HAL
archives-ouvertes.fr

Symmetric monochromatic subsets in colorings of the Lobachevsky plane

Taras Banakh, Artem Dudko, Dusan Repovs

To cite this version:

Taras Banakh, Artem Dudko, Dusan Repovs. Symmetric monochromatic subsets in colorings of the Lobachevsky plane. Discrete Mathematics and Theoretical Computer Science, DMTCS, 2010, 12 (1), pp.12-20. <hal-00990434>

HAL Id: hal-00990434
https://hal.inria.fr/hal-00990434

Submitted on 13 May 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Discrete Mathematics and Theoretical Computer Science DMTCS vol. **12:1**, 2010, 12–20

Symmetric monochromatic subsets in colorings of the Lobachevsky plane

Taras Banakh[1], Artem Dudko[2], and Dušan Repovš[3]

[1]*Department of Mathematics, Lviv National University, Lviv, Ukraine,*
and Instytut Matematyki, Uniwersytet Humanistyczno-Przyrodniczy J.Kochanowskiego, Kielce, Poland
E-mail: tbanakh@yahoo.com
[2]*University of Toronto, 40 St. George Street Toronto, Ontario, Canada M5S 2E4*
E-mail: artemdudko@rambler.ru, artem.dudko@utoronto.ca
[3]*Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia*
E-mail: dusan.repovs@guest.arnes.si

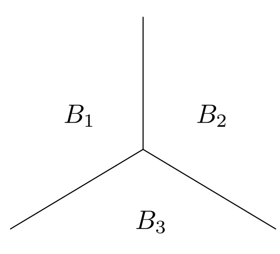
received 13 Dec 2007, *revised 26 May 2008, 12 Jan 2010*, *accepted 15 Jan 2010*.

We prove that for each partition of the Lobachevsky plane into finitely many Borel pieces one of the cells of the partition contains an unbounded centrally symmetric subset.

Keywords: Partition, central symmetry, monochromatic set, Borel piece, Lobachevsky plane, Poincaré model,Borel k-partition, coloring

1 Introduction

It follows from [B_1] (see also [BP_1, Theorem 1]) that for each partition of the n-dimensional space $\mathbb{R}^n$ into n pieces one of the pieces contains an unbounded centrally symmetric subset. On the other hand, $\mathbb{R}^n$ admits a partition $\mathbb{R}^n = B_0 \cup \cdots \cup B_n$ into $(n+1)$ Borel pieces containing no unbounded centrally symmetric subset. For $n = 2$ such a partition is drawn at the picture:



1365–8050 © 2010 Discrete Mathematics and Theoretical Computer Science (DMTCS), Nancy, France

Taking the same partition of the Lobachevsky plane H^2, we can see that each piece B_i does contain an unbounded centrally symmetric subset (for such a set just take any hyperbolic line lying in B_i).

We call a subset S of the hyperbolic plane H^2 *centrally symmetric* or else *symmetric with respect to a point* $c \in H^2$ if $S = f_c(S)$ where $f_c : H^2 \to H^2$ is the involutive isometry of H^2 assigning to each point $x \in H^2$ the unique point $y \in H^2$ such that c is the midpoint of the segment $[x, y]$. The map f_c is called the *central symmetry* of H^2 with respect to the point c.

By a *partition* of a set X we understand a decomposition $X = B_1 \cup \cdots \cup B_n$ of X into pairwise disjoint subsets called the *pieces* of the partition.

The following theorem shows that the Lobachevsky plane differs dramatically from the Euclidean plane from the Ramsey point of view.

Theorem 1.1 *For any partition $H^2 = B_1 \cup \cdots \cup B_m$ of the Lobachevsky plane into finitely many Borel pieces one of the pieces contains an unbounded centrally symmetric subset.*

2 Proof of Theorem 1.1

We shall prove a bit more: given a partition $H^2 = B_1 \cup \cdots \cup B_m$ of the Lobachevsky plane into m Borel pieces we shall find $i \le m$ and an unbounded subset $S \subset B_i$ symmetric with respect to some point c in an arbitrarily small neighborhood of some finite set $F \subset H^2$ depending only on m.

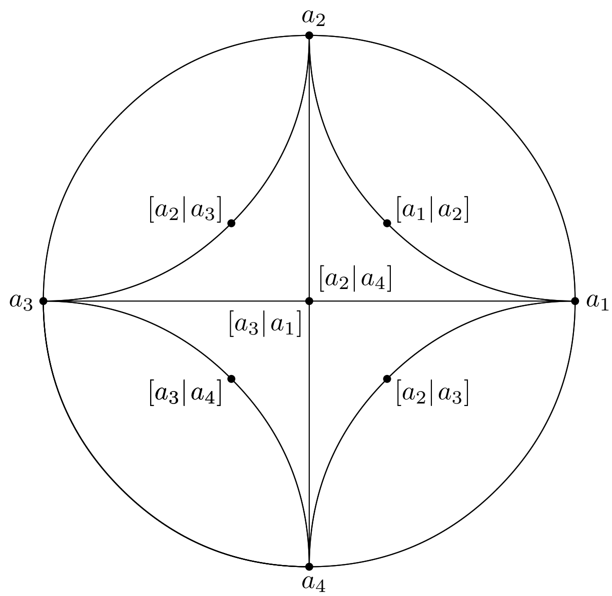
To define this set F it will be convenient to work in the Poincaré model of the Lobachevsky plane H^2. In this model the hyperbolic plane H^2 is identified with the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane and hyperbolic lines are just segments of circles orthogonal to the boundary of $\mathbb{D}$. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$ be the hyperbolic plane $\mathbb{D}$ with attached ideal line. For a real number $R > 0$ the set $\mathbb{D}_R = \{z \in \mathbb{C} : |z| \le 1 - 1/R\}$ can be thought as a hyperbolic disk of increasing radius as R tends to ∞.

On the boundary of the unit disk $\mathbb{D}$ consider the $(m+1)$-element set

$$A = \{z \in \mathbb{C} : z^{m+1} = 1\}.$$

For two distinct points $x, y \in A$ let $[x|y] \in \mathbb{D}$ denote the midpoint of the arc in $\overline{\mathbb{D}}$ that connects the points x, y and lies on a hyperbolic line in $H^2 = \mathbb{D}$. Then $F = \{[x|y] : x, y \in A,\ x \ne y\}$ is a finite subset of cardinality $|F| \le m(m+1)/2$ in the unit disk $\mathbb{D}$.

For $m = 3$ the set A consists of four points $a_1 = 1$, $a_2 = i$, $a_3 = -1$ and $a_4 = -i$ while F consists of five points $[a_1|a_2]$, $[a_2|a_3]$, $[a_3|a_4]$, $[a_4|a_1]$, $[a_1|a_3] = [a_2|a_4]$ as shown at the following picture:



We claim that for any open neighborhood W of F in $\mathbb{C}$ one of the pieces of a partition $H^2 = B_1 \cup \cdots \cup B_m$ contains an unbounded subset symmetric with respect to some point $c \in W$. To derive a contradiction we assume the converse: for every point $c \in W$ and every $i \le m$ the set $B_i \cap f_c(B_i)$ is bounded in H^2.

For every $n \in \mathbb{N}$ consider the set

$$C_n = \{c \in W : \bigcup_{i=1}^{m} B_i \cap f_c(B_i) \subset \mathbb{D}_n\}.$$

We claim that C_n is a coanalytic subset of W, that is, the complement $W \setminus C_n$ is analytic, which in its turn means that $W \setminus C_n$ is the continuous image of a Polish space. Observe that

$$W \setminus C_n = \{c \in W : \text{there are } i \le m \text{ and } x \in B_i \cap f_c(B_i) \setminus \mathbb{D}_n\} = \mathrm{pr}_2(E),$$

where $\mathrm{pr}_2 : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ is the projection on the second factor and

$$E = \bigcup_{i=1}^{m} \{(x, c) \in \mathbb{D} \times W : x \in B_i \cap f_c(B_i) \setminus \mathbb{D}_n\}$$

is a Borel subset of $\mathbb{D} \times W$. Being a Borel subset of the Polish space $\mathbb{D} \times W$, the space E is analytic and so is its continuous image $\mathrm{pr}_2(E) = W \setminus C_n$. Then C_n is coanalytic and hence has the Baire property [Ke, 21.6], which means that C_n coincides with an open subset U_n of W modulo some meager set. The latter means that the symmetric difference $U_n \triangle C_n$ is meager (i.e., is of the first Baire category) in W. Replacing U_n by the interior of the closure $\overline{U}_n$ of U_n in W, if necessary, we may additionally assume that U_n is regular open, that is, U_n coincides with the interior of its closure in W.

We claim that $C_n \subset C_{n+1}$ implies $U_n \subset U_{n+1}$. First we check that

$$U_n \setminus U_{n+1} \subset (U_n \triangle C_n) \cup (U_{n+1} \triangle C_{n+1})$$

is meager. Indeed, for every $x \in U_n \setminus U_{n+1}$ we get $x \in U_n \setminus C_n \subset U_n \triangle C_n$ if $x \notin C_n$ and $x \in C_{n+1} \setminus U_{n+1} \subset U_{n+1} \triangle C_{n+1}$ if $x \in C_n \subset C_{n+1}$. Therefore, the set $U_n \setminus U_{n+1}$ is meager, which implies $U_n \subset \overline{U}_{n+1}$ and hence $U_n \subset U_{n+1}$ because the set U_{n+1} is regular open.

Let $U = \bigcup_{n=1}^{\infty} U_n$ and $M = \bigcup_{n=1}^{\infty} U_n \triangle C_n$. Taking into account that $W = \bigcup_{n=1}^{\infty} C_n$, we conclude that

$$W \setminus U = \bigcup_{n=1}^{\infty} C_n \setminus \bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n=1}^{\infty} C_n \setminus U_n \subset \bigcup_{n=1}^{\infty} C_n \triangle U_n = M$$

which implies that the open set U has meager complement and thus is dense in W.

We claim that $F \subset h^{-1}(U)$ for some isometry h of the hyperbolic plane $H^2 = \mathbb{D}$.

For this consider the natural action

$$\mu : \mathrm{Iso}(H^2) \times \mathbb{D} \to \mathbb{D}, \;\; \mu : (h, x) \mapsto h(x)$$

of the isometry group $\mathrm{Iso}(H^2)$ of the hyperbolic plane $H^2 = \mathbb{D}$. It is easy to see that for every $x \in \mathbb{D}$ the map $\mu_x : \mathrm{Iso}(H^2) \to \mathbb{D}$, $\mu_x : h \mapsto h(x)$, is continuous and open (with respect to the compact-open topology on $\mathrm{Iso}(H^2)$). It follows that the set

$$\bigcap_{x \in F} \mu_x^{-1}(W) = \{h \in \mathrm{Iso}(H^2) : h(F) \subset W\}$$

is an open neighborhood of the neutral element of the group $\mathrm{Iso}(H^2)$.

Taking into account that U is open and dense in W, and that for every $x \in F$ the map $\mu_x : \mathrm{Iso}(H^2) \to \mathbb{D}$ is open, we conclude that the preimage $\mu_x^{-1}(U)$ is open and dense in $\mu_x^{-1}(W) \subset \mathrm{Iso}(H^2)$. Then the intersection $\bigcap_{x \in F} \mu_x^{-1}(U)$, being an open dense subset of $\bigcap_{x \in F} \mu_x^{-1}(W)$, is not empty and hence contains some isometry h having the desired property: $F \subset h^{-1}(U)$. Since F is finite, there is $n \in \mathbb{N}$ with $F \subset h^{-1}(U_n)$. For a complex number $r \in \mathbb{D}$ consider the set $rA = \{rz : z \in A\} \subset \mathbb{D}$ and let

$$F_r = \{[x|y] : x, y \in rA, \; x \neq y\} \subset \mathbb{D},$$

where $[x|y]$ stands for the midpoint of the hyperbolic segment connecting x and y in H^2. It can be shown that for any distinct points $x, y \in A$ the midpoint $[rx|ry]$ tends to the midpoint $[x|y] \in F$ as r tends to 1. Such a continuity yields a neighborhood O_1 of 1 such that $F_r \subset h^{-1}(U_n)$ for all $r \in O_1 \cap \mathbb{D}$.

It is clear that for any points $x, y \in A$ the map

$$f_{x,y} : \mathbb{D} \to \mathbb{D}, \; f_{x,y} : r \mapsto [rx|ry]$$

is open and continuous. Consequently, the preimage $f_{x,y}^{-1}(h^{-1}(M))$ is a meager subset of $\mathbb{D}$ and so is the union $M' = \bigcup_{x,y \in A} f_{x,y}^{-1}(h^{-1}(M))$. So, we can find a non-zero point $r \in O_1 \setminus M'$ so close to 1 that the set rA is disjoint with the hyperbolic disk $h^{-1}(\mathbb{D}_n)$ (observe that for a complex number r close to 1 the set rA is close to the set A lying in the boundary circle of $\mathbb{D}$ and thus rA can be made disjoint with the compact subset $h^{-1}(\mathbb{D}_n)$ of $\mathbb{D}$). For this point r we shall get $F_r \cap h^{-1}(M) = \emptyset$.

The set rA consists of $m+1$ points. Consequently, some cell $h^{-1}(B_i)$ of the partition $\mathbb{D} = h^{-1}(B_1) \cup \cdots \cup h^{-1}(B_m)$ contains two distinct points rx, ry of rA. Those points are symmetric with respect to the point

$$[rx|ry] \in F_r \subset h^{-1}(U_n) \setminus h^{-1}(M).$$

Then the images $a = h(rx)$ and $b = h(ry)$ belong to B_i and are symmetric with respect to the point $c = h([rx|ry]) \in U_n \setminus M \subset C_n$. It follows from the definition of C_n that $\{a, b\} \subset B_i \cap f_c(B_i) \subset \mathbb{D}_n$, which is not the case because $rx, ry \notin h^{-1}(\mathbb{D}_n)$.

3 Concerning partitions of H^2

We do not know if Theorem 1.1 is true for any finite (not necessarily Borel) partition of the Lobachevsky plane H^2. For partitions of H^2 into two pieces, the Borel assumption is superfluous.

Theorem 3.1 *There is a subset $T \subset H^2$ of cardinality $|T| = 3$ such that for any partition $H^2 = A_1 \cup A_2$ of H^2 into two pieces either A_1 or A_2 contains an unbounded subset, symmetric with respect to some point $c \in T$.*

Proof: Lemma 3.2 below allows us to find an equilateral triangle $\triangle c_0c_1c_2$ on the Lobachevsky plane H^2 such that the composition $f_{c_2} \circ f_{c_1} \circ f_{c_0}$ of the symmetries with respect to the points c_0, c_1, c_2 coincides with the rotation by the angle $2\pi/3$ about some point $o \in H^2$. Consequently $(f_{c_2} \circ f_{c_1} \circ f_{c_0})^3$ is the identity isometry of H^2.

We claim that for any partition $H^2 = A_1 \sqcup A_2$ of the Lobachevsky plane into two pieces one of the pieces contains an unbounded subset symmetric with respect to some point in the triangle $T = \{c_0, c_1, c_2\}$. Assuming the converse, we conclude that the set

$$B = \bigcup_{c \in T} \bigcup_{i=1}^{2} A_i \cap f_c(A_i)$$

is bounded. It follows that two points $x, y \in H^2 \setminus B$, symmetric with respect to a center $c \in T$ cannot belong to the same cell A_i of the partition.

Let $B_0 = B$ and $B_{i+1} = B_i \cup \bigcup_{j=0}^{2} f_{c_j}^{-1}(B_i)$ for $i \ge 0$. By induction it can be shown that each set B_i, $i \ge 0$, is bounded in H^2.

Fix any point $x_0 \in H^2 \setminus B_9$ and consider the sequence of points $x_1, \dots x_9$ defined by the recursive formula: $x_{i+1} = f_{c_{i \bmod 3}}(x_i)$. It follows that

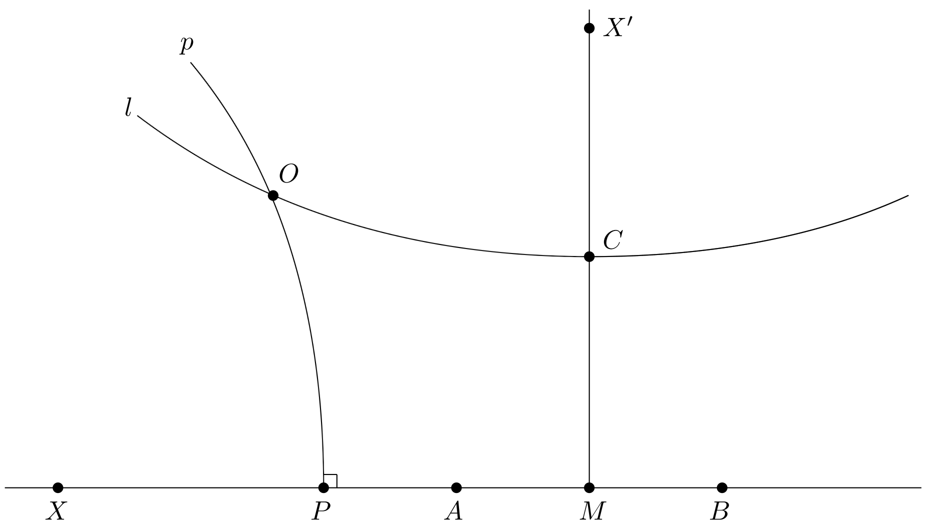
$$x_9 = (f_{c_2} \circ f_{c_1} \circ f_{c_0})^3(x_0) = x_0.$$

We claim that for every $i \le 9$ the point x_i does no belong to the set B. Assuming by contradiction that $x_i \in B$, we would conclude that $x_{i-1} \in \bigcup_{j=0}^{2} f_{c_j}^{-1}(B) \subset B_1$. Continuing by induction, for every $k \le i$ we would get $x_{i-k} \in B_k$. In particular, $x_0 \in B_i \subset B_9$, which contradicts the choice of x_0.

The point x_0 belongs either to A_1 or to A_2. We loose no generality assuming that $x_0 \in A_2$. Since the points $x_0, x_1 \notin B$ are symmetric with respect to c_0 and $x_0 \in A_2$, we get that $x_1 \in H^2 \setminus A_2 = A_1$. By the same reason x_1, x_2 cannot simultaneously belong to A_1 and hence $x_2 \in A_2$. Continuing in this fashion we conclude that x_i belongs to A_1 for odd i and to A_2 for even i. In particular, $x_9 \in A_1$, which is not possible because $x_9 = x_0 \in A_2$. □

Lemma 3.2 *There is an equilateral triangle $\triangle ABC$ on the Lobachevsky plane such that the composition $f_C \circ f_B \circ f_A$ of the symmetries with respect to the points A, B, C coincides with the rotation by the angle $2\pi/3$ about some point O.*

Proof: For a positive real number t consider an equilateral triangle $\triangle ABC$ with side t the on the Lobachevsky plane. Let M be the midpoint of the side AB and l be the line through C that is orthogonal to the line CM. Consider also the line p that is orthogonal to the line AB and passes through the point P such that A is the midpoint between P and M. Observe that $|PM| = |AB| = t$ and for sufficiently small t the lines p and l intersect at some point O.



It is easy to see that the composition $f_B \circ f_A$ is the shift along the line AB by the distance $2t$ and hence the image $f_B \circ f_A(O)$ of the point O is the point symmetric to O with respect to the point C. Consequently, $f_C \circ f_B \circ f_A(O) = O$, which means that the isometry $f_C \circ f_B \circ f_A$ is a rotation of the Lobachevsky plane about the point O by some angle φ_t.

To estimate this angle, consider the point X such that P is the midpoint between X and M. Then $|XM| = 2t$ and consequently, $f_B \circ f_A(X) = M$ while $X' = f_C \circ f_B \circ f_A = f_C(M)$ is the point on the line CM such that C is the midpoint between X' and M. It follows that $|X'X| \le |XM| + |MX'| < 2t + 2t = 4t$.

Observe that for small t the point X' is near to the point, symmetric to X with respect to O, which means that the angle $\varphi_t = \angle XOX'$ is close to π for t close to zero. On the other hand, for very large t the lines p and l on the Lobachevsky plane do not intersect. So we can consider the smallest upper bound t_0 of numbers t for which the lines l and p meet. For values $t < t_0$ near to t_0 the point O tends to infinity as t tends to t_0. Since the length of the side XX' of the triangle $\triangle XOX'$ is bounded by $4t_0$ the angle $\varphi_t = \angle XOX'$ tends to zero as O tends to infinity. Since the angle φ_t depends continuously on t and decreases from π to zero as t increases from zero to t_0, there is a value t such that $\varphi_t = 2\pi/3$. For such t the composition $f_C \circ f_B \circ f_A$ is the rotation around O on the angle $2\pi/3$. $\square$

4 Some comments and open problems

In contrast with Theorem 1.1, Theorem 3.1 is true for the Euclidean plane E^2 even in a stronger form: for any subset $C \subset E^2$ not lying on a line and any partition $E^2 = A_1 \cup A_2$ one of the cells of the partition contains an unbounded subset symmetric with respect to some center $c \in C$, see [B$_2$].

Having in mind this result let us call a subset C of a Lobachevsky or Euclidean space X *central for (Borel) k-partitions* if for any partition $X = A_1 \cup \cdots \cup A_k$ of X into k (Borel) pieces one of the pieces contains an unbounded monochromatic subset $S \subset X$, symmetric with respect to some point $c \in C$. By $c_k(X)$ (resp. $c_k^B(X)$) we shall denote the smallest size of a subset $C \subset X$, central for (Borel) k-partitions of X. If no such set C exists, then we put $c_k(X) = \infty$ (resp. $c_k^B(X) = \infty$) where ∞ is assumed to be greater than any cardinal number. It follows from the definition that $c_k^B(X) \leq c_k(X)$.

We have a lot of information about the numbers $c_k^B(E^n)$ and $c_k(E^n)$ for Euclidean spaces E^n, see [B$_2$]. In particular, we known that

1. $c_2(E^n) = c_2^B(E^n) = 3$ for all $n \geq 2$;
2. $c_3(E^3) = c_3^B(E^3) = 6$;
3. $12 \leq c_4^B(E^4) \leq c_4(E^4) \leq 14$;
4. $n(n+1)/2 \leq c_n^B(E^n) \leq c_n(E^n) \leq 2^n - 2$ for every $n \geq 3$.

Much less is known about the numbers $c_k^B(H^n)$ and $c_k^B(H^n)$ in the hyperbolic case. Theorem 3.1 yields the upper bound $c_2(H^2) \leq 3$. In fact, 3 is the exact value of $c_2(H^n)$ for all $n \geq 2$.

Proposition 4.1 $c_2^B(H^n) = c_2(H^n) = 3$ *for all* $n \geq 2$.

Proof: The upper bound $c_2(H^n) \leq c_2(H^2) \leq 3$ follows from Theorem 3.1. The lower bound $3 \leq c_2^B(H^n)$ will follow as soon as for any two points $c_1, c_2 \in H^n$ we construct a partition $H^n = A_1 \cup A_2$ in two Borel pieces containing no unbounded set, symmetric with respect to a point c_i. To construct such a partition, consider the line l containing the points c_1, c_2 and decompose l into two half-lines $l = l_1 \sqcup l_2$. Next, let H be an $(n-1)$-hyperplane in H^n, orthogonal to the line l. Let S be the unit sphere in H centered at the intersection point of l and H. Let $S = B_1 \cup B_2$ be a partition of S into two Borel pieces such that no antipodal points of S lie in the same cell of the partition. For each point $x \in H^n \setminus l$ consider the hyperbolic plane P_x containing the points x, c_1, c_2. The complement $P_x \setminus l$ decomposes into two half-planes $P_x^+ \cup P_x^-$ where P_x^+ is the half-plane containing the point x. The plane P_x intersects the hyperplane H by a hyperbolic line containing two points of the sphere S. Finally put

$$A_i = l_i \cup \{x \in H^2 \setminus l : P_x^+ \cap B_i \neq \emptyset\}$$

for $i \in \{1, 2\}$. It is easy to check that $A_1 \sqcup A_2 = H^n$ is the desired partition of the hyperbolic space into two Borel pieces none of which contains an unbounded subset symmetric with respect to one of the points c_1, c_2. □

The preceding proposition implies that the cardinal numbers $c_2(H^n)$ are finite.

Problem 4.2 *For which numbers k, n are the cardinal numbers $c_k(H^n)$ and $c_k^B(H^n)$ finite? Is it true for all $k \leq n$?*

Except for the equality $c_2(E^n) = 3$, we have no information on the numbers $c_k(E^n)$ with $k < n$.

Problem 4.3 *Calculate (or at least evaluate) the numbers $c_k(E^n)$ and $c_k(H^n)$ for $2 < k < n$.*

In all the cases where we know the exact values of the numbers $c_k(E^n)$ and $c_k^B(E^n)$ we see that those numbers are equal.

Problem 4.4 *Are the numbers $c_k(E^n)$ and $c_k^B(E^n)$ (resp. $c_k(H^n)$ and $c_k^B(H^n)$) equal for all k, n?*

Having in mind that each subset not lying on a line is central for 2-partitions of the Euclidean plane, we may ask about the same property of the Lobachevsky plane.

Problem 4.5 *Is any subset $C \subset H^2$ not lying on a line central for (Borel) 2-partitions of the Lobachevsky plane H^2?*

Finally, let us ask about the numbers $c_k^B(H^2)$ and $c_k(H^2)$. Observe that Theorem 1.1 guarantees that $c_k^B(H^2) \le \mathfrak{c}$ for all $k \in \mathbb{N}$. Inspecting the proof we can see that this upper bound can be improved to $c_k^B(H^2) \le \mathrm{non}(\mathcal{M})$ where $\mathrm{non}(\mathcal{M})$ is the smallest cardinality of a non-meager subset of the real line. It is clear that $\aleph_1 \le \mathrm{non}(\mathcal{M}) \le \mathfrak{c}$. The exact location of the cardinal $\mathrm{non}(M)$ on the interval $[\aleph_1, \mathfrak{c}]$ depends on axioms of Set Theory, see [Bl]. In particular, the inequality $\aleph_1 = \mathrm{non}(\mathcal{M}) < \mathfrak{c}$ is consistent with ZFC.

Problem 4.6 *Is the inequality $c_k^B(H^2) \le \aleph_1$ provable in ZFC? Are the cardinals $c_k^B(H^2)$ countable? finite?*

The last problem asks if H^2 contains a countable (or finite) central set for Borel k-partitions of the Lobachevsky plane. Inspecting the proof of Theorem 1.1 we can see that it gives an "approximate" answer to this problem:

Proposition 4.7 *For any $k \in \mathbb{N}$ there is a finite subset $C \subset H^2$of cardinality $|C| \le k(k+1)/2$ such that for any partition $H^2 = B_1 \cup \dots \cup B_k$ of H^2 into k Borel pieces and for any open neighborhood $O(C) \subset H^2$ of C one of the pieces B_i contains an unbounded subset $S \subset B_i$ symmetric with respect to some point $c \in O(C)$.*

Remark 4.8 *For further results and open problems related to symmetry and colorings see the surveys [BP$_2$], [BVV] and the list of problems [BBGRZ, §4].*

Acknowledgements

This reseach was supported the Slovenian Research Agency grants P1-0292-0101, J1-2057-0101 and BI-UA/09-10-002. The authors express their sincere thanks to Christian Krattenthaler for several comments and suggestions.

References

[B_1] T. Banakh, *Solutions of certain problems of I.V.Protasov from the combinatorics of colorings*, U Sviti Mat. **3**:1 (1997), 8–11 (in Ukrainian).

[B_2] T. Banakh, *On a cardinal group invariant related to partition of abelian groups*, Mat. Zametki. 64:3 (1998), 341–350.

[BBGRZ] T. Banakh, B. Bokalo, I. Guran, T. Radul, M. Zarichnyi, *Problems from the Lviv topological seminar*, in: Open Problems in Topology, II (E.Pearl, Ed.), Elsevier North-Holland, Amsterdam, 2007, pp. 655–667.

[BP_1] T. Banakh, I. Protasov, *Asymmetric partitions of abelian groups*, Mat. Zametki, **66** (1999), 10–19.

[BP_2] T. Banakh, I. Protasov, *Symmetry and colorings: some results and open problems*, Izv. Gomel Univ. Voprosy Algebry. **4** (2001), 5–16 (arXiv:0901.3356).

[BVV] T. Banakh, O. Verbitsky, Ya. Vorobets, *A Ramsey treatment of symmetry*, Electron. J. Combin. **7** (2000), R52, 25 pp.

[Bl] A. Blass, *Combinatorial cardinal characteristics of the continuum*, in: Handbook of Set Theory (M. Foreman, M. Magidor, A. Kanamoried, eds.), Springer-Verlag, Berlin, 2007.

[Ke] A. Kechris, Classical Descriptive Set Theory, Springer-Verlag, Berlin, 1995.