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Discrete Mathematics and Theoretical Computer Science DMTCS vol. **12:5**, 2010, 1–24

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received 15th December 2009, *revised 13th August 2010*, *accepted 24th September 2010*.

The split-coloring problem is a generalized vertex coloring problem where we partition the vertices into a minimum number of split graphs. In this paper, we study some notions which are extensively studied for the usual vertex coloring and the cocoloring problem from the point of view of split-coloring, such as criticality and the uniqueness of the minimum split-coloring. We discuss some properties of split-critical and uniquely split-colorable graphs. We describe constructions of such graphs with some additional properties. We also study the effect of the addition and the removal of some edge sets on the value of the split-chromatic number. All these results are compared with their cochromatic counterparts. We conclude with several research directions on the topic.

Keywords: Split-coloring, split-critical, uniquely split-colorable, split-decreasing.

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1 Introduction

In this paper, we deal with a generalization of the graph coloring problem, called *split-coloring*. The minimum split-coloring problem in an undirected graph $G = (V, E)$, defined first in [EdW05], consists of minimizing the integer $\max(p, k)$ such that the vertices of G can be partitioned into p cliques and k stable sets. Since empty stable sets or cliques are also allowed, it can be easily seen that this problem is equivalent to partitioning the vertices of G into a minimum number of *split graphs* (defined as graphs whose vertex set can be partitioned into a clique and a stable set). Thus, here a color stands for a subset of vertices inducing a split graph. The optimal value is denoted by $\chi_S(G)$ and called *split-chromatic number*. Clearly, minimum split-coloring is $\mathcal{NP}$-hard in general [EdW05]. Several papers considered polynomially solvable cases of minimum split-coloring with respect to various graph classes; in cacti

†The research of Tınaz Ekim is funded by the B.U. Research Fund, Grant 09A302P, whose support is greatly acknowledged.
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[EdW05], in chordal graphs [HKNP04] and in cographs [DEdW05a]. On the other hand, it is shown that minimum split-coloring remains $\mathcal{NP}$-hard in line graphs of bipartite graphs [DEdW05b] and in permutation graphs [DEdW06].

In some papers, minimum split-coloring is studied in parallel with the so-called *minimum cocoloring*; given a graph G, minimum cocoloring consists of partitioning its vertex set into a minimum total number of cliques or stable sets. The related optimal value is called *cochromatic number* of G and is denoted by $z(G)$. This problem was first introduced by Lesniak et al. in [LS77] and extensively studied since then [DEdW05b, DEdW05a, EGK91, GKS94]. Minimum split-coloring is proved to be at least as difficult as minimum cocoloring from both the complexity point of view [DEdW05b, EG09] and the approximation point of view [DEdW06]. In [EG09], graphs for which one of the inequalities in $\chi_S(G) \leq z(G) \leq 2\chi_S(G)$ is satisfied with equality are studied. In [DEdW09], some applications of split-coloring and cocoloring (as well as some other generalized coloring problems) related to robotics are discussed.

This paper discusses some very natural but unexplored topics for split-coloring such as criticality, the uniqueness of the optimal split-coloring and the effect of the addition/removal of some edges on the split-chromatic number . These notions are studied for the usual coloring [Bac07, Dan01, Tru84, Zhu99] as well as the cocoloring [BB86, BB89, GS87, Jor95]. We note that uniquely colorable graphs have a crucial importance in the construction of gadgets for various $\mathcal{NP}$-hardness proofs. Besides, they also play an important role in the study of minimally imperfect graphs, namely as a possible alternative way to prove the Strong Perfect Graph Theorem [RRT09]. Our results will be sometimes similar and sometimes different from their cochromatic counterparts.

We write $G-v$ (resp. $G+v$) for the subgraph obtained by deleting (resp. adding) a vertex v. Similarly, we denote by $G-e$ (resp. $G+e$) the subgraph obtained by deleting (resp. adding) an edge e. A graph $G=(V,E)$ is said to be k*-split-critical* if $\chi_S(G)=k$ and for every $v \in V$, $\chi_S(G-v)=k-1$. Note that k-split-colorability has been defined in [EdW05] where some simple k-split-critical graphs are mentioned. In Section 2, we give some additional properties of k-split-critical graphs and describe a construction of triangle-free k-split-critical graphs with p connected components for $1 \leq p \leq k$.

We say that a graph G is *uniquely split-colorable* if every optimal split-coloring of G has the same collection of cliques and stable sets. In Section 3, after discussing some properties of uniquely split-colorable graphs, we give some necessary conditions for a graph to be uniquely split-colorable. We show that these conditions are also sufficient if the graph has split-chromatic number equal to 1, that is, if it is a split graph. We also construct a family of uniquely split-colorable graphs with split-chromatic number k for any $k \geq 2$.

Lastly, in Section 4, we study the variation of χ_S when we add or remove an edge (or the set of edges of a $2K_2$, a C_4 or a C_5). We say that a graph $G=(V,E)$ is split-decreasing if $\chi_S(G)=k$, $\chi_S(G+e)=k-1$ for any missing edge e in G, and $\chi_S(G-e')=k-1$ for any edge $e' \in E$. We give a description of split-decreasing graphs with two or three connected components. The cochromatic counterpart of such a study has been carried out in [GS87]. We conclude with some open questions.

In this paper all graphs are undirected simple finite graphs. Let G and H be two disjoint graphs. We denote by $G \oplus H$ the graph obtained by making every vertex in G adjacent to every vertex in H, and we denote by $G \cup H$ the graph obtained by taking the disjoint union of G and H such that there are no edges between $V(G)$ and $V(H)$. We denote by kG, for $k \geq 1$, the graph consisting in k disjoint copies of G such that there are no edges between any two distinct copies. For a graph $G=(V,E)$ and a set $V' \subseteq V$, we denote by $G-V'$ the subgraph obtained by deleting all vertices in V'. As usual K_k will denote a clique with k vertices while S_k will be a stable set with k vertices. A k-split-coloring of a graph G will be denoted by $(K^1, \ldots, K^k; S^1, \ldots, S^k)$, where K^i is a clique, S^j is a stable set and $K^i \cap S^j = \emptyset$, for $i, j = 1, \ldots, k$. Throughout the paper, whenever we say that a split-coloring *uses/needs/contains* k stable sets (resp. k cliques), these stable sets (resp. cliques) are supposed to be non-empty. For all graph theoretical notions that are not defined here, the reader is referred to [Ber73].

2 Split-critical graphs

In this section we will present some properties of k-split-critical graphs. In [FH77], it is shown that split graphs are precisely graphs which do not contain any induced subgraph isomorphic to $2K_2, C_4$ or C_5. Hence $2K_2, C_4$ and C_5 are the only 2-split-critical graphs. Now, we study some properties concerning the connected components of k-split-critical graphs for $k \geq 3$.

Proposition 2.1 *Let G be a k-split-critical graph, with $k \geq 3$. Then every connected component C of G satisfies $\chi(C) \geq k$.*

Proof: Let $C^1, \ldots, C^p$ be the connected components of G. Suppose that there exists C^i, $i \in \{1, \ldots, p\}$, which is $(k-1)$-colorable, i.e., $\chi(C^i) \leq k-1$, and let $S^{1,i}, \ldots, S^{k-1,i}$ be a $(k-1)$-coloring of C^i. Since G is k-split-critical, it follows that $G - V(C^i)$ is $(k-1)$-split-colorable. Let $(K^1, \ldots, K^{k-1}; S^1, \ldots, S^{k-1})$ be a $(k-1)$-split-coloring of $G - V(C^i)$. Then by adding $S^{j,i}$ to S^j, for $j = 1, \ldots, k-1$, we obtain a feasible $(k-1)$-split-coloring of G, a contradiction. □

The following result is an imediate consequence of Proposition 2.1.

Corollary 2.2 *Let G be a 3-split-critical graph. Then every connected component C of G contains at least one odd cycle.*

Proposition 2.3 *Let G be a k-split-critical graph, with $k \geq 3$. Then G has at most k connected components.*

Proof: Suppose that G has $k+1$ connected components $C^1, \ldots, C^{k+1}$. Let $v \in C^{k+1}$. Then $G - v$ is $(k-1)$-split-colorable. From Proposition 2.1, it follows that every connected component C of G satisfies $\chi(C) \geq k$. Now if there is a connected component C^i, $i \in \{1, \ldots, k\}$, such that the vertices of C^i can be colored using only stable sets in a $(k-1)$-split-coloring of $G - v$, we would need at least k stable sets, which contradicts the fact that $G - v$ is $(k-1)$-split-colorable. Thus the vertices of each connected component $C^1, \ldots, C^k$ are colored using at least one clique in every $(k-1)$-split-coloring of $G - v$. But this means that we have at least k disjoint cliques (hence k different colors) which contradicts the fact that $G - v$ is $(k-1)$-split-colorable. □

Proposition 2.4 *Let G be a k-split-critical graph, with $k \geq 3$. If G contains exactly $k-1$ connected components $C^1, \ldots, C^{k-1}$, then there exists C^i, $i \in \{1, \ldots, k-1\}$, which is k-critical.*

Proof: From Proposition 2.1 we know that each connected component C^i satisfies $\chi(C^i) \geq k$. Suppose for a contradiction that no C^i is k-critical, i.e., for every $i = 1, \ldots, k-1$ there exists v_i such that $\chi(C^i - v_i) \geq k$. Since $G - v_1$ is $(k-1)$-split-colorable, denote by $(K^1, \ldots, K^{k-1}; S^1, \ldots, S^{k-1})$ a

$(k-1)$-split-coloring of $G - v_1$. Since $\chi(C^1 - v_1) \geq k$, with an appropriate labeling we have without loss of generality $V(K^1) \subseteq V(C^1)$. By repeating this argument for a vertex $v_2 \in V(C^2)$, we deduce that each connected component is $(k-1)$-split-colorable by partitioning its vertex set into exactly one clique and at most $k-1$ stable sets. But then G is $(k-1)$-split-colorable, which is a contradiction. So there exists at least one connected component C^i, $i \in \{1, \dots, k-1\}$, which is k-critical. □

As an immediate consequence we get the following result.

Corollary 2.5 *Let G be a 3-split-critical graph. If G contains exactly two connected components, then one of them is an odd cycle.*

Proof: This follows from Proposition 2.4 and from the fact that the only 3-critical graphs are odd cycles. □

Now using a similar argument as in the proof of Proposition 2.4, one can show the following.

Proposition 2.6 *Let G be a k-split-critical graph, with $k \geq 3$. If G contains exactly k connected components $C^1, \dots, C^k$, then each of them is k-critical.*

Corollary 2.7 *A graph G is 3-split-critical with 3 connected components if and only if each connected component is an odd cycle.*

We will give now a necessary and sufficient condition for a graph to be k-split-critical, with $k \geq 3$. First we mention an easy observation which we will need in our proof.

Proposition 2.8 *If G is k-split-critical, then G admits a k-split-coloring using exactly k cliques (resp. k stable sets) (with one of them of size one) and at most $k-1$ stable sets (resp. at most $k-1$ cliques), with $k \geq 2$.*

Proof: Suppose that there is a k-split-coloring of G using exactly k cliques and k stable sets. Now consider a vertex $v \in V(G)$. By criticality of G we know that $G - v$ is $(k-1)$-split-colorable. By considering v as a clique (resp. a stable set) we obtain a k-split-coloring of G with at most $k-1$ stable sets (resp. $k-1$ cliques). □

Proposition 2.9 *A graph G is k-split-critical if and only if $G' = (G \oplus S_k) \cup K_{k+1}$ is $(k+1)$-split-critical, with $k \geq 2$.*

Proof: First assume that G is k-split-critical. Then it follows from Proposition 2.8 that there exists a k-split-coloring of G using k cliques and at most $k-1$ stable sets. This can be extended to a $(k+1)$-split-coloring of G' by adding the clique K_{k+1} and the stable set S_k; hence $\chi_S(G') \leq k+1$. Suppose $\chi_S(G') < k+1$, then there is a k-split-coloring of G' using at most k stable sets, which can cover at most k vertices of K_{k+1}. Consequently, one clique which is completely in K_{k+1} is used. The remaining (at most) $k-1$ cliques can cover at most $k-1$ vertices of S_k; hence one stable set which is completely in S_k is used. It follows that G is $(k-1)$-split-colorable, a contradiction. Thus, $\chi_S(G') = k+1$.

Let us show now that for all $v \in G'$, we have $\chi_S(G' - v) = k$. If $v \in G$, then take a $(k-1)$-split-coloring of $G - v$, add clique K_{k+1} and stable set S_k to obtain a k-split-coloring of $G' - v$. If $v \in S_k$ then take a k-split-coloring of G with k stable sets and at most $k-1$ cliques (such a k-split-coloring exists by Proposition 2.8). This split-coloring can be extended to cover all vertices of $S_k - v$ and all vertices of K_{k+1} but one; take this uncovered vertex as a new clique, giving a k-split-coloring of $G' - v$. Lastly, if $v \in K_{k+1}$, the same k-split-coloring of G can be extended to cover all vertices of $K_{k+1} - v$ and all vertices of S_k but one; take this uncovered vertex as a new clique to obtain a k-split-coloring of $G' - v$.

Now let us assume that G' is $(k+1)$-split-critical. If $\chi_S(G) < k$ then take a $(k-1)$-split-coloring of G, add clique K_{k+1} and stable set S_k to obtain a k-split-coloring of G', a contradiction. Therefore $\chi_S(G) \geq k$. Moreover G' is $(k+1)$-split-critical and G is an induced subgraph of G'; thus $\chi_S(G) = k$. Let us show now that for all $v \in G$, we have $\chi_S(G - v) = k - 1$. For $v \in G$, take a k-split-coloring of $G' - v$; it contains at most k stable sets which can cover at most k vertices of K_{k+1}; hence it has a clique completely contained in K_{k+1}. The remaining (at most) $k-1$ cliques can cover at most $k-1$ vertices of S_k; hence it has a stable set completely contained in S_k. The remaining (at most) $k-1$ cliques and (at most) $k-1$ stable sets give a $(k-1)$-split-coloring of $G - v$; this concludes the proof. $\square$

A construction of k-split-critical graphs for $k \geq 2$ follows from Proposition 2.9. Take one of the three 2-split-critical graphs, that is, $2K_2, C_4$ or C_5, and apply recursively $k-2$ times the transformation described in Proposition 2.9 to obtain a k-split-critical graph. Note that all graphs for $k > 2$ obtained in this way have two connected components.

The following result gives a construction of triangle-free k-split-critical graphs with $p \leq k$ connected components.

Theorem 2.10 *For every p, $1 \leq p \leq k$, there exists a k-split-critical triangle-free graph $G_{p,k}$ with p connected components for $k \geq 3$.*

Proof: We know that the Mycielski graph M_k of order k is a triangle-free k-critical graph [Myc95]. $G_{p,k}$ for $p = 1, \ldots, k$ is obtained as follows. If $p \neq k$, we take a cycle C with edges $e_1, e_2, \ldots, e_l$, where $l = 4(k-p) + 2$, such that every even indexed edge belongs to a distinct copy of M_k. We also add $(p-1)M_k$ (see Figure 1 for an example, where $p = 3$ and $k = 5$). If $p = k$, we simply take kM_k. Clearly, $G_{p,k}$ has p connected components.

Notice that $G_{p,k}$ is not $(k-1)$-split-colorable, otherwise every Mycielski graph must have at least one vertex in a clique of that $(k-1)$-split-coloring. But clearly we need at least k cliques to do so, a contradiction. Thus $\chi_S(G_{p,k}) \geq k$. Next we show that $G_{p,k}$ is k-split-colorable by taking as cliques (if $p \neq k$) one vertex from each M_k in the subgraph $(p-1)M_k$, edges e_i, $i = 1, 5, 9, \ldots, l-5$, and the vertex v_{l-1} which is the common endpoint of e_{l-2} and e_{l-1}, summing up to $(p-1) + k - p + 1 = k$ cliques (if $p = k$ take simply k cliques consisting of one vertex from each M_k); then the remaining graph (after removal of these cliques) is $(k-1)$-colorable by criticality of the M_k's.

Moreover, for every vertex v, $G_{p,k} - v$ is $(k-1)$-split-colorable. This clearly holds for kM_k (in case $p = k$) by criticality of the M_k's. Now, assume that $p \neq k$. Indeed, if v belongs to one of the $p-1$ disjoint copies of M_k, then clearly we need one clique less to split-color $G_{p,k} - v$ and thus $\chi_S(G_{p,k}) = k-1$. Now if v does not belong to any of the $p-1$ disjoint copies of M_k, then we may assume w.l.o.g. that v belongs to the Mycielski graph M_k such that $v_{l-1} \in M_k$. Now in $G_{p,k} - v$, take the same cliques as before except v_{l-1}. These $k-1$ cliques of $G_{p,k} - v$ touch (i.e., contain at least one vertex of) all

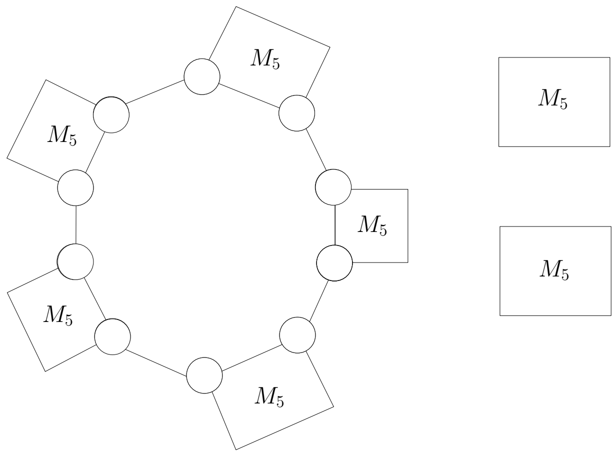

Fig. 1: The graph $G_{3,5}$.

induced M_k's; again by criticality, the remaining graph is $(k-1)$-colorable. It follows that $G_{p,k} - v$ is $(k-1)$-split-colorable. □

Note that if triangle-freeness is not required, any k-critical graph can play the role of M_k in Theorem 2.10. Finally let us observe the following result.

Proposition 2.11 *Let G be a k-critical graph. Then $\chi_S(G) \leq k-1$.*

Proof: Suppose G is k-critical and $\chi_S(G) \geq k$. It means that for any vertex v, $G-v$ is $(k-1)$-colorable. But then there is a $(k-1)$-split-coloring of G using one clique (vertex v) and $(k-1)$ stable sets, a contradiction. □

Note that the cochromatic counterpart of Proposition 2.11 does not hold; that is, there are k-critical graphs which are also critically k-cochromatic. In fact, it is shown in [LS77] that every triangle-free graph G with at least 3 vertices has $\chi(G) = z(G)$; hence every triangle-free k-critical graph is also critically k-cochromatic; take for instance the Mycielski graphs [Myc95]. Also, in [BB89], for all $k \geq 4$, a construction of (not triangle-free) graphs which are both k-critical and critically k-cochromatic is described.

3 Uniquely split-colorable graphs

In this section we give some properties of uniquely split-colorable graphs. We recall that a graph G is uniquely split-colorable if every optimal split-coloring of G has the same collection of cliques and stable sets. Let us first notice the following result whose cochromatic counterpart is also true [GS87].

Lemma 3.1 *If G is uniquely k-split-colorable then G is not k-split-critical.*

Proof: Suppose there is a graph G which is uniquely k-split-colorable and also k-split-critical. By criticality, we have that for every vertex v, there is a k-split-coloring of G where v is a color class by itself. No two such k-split-colorings are the same. Indeed, if two such k-split-colorings are the same, it follows that there exist two vertices, say v and w, such that each of them is a color class by itself. If v and w are non-adjacent (resp. adjacent), we would get a $(k-1)$-split-coloring by considering $\{v, w\}$ as a stable set (resp. vw as a clique). Thus we get n different optimal split-colorings of G (where n is the number of vertices in G), contradicting the uniqueness of the split-coloring. $\square$

The next result gives some necessary conditions for a graph to be uniquely split-colorable.

Proposition 3.2 *Let G be a uniquely k-split-colorable graph and let $\mathcal{U}$ be its unique partition into stable sets and cliques. Then the following conditions are satisfied:*

(1) the union of any two stable sets in $\mathcal{U}$ is connected;

(2) the union of any two cliques in $\mathcal{U}$ is anticonnected, i.e., connected in the complement;

(3) for every $v \in K$, where K is a clique of $\mathcal{U}$, v has a neighbor in every stable set of $\mathcal{U}$;

(4) for every $w \in S$, where S is a stable set of $\mathcal{U}$, w has a non-neighbor in every clique of $\mathcal{U}$;

(5) the number of stables sets in $\mathcal{U}$ is equal to the number of cliques in $\mathcal{U}$.

Proof: The first two conditions respectively follow from the fact that the subgraph induced by the stable sets in $\mathcal{U}$ and the complement of the subgraph induced by the cliques in $\mathcal{U}$ have unique partitions. The third and forth conditions express that no vertex can be moved from a clique to a stable set or vice versa. The fifth condition is trivially necessary. $\square$

In Figure 2, we can observe that the five conditions given in Proposition 3.2 are not sufficient. The 2-split-coloring $(\{a, d, 4\}, \{f, 3, 6\}; \{5, 1, b\}, \{e, 2, c\})$ of this graph satisfies the five conditions of Proposition 3.2 but the graph is not uniquely split-colorable; another 2-split-coloring is obtained by taking the sugbraph induced by vertices a, b, c, d, e, f as one split graph, and the subgraph induced by vertices $1, 2, 3, 4, 5, 6$ as the second split graph.

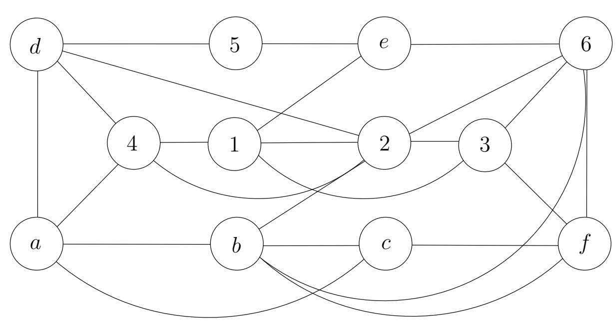

Fig. 2: The five conditions in Proposition 3.2 are not sufficient.

We note with the following proposition that conditions (3) and (4) of Proposition 3.2 are also sufficient for a graph G such that $\chi_S(G) = 1$, i.e., for a split graph.

Proposition 3.3 *Let $G = (V, E)$ be a split graph. Then G has a unique split-partition if and only if its vertices can be partitioned into a clique K and a stable set S such that*

(1) for every $v \in K$, v has a neighbor in S;

(2) for every $w \in S$, w has a non-neighbor in K.

Proof: From Proposition 3.2 we immediately deduce the necessity of both conditions. Let us show now that (1) and (2) are also sufficient conditions. Let (K^1, S^1) be a split-partition of G for which (1) and (2) are satisfied, and let us assume that there exists another split-partition, denoted by (K^2, S^2), which is different from (K^1, S^1). Then clearly $|V(K^1) \setminus V(K^2)| = |V(K^2) \setminus V(K^1)| = 1$. So let $\{v\} = V(K^1) \setminus V(K^2)$ and $\{v'\} = V(K^2) \setminus V(K^1)$. From (1) we know that v must have a neighbor in S^1. But this neighbor is necessarily vertex v', since otherwise we contradict the fact that S^2 is a stable set. So $vv' \in E$. But now we have a contradiction with (2) since v, which belongs to S^2, is adjacent to all vertices in K^2. □

Now, let us describe a construction of graphs U_k which are uniquely k-split-colorable for $k \geq 2$.

Construction: For any $k \geq 2$, we construct a graph U_k as follows. First, take k graphs $G_1, \ldots, G_k$ where each G_i is isomorphic to a split graph with unique split-partition (K^i_{k+1}, S^i_{k+1}) where K^i_{k+1} is a clique on vertices $v^i_1, \ldots, v^i_{k+1}$ and S^i_{k+1} is a stable set on vertices $u^i_1, \ldots, u^i_{k+1}$, and the only edges between K^i_{k+1} and S^i_{k+1} are $v^i_1 u^i_1, v^i_2 u^i_2, \ldots, v^i_{k+1} u^i_{k+1}$. Now, we also add for $i, i' = 1, \ldots, k$ and $j = 1, \ldots, k+1$, edges $v^i_j u^{i'}_j$ with $i' \neq i$ and all possible edges between every pair of stable sets S^i_{k+1}, S^j_{k+1}, $i \neq j$ (hence $S^1_{k+1}, \ldots, S^k_{k+1}$ induce a complete k-partite graph). See Figure 3 for an example with $k = 2$.

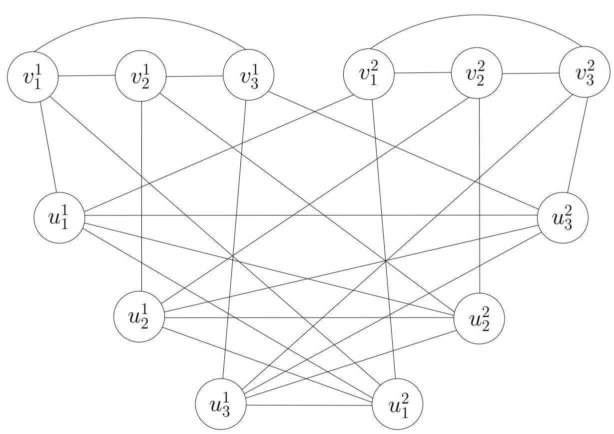

Fig. 3: The graph U_2.

Theorem 3.4 *For $k \geq 2$, U_k is uniquely k-split-colorable.*

Proof: A k-split coloring $\mathcal{U}$ of U_k is obtained by taking each G_i as a split graph. Assume there exists a different k-split-coloring $\mathcal{U}'$ of U_k. Now, every clique in $\mathcal{U}$ contains at least one vertex in some clique of $\mathcal{U}'$ since otherwise we would need $k+1$ stable sets to cover all vertices of some clique K^i_{k+1} of $\mathcal{U}$. Thus each one of the k cliques in $\mathcal{U}'$ contains at least one vertex from a clique K^i_{k+1} of $\mathcal{U}$. Moreover, there is at least one clique K in $\mathcal{U}'$ containing some vertices of K^i_{k+1} for some i, say K^I_{k+1}, which also contains at least one vertex from a stable set of $\mathcal{U}$. Indeed if all cliques of $\mathcal{U}'$ are subsets of K^i_{k+1}, $i = 1, \dots, k$, with at least one proper subset (since $\mathcal{U}$ is different from $\mathcal{U}'$) then k stable sets cannot cover the remaining vertices (the only way to partition vertices $u^i_j, i = 1, \dots, k, j = 1, \dots, k+1$, into stable sets is to take $S^i_{k+1}, i = 1, \dots, k$, which does not include any vertex from the cliques K^i_{k+1}, $i = 1, \dots, k$). Now we can assume without loss of generality that the clique K is maximal and hence is formed by $v^I_J, u^1_J, u^2_J, \dots, u^k_J$ for some $J \in \{1, \dots, k+1\}$. Indeed, if K is not maximal then one can make it maximal by adding new vertices, and this would still give us a different k-split-coloring than $\mathcal{U}$. Also, as already pointed out, the remaining $k-1$ cliques of $\mathcal{U}'$ contain at least one vertex from each clique K^i_{k+1}, $i = 1, \dots, k$, $i \neq I$. Then, since K is different from all cliques in $\mathcal{U}$ (and thus $\mathcal{U}'$ is different from $\mathcal{U}$), we may assume without loss of generality that each of these cliques (except K) is either K^i_{k+1}, for some $i \neq I$, or formed by one vertex in K^i_{k+1}, for some $i \neq I$, and one vertex from each stable set S^J_{k+1} for $J = 1, \dots, k$. Notice that $((U_k - V(K)) - \cup_{i \neq I} V(K^i_{k+1}))$ contains k vertex disjoint cliques formed by one vertex in K^I_{k+1} different from v^I_J and one vertex from each stable set S^J_{k+1}, $J = 1, \dots, k$; at least one of these cliques of size $k+1$ remains untouched (i.e., none of its vertices is contained in some clique of $\mathcal{U}'$). Clearly the vertices of this clique cannot be partitioned into k stable sets. Therefore, $\mathcal{U}'$ cannot be a k-split-coloring. $\Box$

4 Adding/deleting edges

In [GS87], the variation of the cochromatic number when we remove an edge is studied. Here, instead of a single edge, we first consider the removal of the edges of a C_4, $2K_2$ or a C_5 since these are the only minimal forbidden induced subgraphs for split graphs. Let G be a graph and let H be an induced subgraph of G. We denote by $V(H)$ (resp. $E(H)$) the vertex set (resp. edge set) of H, and by $G - E(H)$ the graph obtained from G by deleting the edges of H. Let H' be a graph and let $\mathcal{S}$ be the set of stable sets in G of size $|V(H')|$. We denote by $G + E(H')$ the graph obtained from G by adding the edges of a graph isomorphic to H' using the vertices of some $S \in \mathcal{S}$. If two vertices u, v are not adjacent in a graph G, we say that $e = uv$ is a *missing edge* of G.

Proposition 4.1 *Let $G = (V, E)$ be a graph and let H be an induced subgraph of G isomorphic to either C_4, $2K_2$ or C_5. Then $\chi_S(G) - 1 \leq \chi_S(G - E(H)) \leq \chi_S(G) + 1$.*

Proof: Consider an optimal split-coloring of G. Now in $G - E(H)$, where H is isomorphic to either C_4 or $2K_2$ or C_5, $V(H)$ forms a stable set. Thus by introducing a new color which corresponds to this stable set, we get a feasible split-coloring of $G - E(H)$ with $\chi_S(G) + 1$ colors and hence $\chi_S(G - E(H)) \leq \chi_S(G) + 1$.

Consider now $G - E(H)$ with H being isomorphic to a C_5 (the proofs are similar for the case where H is isomorphic to C_4 or $2K_2$). Let $E(H) = \{ab, bc, cd, de, ae\}$. Suppose that $G - E(H)$ admits a split-coloring $\mathcal{U}$ using at most $\chi_S(G) - 2$ colors. Then we obtain a $(\chi_S(G) - 1)$-split-coloring of G by first uncoloring vertices a, b and d and by adding to $\mathcal{U}$ one new color consisting of the stable set $\{d\}$ and the clique $\{ab\}$, a contradiction. Thus $\chi_S(G) - 1 \leq \chi_S(G - E(H))$. □

Proposition 4.2 *Let $G = (V, E)$ be a graph and let H be a graph isomorphic to either C_4, $2K_2$ or C_5. Then $\chi_S(G) - 1 \leq \chi_S(G + E(H)) \leq \chi_S(G) + 1$.*

Proof: First notice that $\chi_S(G) = \chi_S(\overline{G})$, where $\overline{G}$ denotes the complement of G. Indeed, this follows from the fact that the complement of a clique is a stable set and vice versa. Next observe that $\overline{G + E(H)} = \overline{G} - E(H)$. Thus $\chi_S(G + E(H)) = \chi_S(\overline{G + E(H)}) = \chi_S(\overline{G} - E(H))$. It follows from Proposition 4.1 that $\chi_S(\overline{G}) - 1 \leq \chi_S(G + E(H)) \leq \chi_S(\overline{G}) + 1$. Now we conclude using the fact that $\chi_S(G) = \chi_S(\overline{G})$. □

In what follows we will concentrate on the properties of adding any missing edge or removing any existing edge in a graph. A vertex u of a graph G is called *universal* if u is adjacent to all the vertices in $G - u$.

Proposition 4.3 *Let G be a graph with no universal vertex and such that $\chi_S(G + e) = \chi_S(G) - 1$ for any missing edge e in G. Then G is k-split-critical, where $k = \chi_S(G)$.*

Proof: Let u, v be two non-adjacent vertices in G. Consider the missing edge $e = vu$. $G - v$ is an induced subgraph of $G + e$, hence $\chi_S(G - v) \leq \chi_S(G + e) = \chi_S(G) - 1$. Since G has no universal vertex, this is true for every vertex $v \in V(G)$. □

Since $\chi_S(G) = \chi_S(\overline{G})$ and $\overline{G + e}$ is isomorphic to $\overline{G} - e$, Proposition 4.3 can also be expressed as follows: if G has no universal vertex and if G is such that the removal of any edge decreases $\chi_S(G) = k$ then G is k-split-critical. This result is in parallel with its cochromatic counterpart [GS87]. We also note that the converse of Proposition 4.3 is not true as can be observed in Figure 4.

One can easily observe the following since the only 2-split-critical graphs are $2K_2, C_4$ and C_5.

Corollary 4.4 *A graph G with $\chi_S(G) = 2$ is such that the addition of any missing edge decreases its split-chromatic number if and only if it is isomorphic to a $2K_2$, a C_4 or it is obtained from $2K_2$ or C_4 by adding universal vertices.*

Unlike to its cochromatic counterpart, we notice the following.

Proposition 4.5 *There is no graph G such that addition of any missing edge increases $\chi_S(G)$.*

Proof: Suppose G is such that the addition of any missing edge increases $\chi_S(G) = k$. This implies that for every pair of non-adjacent vertices u, v in G, all k-split-colorings of G have a stable set containing both u and v. Indeed, if there exists a k-split-coloring $\mathcal{U}$ of G in which u, v do not belong to the same stable set, then $\mathcal{U}$ is also a feasible k-split-coloring of $G + e$, a contradiction. It follows that G is $\overline{P}_3$-free,

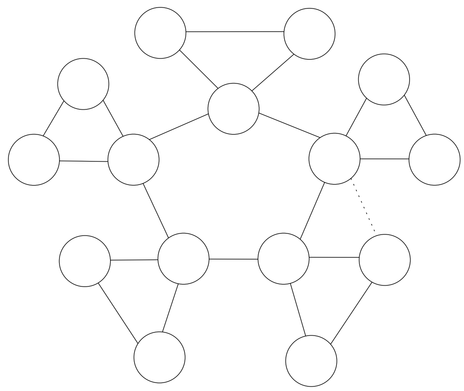

Fig. 4: A connected 3-split-critical graph where adding the dotted missing edge does not decrease χ_S.

where P_3 is an induced path on three vertices. In fact, if x_1, x_2, x_3 induce a $\overline{P}_3$ with x_1 adjacent to x_2, then x_1, x_3 and x_2, x_3 are in the same stable set in all k-split-colorings by the previous observation, a contradiction since x_1 is adjacent to x_2. Thus G is necessarily a complete k-partite graph. But now one can see that adding any missing edge $e = uv$ does not increase $\chi_S(G)$. Indeed we get a k-split-coloring of $G + e$ by taking u, v as a clique and partitioning the remaining vertices into k stable sets. □

Let $G = (V, E)$ be a graph with $\chi_S(G) = k$ and such that $\chi_s(G + e) = k - 1$ for any missing edge e in G and $\chi_S(G - e') = k - 1$ for any edge $e' \in E$. Then G is called *split-decreasing*.

The following result gives a construction of split-decreasing graphs.

Theorem 4.6 *For every p, $1 \leq p \leq k$, there exists a split-decreasing graph $G_{k,p} = (V, E)$ with p connected components for $k \geq 2$ and $\chi_S(G_{k,p}) = k$. This graph $G_{k,p}$ is isomorphic to $(p-1)K_k \cup \overline{kK_{k-p+1}}$.*

Proof: Consider the graph $G_{k,p}$, for $1 \leq p \leq k$, as described above. Denote by $K_k^1, \ldots, K_k^{p-1}$ the cliques of the subgraph $(p-1)K_k$. Clearly $\chi_S(G_{k,p}) = k$. Indeed, $\chi_S(G_{k,p}) \leq k$ since k cliques can partition $G_{k,p}$. Moreover, $\chi_S(G_{k,p}) > k - 1$ since $k - 1$ stable sets leave uncovered at least one vertex from each K_k^i, $i = 1, \ldots, p - 1$, which induce a stable set of size k that cannot be partitioned with $k - 1$ cliques.

Now assume we add a missing edge e to $\overline{kK_{k-p+1}}$. Notice that $\overline{kK_{k-p+1}}$ is a complete k-partite graph, each of the stable sets $S^1, \ldots, S^k$ having size $k - p + 1$. Denote by $u_i^1, \ldots, u_i^{k-p+1}$ the vertices of stable set S^i, $i = 1, \ldots, k$. We may assume that $e = u_1^1 u_1^2$. Then we get a $(k-1)$-split coloring of $G_{k,p} + e$ by taking $p - 1$ cliques $K_k^1, \ldots, K_k^{p-1}$, a clique corresponding to e as well as $k - p - 1$ cliques each of them being a vertex of $S^1 - \{u_1^1, u_1^2\}$, and by taking $k - 1$ stable sets $S^2, \ldots, S^k$.

If we add a missing edge $e = uv$ between two cliques, say K_k^1 and K_k^2 with $u \in K_k^1$ and $v \in K_k^2$, then we obtain a $(k-1)$-split-coloring of $G_{k,p} + e$ by taking $p - 3$ cliques $K_k^3, \ldots, K_k^{p-1}$, a clique corresponding to edge e as well as $k - p + 1$ cliques in $\overline{kK_{k-p+1}}$ (each such clique is induced by $u_1^j, \ldots, u_k^j$, for $j = 1, \ldots, k - p + 1$), and finally $k - 1$ stable sets, each of them corresponding to two vertices, one in $K_k^1 - u$ and the other in $K_k^2 - v$.

Suppose we add an edge e between a clique, say K_k^1, and $\overline{kK_{k-p+1}}$. We may assume that $e = vu_1^1$, where $v \in K_k^1$. Then take $p-2$ cliques $K_k^2, \dots, K_k^{p-1}$, a clique corresponding to edge e as well as $k-p$ cliques each corresponding to one vertex of $S^1 - u_1^1$, and $k-1$ stable sets $S^2, \dots, S^k$, each one together with one vertex from $K_k^1 - v$. This gives us a $(k-1)$-split coloring of $G_{k,p} + e$.

If we delete an edge $e = uv$ from a clique, say K_k^1, then a $(k-1)$-split coloring of $G_{k,p} - e$ is obtained by taking $p-2$ cliques $K_k^2, \dots, K_k^{p-1}$, $k-p+1$ cliques partitioning $\overline{kK_{k-p+1}}$, and $k-1$ stable sets, one corresponding to the vertices u, v, and $k-2$ corresponding each to a different vertex of $K_k^1 - \{u, v\}$.

Finally the last case to consider is when we delete an edge e from $\overline{kK_{k-p+1}}$. W.l.o.g we may assume that $e = u_1^1 u_2^1$. Then by taking $p-1$ cliques $K_k^1, \dots, K_k^{p-1}$, $k-p$ cliques of $\overline{kK_{k-p+1}}$ (each clique is induced by vertices $u_1^j, \dots, u_k^j$ for $j = 2, 3, \dots, k-p+1$), and $k-1$ stable sets, one corresponding to vertices u_1^1, u_1^2 and $k-2$ corresponding each to a vertex u_i^1, $i = 3, \dots, k$, we obtain a $(k-1)$-split coloring of $G_{k,p} - e$. □

We also observe the following two properties.

Proposition 4.7 *A graph G is split-decreasing if and only if $\overline{G}$ is split-decreasing.*

Proof: This follows from the facts that $\chi_S(G+e) = \chi_S(G) - 1 = \chi_S(\overline{G}) - 1$, and $\chi_S(G+e) = \chi_S(\overline{G+e}) = \chi_S(\overline{G} - e)$. □

Proposition 4.8 *Let $G = (V, E)$ be a graph with k connected components and $\chi_S(G) = k$, for $k \geq 2$. Then G is split-decreasing if and only if G is isomorphic to kK_k.*

Proof: From Theorem 4.6 we know that if G is isomorphic to kK_k, then G is split-decreasing ($G_{k,k}$ in Theorem 4.6 is isomorphic to kK_k).

Suppose now that G has k connected components $C^1, \dots, C^k$, $\chi_S(G) = k$, and G is split-decreasing. Clearly if C^i is a clique for $i = 1, \dots, k$, then each C^i has size at least k (otherwise $\chi_S(G) < k$). Moreover, if there exist C^i isomorphic to $K_{k'}$, $k' > k$, then $G - e$, where $e \in E(K_{k'})$, still contains kK_k as a subgraph, hence $\chi_S(G-e) \geq k$, and thus G is not split-decreasing, a contradiction. Now let us assume that there is at least one connected component, say C^1, which is not a clique. Then $uv \notin E$ for some $u, v \in V(C^1)$. Since G is k-split-critical (see Proposition 4.3), we deduce from Proposition 2.1 that $\chi(C^i) \geq k$. Now consider $G + uv$. Since $\chi_S(G+uv) = k-1$, we deduce that in any $(k-1)$-split-coloring we need exactly one clique in each component $C^2, \dots, C^k$. Furthermore uv must be in a clique. In fact if uv is not in a clique then the $(k-1)$-split-coloring of $G + uv$ is also a $(k-1)$-split-coloring of G, a contradiction. But then we get k cliques in the $(k-1)$-split-coloring of $G + uv$, which is also a contradiction. Thus all connected components are cliques of size k. □

We will now concentrate on graphs G with at least two connected components that are split-decreasing and have split-chromatic number $\chi_S(G) = 3$. Since these graphs are 3-split-critical (see Proposition 4.3; clearly G has no universal vertex and there necessarily exist two non-adjacent vertices), it follows from Proposition 2.3 that they have at most three connected components. Proposition 4.8 gives a caracterization of all 3-split-colorable graphs that are split-decreasing with three connected components. In fact, there is a unique such graph: each connected component is isomorphic to K_3. In what follows we will study the case of 3-split-colorable graphs that are split-decreasing and have exactly two connected components.

Lemma 4.9 *Let G be a graph with two connected components and $\chi_S(G) = 3$. If G is split-decreasing, then one of its connected components is isomorphic to K_3.*

Proof: Since G has no universal vertex, we can apply Proposition 4.3 and thus it follows that G is 3-split-critical. Corollary 2.5 implies that one of the two connected components is an induced odd cycle C. Clearly, C cannot have length ≥ 5, since adding a missing edge (a chord) between two vertices of C will not decrease $\chi_S(G)$. In fact, in that case we would still need either one clique and two stable sets or two cliques to cover the vertices of C in any split-coloring. Thus one connected component is isomorphic to K_3. □

In what follows, if G is a split-decreasing graph with two connected components and $\chi_S(G) = 3$, we denote by C^1, C^2 the two connected components of G. Moreover, from Lemma 4.9, we may assume that C^1 is isomorphic to a K_3.

The following two lemmas represent the main tool that we will use in order to prove the main result of this section. We will repeatedly refer to these lemmas in the remaining of the paper.

Lemma 4.10 *Let G be a split-decreasing graph with two connected components and $\chi_S(G) = 3$. Then $C^2 + e$, where $e = xy$ is a missing edge in C^2, contains a clique K such that $x, y \in V(K)$ and $(C^2 + e) - V(K)$ is bipartite (not reduced to a stable set).*

Proof: Since G is split-decreasing, $G + e$ is 2-split-colorable. In addition, any 2-split-coloring of $G + e$ uses two cliques and two stable sets, otherwise G would be 2-split-colorable. Clearly one clique of any 2-split-coloring of $G + e$ must be contained in C^1. Thus, the other clique, denoted by K, is necessarily such that $(C^2 + e) - V(K)$ is bipartite. Moreover, K contains the new edge $e = xy$. In fact, if K does not contain e, then the 2-split-coloring of $G + e$ is also a feasible 2-split-coloring of G, a contradiction. Furthermore $(C^2 + e) - V(K)$ is not a stable set, since any 2-split-coloring of $G + e$ must use two cliques and two stable sets (as mentioned above). □

Lemma 4.11 *Let G be a split-decreasing graph with two connected components and $\chi_S(G) = 3$. Then $C^2 - e$, where $e = xy$ is an edge in C^2, contains a stable set S such that $x, y \in V(S)$ and $(C^2 - e) - V(S)$ is a split graph (which is not reduced to a clique or a stable set).*

Proof: Since G is split-decreasing, $G - e$ is 2-split-colorable. In addition, any 2-split-coloring of $G - xy$ uses two cliques and two stable sets, otherwise G would be 2-split-colorable. Clearly one clique of any 2-split-coloring of $G - e$ must be contained in C^1. Then $C^2 - e$ can be partitioned into one clique and two stable sets. Suppose none of the two stable sets contains both x and y. Then the 2-split coloring of $G - e$ is also a feasible 2-split-coloring of G, a contradiction. So there is a stable set S containing both x and y such that $(C^2 - e) - V(S)$ is a split graph. Moreover, $(C^2 - e) - V(S)$ cannot be a clique or a stable set, since this would mean that G is 2-split-colorable, a contradiction. □

Let C be an induced cycle of a graph G. Let $v_1, \ldots, v_p$ be the vertices of C in clockwise order. A vertex $v \in V(G)$ is called a *center of* C if v is adjacent to all the vertices of C. A vertex $w \in V(G) \setminus V(C)$ is called a *clone of some vertex* $v_i \in V(C)$ *with respect to* C if w is adjacent to v_{i-1}, v_{i+1} and not adjacent to $V(C) \setminus \{v_{i-1}, v_{i+1}\}$; we will simply say *clone of* v_i whenever no confusion on the cycle is possible. Denote by $N(v)$ the set of neighbors of v, and by $\overline{N(v)}$ the set of non-neighbors of v.

We first obtain the following result.

Theorem 4.12 *Let G be a split-decreasing graph with two connected components and $\chi_S(G) = 3$. Then G does not contain any induced cycle of length $2k+1$, $k \geq 2$.*

Proof: Suppose for a contradiction that C^2 contains an induced cycle C of length $2k+1$, $k \geq 2$. Let $v_1, v_2, \ldots, v_{2k+1}$ be the vertices of C in clockwise order. We will prove several claims.

Claim (i) *Let $v \in V(C^2) \setminus V(C)$ such that $N(v) \cap V(C) \neq \emptyset$. Then v has exactly one of the following properties.*

(1) $k = 2$, and v is a center of C;

(2) v is a clone of some vertex $v_i \in V(C)$, $i \in \{1, \ldots, 2k+1\}$;

(3) v is adjacent to exactly one vertex in $V(C)$.

Let $v \in V(C^2) \setminus V(C)$ such that $N(v) \cap V(C) \neq \emptyset$. Suppose that v is not a center of C.

Suppose that v is adjacent to two consecutive vertices of C, say v_1, v_2. It follows from Lemma 4.10 that if we add the edge v_3v_{2k+1}, then there exists a clique K in $C^2 + v_3v_{2k+1}$ containing v_3 and v_{2k+1}, and such that $(C^2 + v_3v_{2k+1}) - V(K)$ is bipartite. This implies that v must be adjacent to both v_3 and v_{2k+1} otherwise K does not contain any of the vertices v, v_1, v_2 and hence $(C^2 + v_3v_{2k+1}) - V(K)$ contains a K_3 induced by v, v_1, v_2, a contradiction. Now we may repeat this argument, by applying Lemma 4.10 to the edge v_1v_4, and we conclude that v must be adjacent to vertex v_4. Continuing this procedure we conclude that v must be adjacent to all the vertices in $V(C)$. But now v is a center of C, a contradiction. It follows that v is not adjacent to two consecutive vertices in $V(C)$. This proves the claim for the case $k = 2$. Thus we may assume now that $k \geq 3$.

Suppose that v is adjacent to at least two vertices in $V(C)$, such that v is not a clone of some vertex of $V(C)$, and v is not adjacent to two consecutive vertices in $V(C)$. Since C has odd length, there necessarily exist two neighbors of v in $V(C)$, say v_i, v_j such that v is non-adjacent to all the vertices in one of the two paths in C joining v_i and v_j, and such that this path together with edges vv_i, vv_j is an induced odd cycle C'. Now consider $x, y \in V(C) \setminus V(C')$, with x being adjacent to v_i and y being adjacent to v_j. Notice that $x \neq y$, otherwise v is a clone of x (resp. y), a contradiction. Furthermore x and y are not adjacent, otherwise C has even length, a contradiction. It follows from Lemma 4.10 that if we add the edge xy, there exists a clique K in $C^2 + xy$ containing both x and y, and such that $(C^2 + xy) - V(K)$ is bipartite. But clearly K does not contain any vertex from $V(C')$. Thus $(C^2 + xy) - V(K)$ contains C', a contradiction. This proves for $k \geq 3$ that if v is a vertex in $V(C^2) \setminus V(C)$ such that $N(v) \cap V(C) \neq \emptyset$ and v is not a center, then either $|N(v) \cap V(C)| = 1$ or v is a clone of some vertex $v_i \in V(C)$.

Finally we need to prove that if $k \geq 3$, v is not a center of C. Suppose for a contradiction that v is a center of C. It follows from Lemma 4.11 that if we delete the edge vv_1, there exists a stable set S in $C^2 - vv_1$ containing both v and v_1, and such that $(C^2 - vv_1) - V(S)$ is a split graph. Since v is a center of C, it follows that $v_2, \ldots, v_{2k+1} \notin S$. But now $v_2, v_3, v_{2k}, v_{2k+1}$ induce a $2K_2$, a contradiction. Thus C does not have a center if $k \geq 3$. This proves Claim (i).

Claim (ii) *If $k = 2$ and there exists a center v of C, then v is not adjacent to any vertex in $V(C^2) \setminus V(C)$.*

Let v be a center of C, and suppose that v is adjacent to some vertex $w \in V(C^2) \setminus V(C)$. It follows from Lemma 4.11 that if we delete the edge vw, there exists a stable set S in $C^2 - vw$ containing both v and w, and such that $(C^2 - vw) - V(S)$ is a split graph. Since v is a center of C, it follows that $v_1, \ldots, v_5 \notin S$. But then $(C^2 - vw) - V(S)$ contains an induced C_5, namely C, a contradiction. This proves Claim (ii).

Claim (iii) *If $k = 2$, and there exists a center v of C, then all vertices in $V(C^2) \setminus V(C)$ are centers.*

Suppose the claim is false. Then since C^2 is connected, there exists $w \in V(C^2) \setminus V(C)$, $w \neq v$, such that w is not a center and $N(w) \cap V(C) \neq \emptyset$. It follows from Claim (i) that either w is a clone or w has exactly one neighbor in $V(C)$. Thus w is non-adjacent to at least two consecutive vertices in $V(C)$, say v_1, v_2. From Claim (ii) it follows that v and w are not adjacent. It follows from Lemma 4.10 that if we add the edge v_1w, there exists a clique K in $C^2 + v_1w$ such that K contains both v_1 and w, and such that $(C^2 + vw) - V(K)$ is bipartite. Clearly K does not contain any of the vertices v, v_2, v_3. But this implies that $(C^2 + v_1w) - V(K)$ contains the clique induced by v, v_2, v_3, a contradiction. This proves Claim (iii).

Claim (iv) *If there exists a clone v of some vertex $v_i \in V(C)$, then every vertex in $V(C^2) \setminus V(C)$ is adjacent to at least one of the vertices v_{i-1}, v_i, v_{i+1}, for $i \in \{1, \ldots, 2k+1\}$ taken modulo $2k+1$.*

Suppose the claim is false. Then there exists a vertex $w \in V(C^2) \setminus V(C)$ non-adjacent to v_{i-1}, v_i, v_{i+1}. It follows from Lemma 4.10 that if we add the edge wv_i, there exists a clique K in $C^2 + wv_i$ containing w and v_i, and such that $(C^2 + wv_i) - K$ is bipartite. But clearly K does not contain any vertex from the odd cycle induced by $(V(C) \setminus \{v_i\}) \cup \{v\}$. Thus $(C^2 + wv_i) - V(K)$ contains an odd cycle, a contradiction. This proves Claim (iv).

Claim (v) *There exists at most two consecutive vertices in $V(C)$ for which there exist clones in C^2.*

Suppose that there exist three consecutive vertices in $V(C)$, say v_1, v_2, v_3, for which there exist clones in C^2. Let w_i be a clone of v_i, for $i = 1, 2, 3$. First notice that w_1, w_3 are both adjacent to w_2. Indeed, if for instance w_1 is not adjacent to w_2, then it follows from Lemma 4.10 that if we add the edge w_1w_2, there exists a clique K in $C^2 + w_1w_2$ containing w_1 and w_2 and such that $(C^2 + w_1w_2) - V(K)$ is bipartite. But K clearly does not contain any vertex from C. It follows that $(C^2 + w_1w_2) - V(K)$ contains C, a contradiction. Thus w_1, w_3 are adjacent to w_2. Moreover we claim that w_1 and w_3 are not adjacent. Suppose for a contradiction that w_1 is adjacent to w_3. From Lemma 4.10 we deduce that if we add the edge v_1v_{2k}, there exists a clique K in $C^2 + v_1v_{2k}$ containing v_1 and v_{2k}, and such that $(C^2 + v_1v_{2k}) - V(K)$ is bipartite. But K clearly does not contain any of the vertices w_1, w_2, w_3, thus $(C^2 + v_1v_{2k}) - V(K)$ contains a K_3 induced by w_1, w_2, w_3, a contradiction. Thus w_1 and w_3 are not adjacent. But now it follows from Lemma 4.10 that if we add the edge w_1w_3, there exists a clique K in $C^2 + w_1w_3$ containing both w_1 and w_3, and such that $(C^2 + w_1w_3) - V(K)$ is bipartite. But K does not contain $v_1, v_3, \ldots, v_{2k+1}$ and either does not contain v_2 or does not contain w_2. Hence $C^2 - V(K)$ contains an induced odd cycle (either C or the odd cycle induced by $v_1, w_2, v_3, \ldots, v_{2k+1}$), a contradiction. This proves Claim (v).

Claim (vi) *There exists at most 2 distinct vertices of $V(C)$ having neighbors in $V(C^2) \setminus V(C)$ which are not centers of C and not clones of some vertices of $V(C)$.*

Suppose the claim is false. Let v_i, v_j, v_l be three distinct vertices of $V(C)$ such that each one has a neighbor in $V(C^2) \setminus V(C)$. Let w_i be such a neighbor of v_i, w_j such a neighbor of v_j, and w_l such a neighbor of v_l. From Claim (i) it follows that w_i, w_j, w_l have each exactly one neighbor in $V(C)$. First we claim that w_i, w_j, w_l induce a K_3. Suppose that they do not induce a K_3. Assume that w_i, w_j are not adjacent. Then from Lemma 4.10 it follows that if we add the edge $w_i w_j$, there exists a clique K in $C^2 + w_i w_j$ containing w_i and w_j, and such that $(C^2 + w_i w_j) - V(K)$ is bipartite. But K clearly does not contain any vertex of $V(C)$. Thus $(C^2 + w_i w_j) - V(K)$ contains C, a contradiction. Thus we may assume that w_i, w_j, w_l induce a K_3. Now from Lemma 4.10 it follows that if we add any edge $v_{i'} v_{j'}$ with $v_{i'}, v_{j'} \in V(C)$, there exists a clique K in $C^2 + v_{i'} v_{j'}$ containing $v_{i'}, v_{j'}$, and such that $(C^2 + v_{i'} v_{j'}) - V(K)$ is bipartite. But clearly K does not contain any of the vertices w_i, w_j, w_l. Hence $C^2 - V(K)$ contains a K_3, a contradiction. This proves Claim (vi).

We will now distinguish two cases: $k \geq 3$ and $k = 2$.

Case $k \geq 3$:

It follows from claims (i), (iv), (v), and (vi) that there exist at least two consecutive vertices in $V(C)$ which have degree exactly two. Indeed, suppose for a contradiction that there exist no two such vertices. From Claim (i) it follows that every vertex $v \in V(C^2) \setminus V(C)$ is either a clone of some vertex in $V(C)$ or adjacent to exactly one vertex in $V(C)$. Suppose that there exists a clone v of some vertex in $V(C)$, say v_1. Then it follows from claims (iv) and (v) that at least one of v_3, v_6 has degree two. Without loss of generality, we may assume that v_3 has degree two. Thus v_4 must have a neighbor in $V(C^2) \setminus V(C)$. It follows from Claim (iv) that this neighbor must be a clone of vertex v_3. Now it also follows from Claim (iv) that neither v_5 nor v_6 can have a neighbor in $V(C^2) \setminus V(C)$. But this contradicts the hypothesis that there exist no two consecutive vertices of degree two. Thus we conlcude that there must exist at least two consecutive vertices in $V(C)$ of degree two. Let v_1, v_2 be these two vertices. This implies that v_1, v_2 belong to the same induced odd cycles in C^2. It follows from Lemma 4.10 that if we add the edge $v_1 v_3$, there exists a clique K in $C^2 + v_1 v_3$ containing v_1, v_3, and such that $(C^2 + v_1 v_3) - V(K)$ is bipartite. Clearly K is isomorphic to K_3 and induced by v_1, v_2, v_3. Thus every induced odd cycle contains at least one of the vertices v_1, v_2, v_3. But since v_1, v_2 belong to the same odd induced cycles, it follows that $C^2 - v_2 v_3$ is bipartite, a contradiction with $\chi_S(G) = 3$.

Case $k = 2$:

If there exists a center w_1 of C, then it follows from claims (ii) and (iii) that all vertices in $V(C^2) \setminus V(C)$ are centers of C, and all these centers are pairwise non-adjacent. Assume there exist at least two centers w_1, w_2. Then it follows from Lemma 4.10 that if we add the edge $v_1 v_3$, there exists a clique K in $C^2 + v_1 v_3$ containing v_1 and v_3, and such that $(C^2 + v_1 v_3) - V(K)$ is bipartite. But K clearly does not contain both w_1 and w_2. So we may assume without loss of generality that K does not contain w_2. Thus there exists a clique isomorphic to K_3 in $(C^2 + v_1 v_3) - V(K)$ induced by w_2, v_4, v_5, a contradiction. So we may assume now that there exists at most one center of C. But then this center together with v_1, v_2 forms a clique K such that $C^2 - V(K)$ is bipartite. This contradicts the fact that $\chi_S(G) = 3$. Thus we may assume now that there exists no center of C. If there exists no clone of some vertex of $V(C)$, then we get a contradiction. Indeed, Claim (vi) implies that there exist at least two consecutive vertices in $V(C)$ having degree two. Now we get the contradiction by applying the same argument as in the case $k \geq 3$. So suppose that there exists at least one clone of some vertex in $V(C)$. Assume that w_1 is a clone of v_1. From Claim (v) it follows that we may assume that there exists no clone for vertex v_2. From Lemma 4.10 it follows that if we add the edge v_1v_3, there exists a clique K in $C^2 + v_1v_3$ containing both v_1 and v_3, and such that $(C^2 + v_1v_3) - V(K)$ is bipartite. Clearly K is isomorphic to K_3, and induced by v_1, v_2, v_3. Thus every induced odd cycle contains at least one of the vertices v_1, v_2, v_3. Repeating the same argument by replacing v_1 by v_4 and v_3 by v_2, we obtain that every induced odd cycle contains at least one of the vertices v_2, v_3, v_4 (resp. one of the vertices v_2, w_3, v_4, where w_3 is a clone for v_3). But now we claim that $C^2 - v_2v_3$ is bipartite. If not, then there necessarily exists an induced odd cycle containing both v_1 and v_4, and not containing v_2, v_3. Clearly this cycle does not contain a clone of vertex v_3, otherwise it must contain v_2, a contradiction. Thus the only vertices adjacent to v_4 which can belong to this cycle are either clones of v_5 or v_5 itself. But these vertices are all adjacent to v_1, contradicting the odd length of the cycle.

This proves Theorem 4.12. □

Next we prove that a split-decreasing graph with two connected components cannot contain any induced cycle of even length at least six.

Theorem 4.13 *Let G be a split-decreasing graph with two connected components and $\chi_S(G) = 3$. Then G does not contain any induced cycle of length $2k$, $k \geq 3$.*

Proof: Suppose for a contradiction that C^2 contains an induced cycle C of length $2k$, for some $k \geq 3$. Let $v_1, v_2, \ldots, v_{2k}$ be the vertices of C in clockwise order. From Theorem 4.12 it follows that C^2 does not contain any induced odd cycles of length $2k' + 1$, $k' \geq 2$. Thus the only odd induced cycles in C^2 are triangles. First notice that there exists no center of C and there exists no vertex $v \in V(C^2) \setminus V(C)$ that is adjacent to 2 consecutive vertices in $V(C)$. This can be shown by using exactly the same arguments as in the proof of Claim (i) of Theorem 4.12 (due to the length of this proof, we do not repeat these same arguments here). Now from Lemma 4.10 it follows that if we add the edge v_3v_5, there exists a clique K in $C^2 + v_3v_5$ containing both v_3 and v_5, and such that $(C^2 + v_3v_5) - V(K)$ is bipartite. Since there is no vertex adjacent to two consecutive vertices in $V(C)$, it follows that K is induced by v_3, v_4, v_5 (resp. by v_3, w_4, v_5, where w_4 is a clone for v_4). Thus every triangle in C^2 contains at least one vertex among v_3, v_4, v_5 (resp. v_3, w_4, v_5). By repeating this argument for vertices v_{2k}, v_2, we deduce that every triangle contains a vertex among v_{2k}, v_1, v_2 (resp. v_{2k}, w_1, v_2, where w_1 is a clone for v_1). But this implies that every triangle contains both v_2, v_3 (or v_5, v_6 in the case where $k = 3$). This contradicts the fact that no vertex in $V(C^2) \setminus V(C)$ is adjacent to two consecutive vertices in $V(C)$. This proves Theorem 4.13. □

Our next result states that a split-decreasing graph with two connected components is not a chordal graph.

Theorem 4.14 *Let G be a split-decreasing graph with two connected components and $\chi_S(G) = 3$. Then G contains an induced cycle of length four.*

Proof: It follows from Lemma 4.9 and Theorems 4.12 and 4.13 that C^1 is isomorphic to K_3, and that C^2 does not contain any induced cycle of length at least 5. Suppose for a contradiction that C^2 does not contain any induced cycle of length four, thus that C^2 is chordal. From [Dir61] we know that every chordal graph has a simplicial vertex, i.e, a vertex whose neighbors induce a clique. Let v be such a simplicial vertex in C^2. Clearly $N(v), \overline{N(v)} \neq \emptyset$ since C^2 is connected and C^2 is not a clique. Let $x_1 \in \overline{N(v)}$. It follows from Lemma 4.10 that if we add the edge vx_1 there exists a clique K in $C^2 + vx_1$ containing both v and x_1, and such that $(C^2 + vx_1) - V(K)$ is bipartite. Clearly K is induced by $D(x_1) \cup \{v, x_1\}$, where $D(x_1) \subseteq N(v)$ is the set of neighbors of x_1 in $N(v)$ (we may assume K maximal without loss of generality). We may assume that $D(x_1) \neq \emptyset$ since there exists at least one vertex in $\overline{N(v)}$ which is adjacent to some vertex in $N(v)$ (otherwise C^2 is not connected). It follows that $|N(v) \setminus D(x_1)| \leq 2$. Since $C^2 - (D(x_1) \cup \{x_1\})$ is not bipartite (otherwise G would be 2-split-colorable) and $(C^2 + vx_1) - V(K)$ is bipartite, it follows that $|N(v) \setminus D(x_1)| = 2$. This holds for any vertex $x \in \overline{N(v)}$ which is adjacent to some vertex in $N(v)$. Let $N(v) \setminus D(x_1) = \{w_1, w_2\}$. Furthermore, since $C^2 - (D(x_1) \cup \{v\})$ is not bipartite (otherwise G would be 2-split-colorable) and since $(C^2 + vx_1) - V(K)$ is bipartite, it follows that there exist $x_2, x_3 \in \overline{N(v)}$ such that x_1, x_2, x_3 induce a graph isomorphic to K_3. Let $d \in D(x_1)$. From Lemma 4.11 it follows that if we delete the edge vd, there exists a stable set S in $C^2 - vd$ containing both v and d, and such that $(C^2 - vd) - V(S)$ is a split graph. Thus the vertices w_1, w_2, x_1, x_2 cannot induce a graph isomorphic to $2K_2$. It follows that x_2 is adjacent to at least one of the vertices w_1, w_2. We may assume without loss of generality that x_2 is adjacent to w_2. We claim that x_2 is adjacent to all the vertices in $D(x_1)$. If not, then let $d \in D(x_1)$ be non-adjacent to x_2. Then x_2, w_2, d, x_1 induce a cycle of length four, a contradiction. Thus x_2 is adjacent to all the vertices in $D(x_1) \cup \{w_2\}$. It follows that x_2 has at most one non-neighbor in $N(v)$, i.e., $|N(v) \setminus D(x_2)| \leq 1$, where $D(x_2)$ is the set of neighbors of x_2 in $N(v)$. But this contradicts the fact that for every vertex $x \in \overline{N(v)}$ which has a neighbor in $N(v)$ we have $|N(v) \setminus D(x)| = 2$. This proves Theorem 4.14. $\square$

We are now in position to prove our main result about split-decreasing graphs.

Theorem 4.15 *Let G be a graph with two connected components C^1, C^2 and $\chi_S(G) = 3$. Then G is split-decreasing if and only if C^1 is isomorphic to K_3 and C^2 is isomorphic either to $\overline{3K_2}$ or to $\overline{C_4 \cup K_2}$ (see Figure 5).*

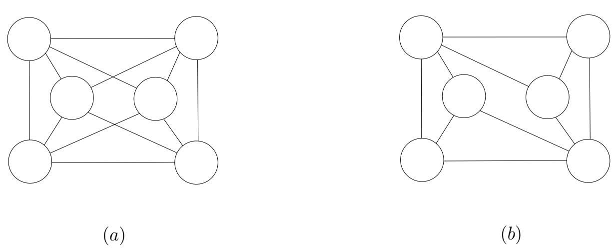

Fig. 5: (a) $\overline{3K_2}$ and (b) $\overline{C_4 \cup K_2}$.

Proof: First suppose that G is split-decreasing. It follows from Lemma 4.9 that C^1 is isomorphic to K_3. Suppose for a contradiction that C^2 is isomorphic neither to $\overline{3K_2}$ nor to $\overline{C_4 \cup K_2}$. It follows from Theorems 4.12 ,4.13 and 4.14 that C^2 contains an induced cycle C of length four, but no induced cycle of length $k \geq 5$. Let $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$ be the edge set of C. In what follows we will prove several claims.

Claim 1 *There exists no center v of C.*

Suppose for a contradiction that there exists a center v of C. We claim that all vertices in $V(C^2)\setminus V(C)$ are non-adjacent to v. If not, then let $w \in V(C^2) \setminus V(C)$ be adjacent to v. Then it follows from Lemma 4.11 that if we delete the edge vw there exists a stable set S in $C^2 - vw$ containing v and w, and such that $(C^2 - vw) - V(S)$ is a split graph. Since v is a center of C, it follows that $v_1, v_2, v_3, v_4 \notin S$. Thus $(C^2 - vw) - V(S)$ contains C, a contradiction. So we may assume that all vertices in $V(C^2) \setminus V(C)$ are non-adjacent to v. Next we claim that these vertices are centers of C. If not, then let $w \in V(C^2) \setminus V(C)$ be non-adjacent to v_i for some $i \in \{1, 2, 3, 4\}$, say v_1. It follows from Lemma 4.10 that if we add the edge wv_1 there exists a clique K in $C^2 + wv_1$ containing w and v_1, and such that $(C^2 + wv_1) - V(K)$ is bipartite. Clearly K does not contain neither v nor v_3, and K contains at most one vertex of v_2, v_4. Thus $(C^2 + wv_1) - V(K)$ contains a graph isomorphic to K_3, a contradiction. Hence all vertices in $V(C^2) \setminus V(C)$ are centers of C. If there exists only one center, then G is clearly 2-split colorable, a contradiction. If there exist exactly two centers of C, then C^2 is isomorphic to $\overline{3K_2}$, a contradiction. Thus we may assume that there exist at least three centers, say v, w, u. It follows from the above that v, w and u are pariwise non-adjacent. Now it follows from Lemma 4.10 that if we add the edge wu there exists a clique K in $C^2 + wu$ containing w and u, and such that $(C^2 + wu) - V(K)$ is bipartite. But clearly $v \notin V(K)$ and at most two adjacent vertices among v_1, v_2, v_3, v_4 belong to K. Thus $(C^2 + wu) - V(K)$ contains a graph isomorphic to K_3, a contradiction. This proves Claim 1. So we conclude that there exists no vertex in $V(C^2)$ which is a center of an induced cycle of length four in C^2.

Claim 2 *There exists no vertex $v \in V(C^2) \setminus V(C)$ adjacent to exactly three vertices of $V(C)$.*

Suppose for a contradiction that there exists a vertex $v \in V(C^2) \setminus V(C)$ adjacent to exactly three vertices of $V(C)$, say v_1, v_2, v_3. We claim that every vertex in $V(C^2) \setminus (V(C) \cup \{v\})$ is adjacent to v_4. If not, then let $w \in V(C^2) \setminus (V(C) \cup \{v\})$ be non-adjacent to v_4. It follows from Lemma 4.10 that if we add the edge wv_4 there exists a clique K in $C^2 + wv_4$ containing both w and v_4, and such that $(C^2 + wv_4) - V(K)$ is bipartite. K clearly does not contain neither v nor v_2. Furthermore K contains at most one of the vertices v_1, v_3. Thus $(C^2 + wv_4) - V(K)$ contains a graph isomorphic to K_3, a contradiction. Now we claim that all vertices in $V(C^2) \setminus (V(C) \cup \{v\})$ are non-adjacent to v. Indeed if there exists a vertex $w \in V(C^2) \setminus (V(C) \cup \{v\})$ adjacent to v, then it follows from Lemma 4.11 that if we delete the edge vw there exists a stable set S in $C^2 - vw$ containing v w, and such that $(C^2 - vw) - V(S)$ is a split graph. But clearly $v_1, v_2, v_3, v_4 \notin S$. Thus $(C^2 - vw) - V(S)$ contains C, a contradiction.

Now consider $w \in V(C^2) \setminus (V(C) \cup \{v\})$. First assume that w is adjacent to vertices v_1, v_2, v_4 (resp. v_2, v_3, v_4). It follows from Lemma 4.11 that if we delete the edge vv_1 (resp. vv_3) there exists a stable set S in $C^2 - vv_1$ (resp. in $C^2 - vv_3$) containing v and v_1 (resp. v, v_3) and such that $(C^2 - vv_1) - V(S)$ (resp. $(C^2 - vv_3) - V(S)$) is a split graph. But clearly $v_2, v_3, v_4, w \notin V(S)$ (resp. $v_1, v_2, v_4, w \notin V(S)$), and thus $(C^2 - vv_1) - V(S)$ (resp. $(C^2 - vv_3) - V(S)$) contains a cycle induced by v_2, v_3, v_4, w (resp. v_1, v_2, v_4, w), a contradiction.

Now let $w \in V(C^2) \setminus (V(C) \cup \{v\})$, and suppose that w is adjacent to v_1, v_4 (resp. v_3, v_4). Then it follows from Lemma 4.10 that if we add the edge wv_3 (resp. wv_1) there exists a clique K in $C^2 + wv_3$ (resp. $C^2 + wv_1$) containing w and v_3 (resp. w, v_1), and such that $(C^2 + wv_3) - V(K)$ (resp. $(C^2 + wv_1) - V(K)$) is bipartite. Clearly K does not contain v, v_1, v_2 (resp. v, v_2, v_3). Thus $(C^2 + wv_3) - V(K)$ (resp. $(C^2 + wv_1) - V(K)$) contains a graph isomorphic to K_3, a contradiction.

Next suppose that some vertex $w \in V(C^2) \setminus (V(C) \cup \{v\})$ is adjacent to v_2, v_4. It follows from Lemma 4.11 that if we delete the edge wv_2 there exists a stable set S in $C^2 - wv_2$ containing w and v_2, and such that $(C^2 - wv_2) - V(S)$ is a split graph. But clearly S does not contain vertices v_1, v_3, v_4, v. Thus $(C^2 - wv_2) - V(S)$ contains an induced cycle of length four, a contradiction.

Now let $w \in V(C^2) \setminus (V(C) \cup \{v\})$ be adjacent to v_4 and non-adjacent to v_1, v_2, v_3, v. But then it follows from Lemma 4.10 that if we add the edge wv_1 there exists a clique K in $C^2 + wv_1$ containing w and v_1, and such that $(C^2 + wv_1) - V(K)$ is bipartite. But K does not contain any of the vertices v, v_2, v_3, and thus $(C^2 + wv_1) - V(K)$ contains a graph isomorphic to K_3, a contradiction.

From the above discussion, it follows that for every vertex $w \in V(C^2) \setminus (V(C) \cup \{v\})$, w is adjacent to v_1, v_3, v_4 and non-adjacent to v, v_2. If only one such vertex exists, then C^2 is isomorphic to $\overline{C_4 \cup K_2}$, a contradiction. So we may assume that at least two such vertices exist, say w_1, w_2. First suppose that w_1, w_2 are non-adjacent. Then it follows from Lemma 4.10 that if we add the edge w_1w_2 there exists a clique K in $C^2 + w_1w_2$ containing w_1, w_2 and such that $(C^2 + w_1w_2) - V(K)$ is bipartite. But clearly $v, v_2 \notin V(K)$ at at most one of v_1, v_3 belongs to K. Thus $(C^2 + w_1w_2) - V(K)$ contains a graph isomorphic to K_3, a contradiction. Thus we may assume that w_1, w_2 are adjacent. But now it follows from Lemma 4.11 that if we delete the edge w_1w_2 there exists a stable set S in $C^2 - w_1w_2$ containing w_1 and w_2, and such that $(C^2 - w_1w_2) - V(S)$ is a split graph. Clearly $v_1, v_3, v_4 \notin V(S)$ and at most one of v, v_2 belongs to S. Thus $C^2 - w_1w_2$ contains a cycle o four vertices, a contradiction.

We conclude $V(C^2) \setminus (V(C) \cup \{v\}) = \emptyset$. But now G is clearly 2-split-colorable, a contradiction. This proves Claim 2. Thus we may assume that no vertex in $V(C^2)$ is adjacent to exactly three vertices of an induced cycle of length four in C^2.

Claim 3 *There exists no vertex $v \in V(C^2) \setminus V(C)$ adjacent to exactly two consecutive vertices in $V(C)$.*

Suppose for a contradiction that there exists a vertex $v \in V(C^2) \setminus V(C)$ adjacent to exactly two consecutive vertices in $V(C)$, say v_1, v_2. We claim that there exists no vertex $w \in V(C^2) \setminus V(C)$ adjacent to v_1, v_4 (resp. v_2, v_3). Indeed if there exists a vertex w adjacent to v_1, v_4 (resp. v_2, v_3), then it follows from Lemma 4.10 that if we add the edge wv_3 (resp. wv_4) there exists a clique K in $C^2 + wv_3$ (resp. $C^2 + wv_4$) containing w and v_3 (resp. w and v_4), and such that $(C^2 + wv_3) - V(K)$ (resp. $(C^2 + wv_4) - V(K)$) is bipartite. But K clearly does not contain any of the vertices v, v_1, v_2. Thus $(C^2 + wv_3) - V(K)$ (resp. $(C^2 + wv_4) - V(K)$) contains a graph isomorphic to K_3, a contradiction.

Furthermore we claim that every vertex $w \in V(C^2) \setminus (V(C) \cup \{v\})$ is adjacent to exactly one vertex of v_2, v_3, and to exactly one vertex of v_1, v_4. Suppose there exists a vertex $w \in V(C^2) \setminus (V(C) \cup \{v\})$ non-adjacent to v_1, v_4 (resp. v_2, v_3). Then it follows from Lemma 4.10 that if we add the edge wv_4 (resp. wv_3) there exists a clique K in $C^2 + wv_4$ (resp. $C^2 + wv_3$) containing w and v_4 (resp. w and v_3), and such that $(C^2 + wv_4) - V(K)$ (resp. $(C^2 + wv_3) - V(K)$) is bipartite. But clearly K does not contain any vertex of v, v_1, v_2. Hence $(C^2 + wv_4) - V(K)$ (resp. $(C^2 + wv_3) - V(K)$) contains a graph isomorphic to K_3, a contradiction.

Thus it follows that the vertices of $V(C^2) \setminus V(C)$ can be partitioned into four sets $V_{12}, V_{34}, V_{13}, V_{24}$, where V_{ij} is the set of vertices adjacent to v_i, v_j and non-adjacent to $\{v_1, v_2, v_3, v_4\} \setminus \{v_i, v_j\}$. We claim that $V_{12} \cup V_{34} \cup V_{13} \cup V_{24}$ induces a stable set. If not, then assume there exist $w, w' \in V_{12}$ (resp. V_{34}) which are adjacent. From Lemma 4.11 it follows that if we delete the edge v_1v_2 (resp. v_3v_4) there exists a stable set S in $C^2 - v_1v_2$ (resp. $C^2 - v_3v_4$) containing v_1 and v_2 (resp. v_3 and v_4), and such that $(C^2 - v_1v_2) - V(S)$ (resp. $(C^2 - v_3v_4) - V(S)$) is a split graph. Clearly $w, w', v_3, v_4 \notin V(S)$

(resp. $w, w, v_1, v_2 \notin V(S)$), but they induce a graph isomorphic to $2K_2$, a contradiction. Now assume that two vertices $w \in V_{12}$ and $w' \in V_{34}$ are adjacent. Then by applying Lemma 4.11 to the edge ww' we conclude that $(C^2 - ww') - V(S)$ is a split graph, where S is a stable set containing w, w'. But clearly $(C^2 - ww') - V(S)$ contains C, a contradiction. Next let us prove that no vertex in V_{12} is adjacent to a vertex in V_{13} (resp. V_{24}). From the symmetry it will follow that no vertex in V_{34} is adjacent to a vertex in V_{13} (resp. V_{24}). For suppose there exists a vertex $w \in V_{12}$ adjacent to some vertex $w' \in V_{13}$ (resp. $w' \in V_{24}$). But now w is adjacent to exactly three vertices of the cycle induced by v_1, v_2, v_3, w' (resp. v_1, v_2, v_4, w'), a contradiction. Similarly if $w, w' \in V_{13}$ (resp. $w, w' \in V_{24}$) are adjacent, then w is adjacent to exactly three vertices of the cycle induced by v_1, v_3, v_4, w' (resp. v_2, v_3, v_4, w'), a contradiction (see Claim 2). Finally we need to prove that no vertex in V_{13} is adjacent to some vertex in V_{24}. Suppose for a contradiction that $w \in V_{13}$ and $w' \in V_{24}$ are adjacent. Then it follows from Lemma 4.11 that if we delete the edge ww' there exists a stable set S in $C^2 - ww'$ containing w and w', and such that $(C^2 - ww') - V(S)$ is a split graph. But since $v_1, v_2, v_3, v_4 \notin S$, it follows that $(C^2 - ww') - V(S)$ contains C, a contradiction. Thus we conclude that $V_{12} \cup V_{34} \cup V_{13} \cup V_{24}$ induces a stable set.

This implies that $C^2 - v_2v_3$ is necessarily bipartite (indeed every trinagle in C^2 contains either v_2 or v_3), a contradiction. This proves Claim 3. So we may assume that there exists no vertex in $V(C^2)$ which is adjacent to exactly two consecutive vertices of an induced cycle of length four in C^2.

Claim 4 *There exists no vertex $v \in V(C^2) \setminus V(C)$ such that v is adjacent to exactly one vertex of $V(C)$.*

Suppose for a contradiction that there exists a vertex $v \in V(C^2) \setminus V(C)$ such that v is adjacent to exactly one vertex of $V(C)$, say v_1. From Lemma 4.10 it follows that if we add the edge vv_3 there exists a clique K in $C^2 + vv_3$ containing v and v_3, and such that $(C^2 + vv_3) - V(K)$ is bipartite. We claim that $|K| = 2$. Indeed, assume there exists a vertex w adjacent to both v and v_3. Clearly $w \neq v_2, v_4$. Furthermore w is non-adjacent to both v_2 and v_4 (see Claims 2 and 3). Thus v_1, v_3, v_4, v, w induce a cycle of length five, a contradiction. Hence $|K| = 2$. We conclude that every induced odd cycle of length three in C^2 necessarily contains either v or v_3. Since $C^2 - v$ is not bipartite (otherwise G would be 2-split-colorable), there exist w_1, w_2 such that v_3, w_1, w_2 induce a graph isomorphic to K_3. Clearly $w_1, w_2 \neq v_2, v_4$. But now it follows from Lemma 4.10 that if we add the edge vv_2 there exists a clique K in $C^2 + vv_2$ containing v and v_2, and such that $(C^2 + vv_2) - V(K)$ is bipartite. But clearly K does not contain any of v_3, w_1, w_2. Thus $(C^2 + vv_2) - V(K)$ contains a graph isomorphic to K_3, a contradiction. This proves Claim 4. Thus there exists no vertex in C^2 that is adjacent to exactly one vertex of an induced cycle of length four.

The above Claims and the facts that C^2 is connected and not 2-split-colorable imply that there exists a vertex $w \in V(C^2) \setminus V(C)$ which is adjacent to two non-consecutive vertices of $V(C)$, say v_1, v_3. First assume that there exists a vertex $w' \in V(C^2) \setminus V(C)$ adjacent to v_2, v_4. Then we claim that w, w' are adjacent. Indeed, if w, w' are non-adjacent, then w' is adjacent to exactly one vertex of the cycle induced by v_1, v_2, v_3, w, contradicting Claim 4. Thus w, w' are adjacent. Now by applying Lemma 4.11 to the edge ww' we conclude that $(C^2 - ww') - V(S)$ is a split graph, where S is a stable set containing w, w'. But $(C^2 - ww') - V(S)$ clearly contains C, a contradiction. So we may assume now that every vertex in $V(C^2) \setminus V(C)$ must be adjacent to both v_1 and v_3. Let $w_1, w_2 \in V(C^2) \setminus V(C)$. Then we claim that w_1, w_2 are non-adjacent. Indeed, if they are adjacent, then w_1 is adjacent to three vertices of the 4-cycle induced by v_1, w_2, v_3, v_4, a contradiction (see Claim 2). Thus $V(C^2) \setminus V(C)$ is a stable set. But now it follows that C^2 is bipartite, a contradiction. This proves the "only if" part.

Let us now prove the "if" part. First suppose that C^2 is isomorphic to $\overline{3K_2}$. Then it follows immediately from Theorem 4.6 that G is split-decreasing by setting $k = 3$ and $p = 2$. So assume now that C^2 is isomorphic to $\overline{C_4 \cup K_2}$. Let v_1, v_2, v_3, v_4 be the vertices inducing a C_4 in C^2 with edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let v, v' be the remaining vertices of C^2 with v being adjacent to v_1, v_2v_3 and v' being adjacent to v_1, v_3, v_4. Notice that v_1v_3 is missing in G. Denote by x, y, z the vertices in C^1. One can easily see that G is 3-split-colorable (take for instance three cliques C^1, K^1, K^2, where K^1 is induced by v, v_1, v_2, and K^2 is induced by v', v_3, v_4) but not 2-split-colorable.

Let us now show that G is split-decreasing. First we suppose that we add a missing edge. By symmetry we only have to consider the following cases. (1) If we add the edge vv', then $(C^1, \{v, v'\}; \{v_2, v_4\}, \{v_1, v_3\})$ is a 2-split-coloring; (2) if we add the edge vx, then $(\{v, x\}, \{v', v_1, v_4\}; \{y, v_2\}, \{z, v_3\})$ is a 2-split-coloring; (3) if we add the edge xv_1, then $(\{x, v_1\}, \{v, v_2, v_3\}; \{y, v_4\}, \{z, v'\})$ is a 2-split-coloring; (4) if we add the edge v_1v_3, then $(C^1, \{v_1, v_3\}; \{v_2, v_4\}, \{v, v'\})$ is a 2-split-coloring. Finally let us suppose that we delete an edge. Again due to symmetry we only have to consider the following cases. (1) If we delete the edge xy, then $(K, K'; \{x, y\}, \{z\})$ is a 2-split-coloring where K is induced by v_1, v_2, v, and K' is induced by v_3, v_4, v'; (2) if we delete the edge v_1v_2, then $(C^1, K; \{v_1, v_2\}, \{v\})$ is a 2-split-coloring where K is induced by v_3, v_4, v'; (3) finally if we delete the edge vv_2, then $(C^1, K; \{v, v_2\}, \{v_3\})$ is a 2-split-coloring where K is induced by v_1, v_4, v'. This proves Theorem 4.15. □

5 Conclusion

We hope that this study will stimulate further research in this topic. For instance, one can try to find graphs for which the bounds in Proposition 4.1 hold with equality. Namely, what are the graphs for which the removal of the edge set of an induced C_4 or $2K_2$ or C_5 decreases/increases the split-chromatic number exactly by one? The characterization of graphs with split-chromatic number two and such that the addition of any edge decreases the split-chromatic number (by one) is given in Corollary 4.4. One can try to characterize graphs with higher split-chromatic number and having the same property. Another interesting question is to consider the connectivity properties of uniquely split-colorable graphs.

Lastly, we showed that a graph is split-decreasing with split-chromatic number three, and with two or three connected components if and only if it is isomorphic to $3K_3$, $K_3 \cup \overline{3K_2}$ or $K_3 \cup (\overline{C_4 \cup K_2})$. We leave as an open question the characterization of connected split-decreasing graphs with split-chromatic number three ; this would complete the characterization of split-decreasing graphs with split-chromatic number three.

Acknowledgements

The authors would like to thank two anonymous referees for their careful reading of this paper and their comments which helped improving the paper considerably.

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