increasing in x_1. We denote $H \prec_{SI} H^*$ this case.

Remark that $H \prec_{SI} H^*$ implies $H \prec_C H^*$ (see Theorem 2.12 in [7]). The natural extension of these definitions to parametric families of copulas is the following.

- A family of copulas $\{C_\theta\}$ is ordered in concordance if $\theta \leq \theta'$ implies $C_\theta \prec_C C_{\theta'}$.
- A family of copulas $\{C_\theta\}$ is SI ordered if $\theta \leq \theta'$ implies $C_\theta \prec_{SI} C_{\theta'}$.

As mentionned in [7], $C_\theta \prec_C C_{\theta'}$ implies $\tau_\theta \leq \tau_{\theta'}$ and $\rho_\theta \leq \rho_{\theta'}$. We now give conditions under which the parametric family of copulas obtained by choosing ϕ according to Corollary 1 is stochastically ordered.

Theorem 4

(i) *$\{C_\theta\}$ is ordered in concordance if and only if ϕ is either positive or negative.*

(ii) *Assume ϕ is twice-differentiable. $\{C_\theta\}$ is SI ordered if ϕ is either convex or concave.*

Proof:

(i) The proof is straigtforward.

(ii) Because of Theorem 2.14 in [7], $\{C_\theta\}$ is SI ordered if for all increasing real-valued functions g,

$$A(u, v, \theta) = \frac{\partial^2 B}{\partial v \partial u} \frac{\partial B}{\partial \theta} - \frac{\partial^2 B}{\partial \theta \partial u} \frac{\partial B}{\partial v} \geq 0,$$

where $B(u, v, \theta) = g(v + \theta \phi'(u)\phi(v))$. We obtain

$$A(u, v, \theta) = -\phi''(u)\phi(v)[g'(v + \theta\phi'(u)\phi(v))]^2.$$

The sign of $A(u, v, \theta)$ is given by $-\phi''(u)\phi(v)$, which is positive whenever ϕ is concave or convex. $\square$

For instance, all families discussed in this document are ordered in concordance, but $\{C_\theta^{[3]}\}$ is not SI ordered.

6 Conclusion

In this paper, a symmetric semiparametric family of copulas is studied and its symmetry and dependence properties are established. As for Archimedean copulas, a parametric subfamily is generated by a univariate function and both symmetry and dependence properties of the family can be geometrically interpreted on the graph of the generating function. Furthermore, the horizontal and vertical sections of the copula are essentially described by the generating function. Finally, the proposed family of copulas provides a simple way to model moderate degrees of dependence. Extension to higher levels of dependence is considered in [3]. Future work will consist of generalizing the family definition to the multivariate case (dimension of the random vector upper than 2). We also refer to [2] for estimation techniques dedicated to this family of copulas.

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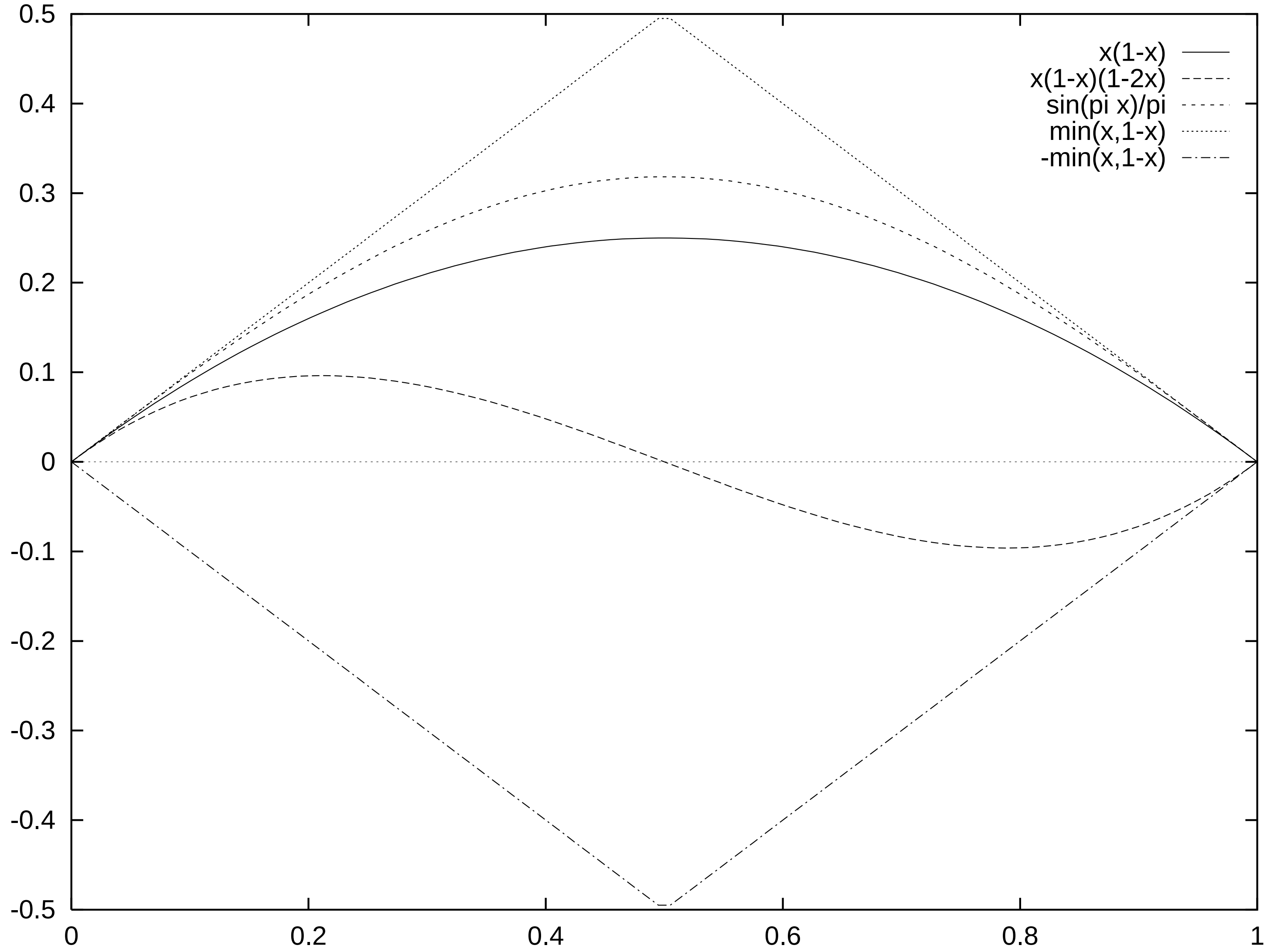
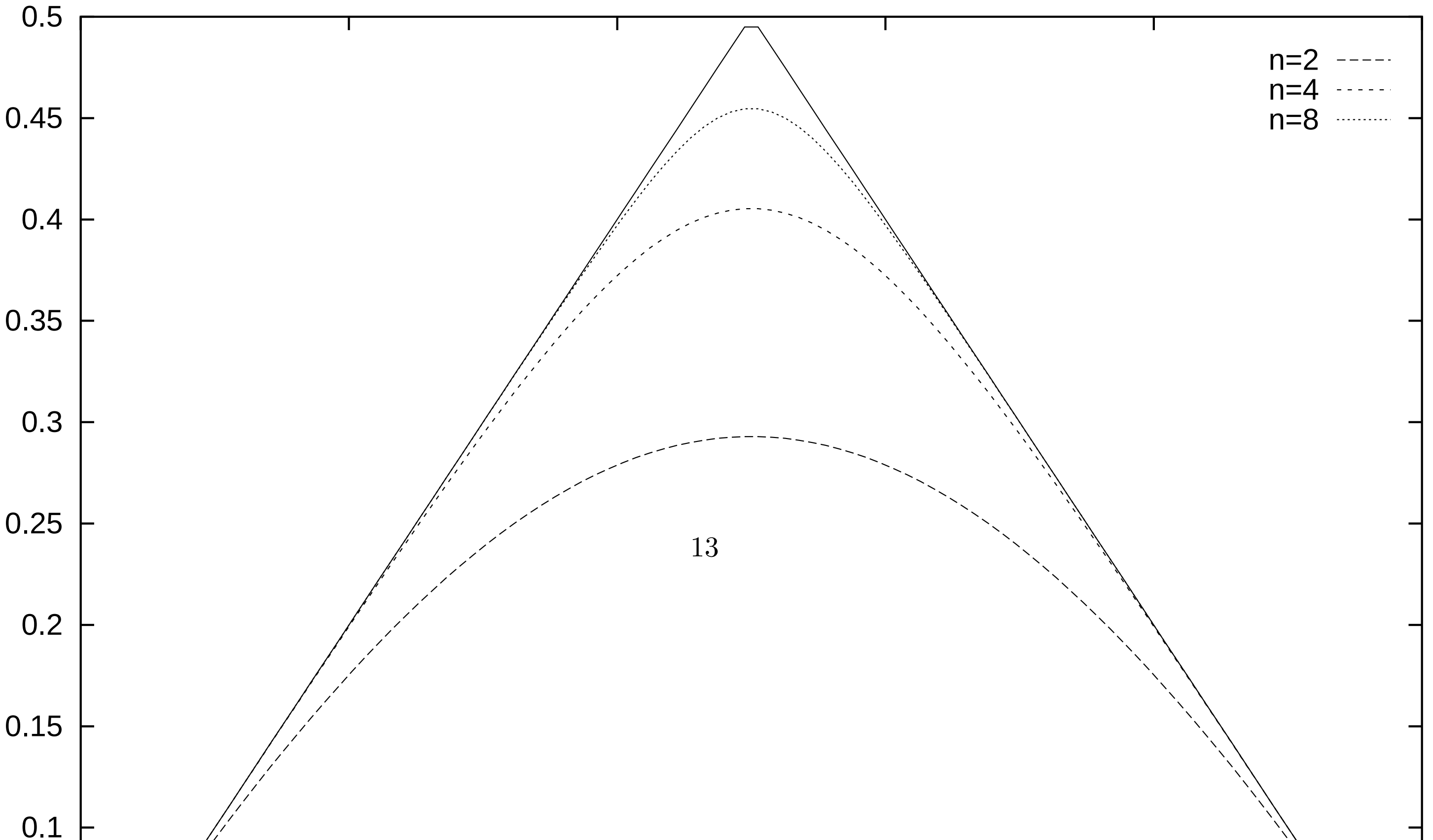


Figure 1: Graphs of the functions $\phi^{[i]}$, $i \in \{1, \ldots, 4\}$ presented Example 1.



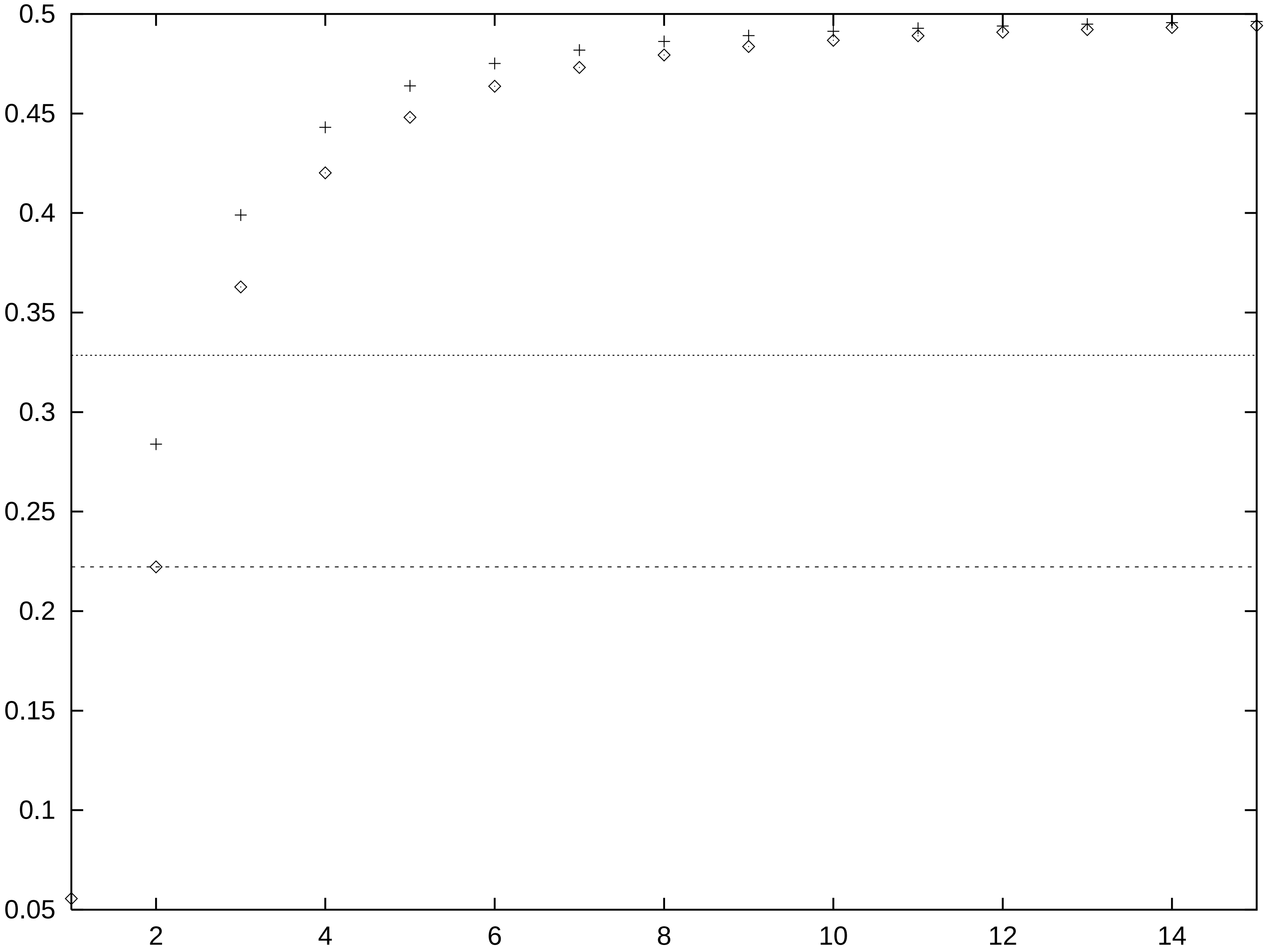


Figure 3: Comparison of several Kendall's Tau sequences. Solid line: $\tau_1^{[2]}$, dashed line: $\tau_1^{[4]}$, circles: $\tau_{n,1}^{[5]}$ and crosses: $\tau_{n,1}^{[6]}$.

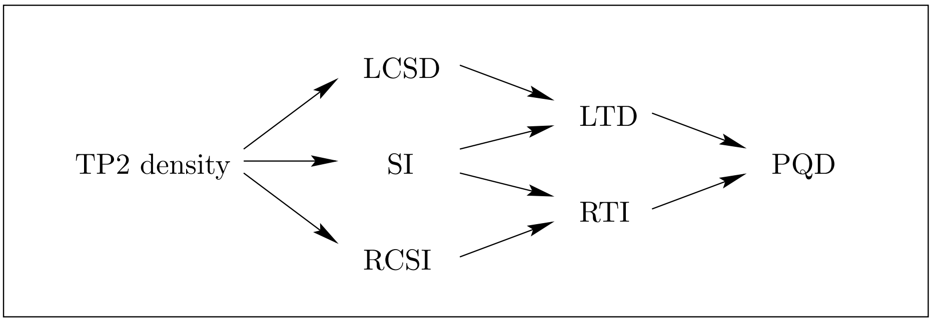


Figure 4: Implications among dependence properties in the general case.

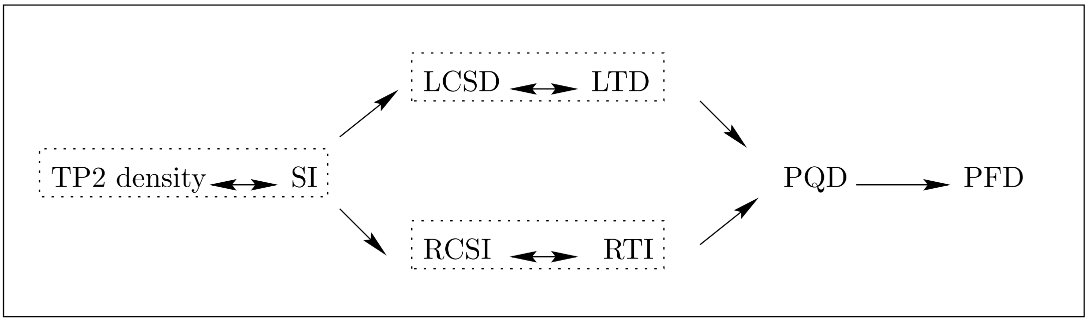


Figure 5: Implications among dependence properties within the $\{C_\theta\}$ family.

$\phi(x)$	σ_θ	τ_θ	ρ_θ
$\phi^{[1]}(x) = \min(x, 1-x)$	$3\|\theta\|/4$	$\theta/2$	$3\theta/4$
$\phi^{[2]}(x) = x(1-x)$	$\|\theta\|/3$	$2\theta/9$	$\theta/3$
$\phi^{[3]}(x) = x(1-x)(1-2x)$	$3\|\theta\|/64$	0	0
$\phi^{[4]}(x) = \frac{1}{\pi}\sin(\pi x)$	$48\|\theta\|/\pi^4$	$32\theta/\pi^4$	$48\theta/\pi^4$

Table 1: Measures of dependence associated to the Example 1 copulas.