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# On the Book Thickness of $k$ -Trees

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Every  $k$ -tree has book thickness at most  $k + 1$ , and this bound is best possible for all  $k \geq 3$ . Vandenbussche et al. [*SIAM J. Discrete Math.*, 2009] proved that every  $k$ -tree that has a smooth degree-3 tree decomposition with width  $k$  has book thickness at most  $k$ . We prove this result is best possible for  $k \geq 4$ , by constructing a  $k$ -tree with book thickness  $k + 1$  that has a smooth degree-4 tree decomposition with width  $k$ . This solves an open problem of Vandenbussche et al.

MSC: 05C62, 68R10

**Keywords:** graph, book embedding, book thickness, pagenumber, stacknumber, treewidth, tree decomposition

## 1 Introduction

Consider a drawing of a graph<sup>(i)</sup>  $G$  in which the vertices are represented by distinct points on a circle in the plane, and each edge is a chord of the circle between the corresponding points. Suppose that each edge is assigned one of  $k$  colours such that crossing edges receive distinct colours. This structure is called a  $k$ -page book embedding of  $G$ : one can also think of the vertices as being ordered along the spine of a book, and the edges that receive the same colour being drawn on a single page of the book without crossings. The *book thickness* of  $G$ , denoted by  $\text{bt}(G)$ , is the minimum integer  $k$  for which there is a  $k$ -page book embedding of  $G$ . Book embeddings, first defined by Ollmann (1973), are ubiquitous structures with a variety of applications; see (Dujmović and Wood, 2004) for a survey with over 50 references. A book embedding is also called a *stack layout*, and book thickness is also called *stacknumber*, *pagenumber* and *fixed outerthickness*.

This paper focuses on the book thickness of  $k$ -trees. A vertex  $v$  in a graph  $G$  is  $k$ -simplicial if its neighbourhood,  $N_G(v)$ , is a  $k$ -clique. For  $k \geq 1$ , a  $k$ -tree is a graph  $G$  such that either  $G \simeq K_{k+1}$ , or  $G$  has a  $k$ -simplicial vertex  $v$  and  $G - v$  is a  $k$ -tree. In the latter case, we say that  $G$  is obtained from  $G - v$  by *adding  $v$  onto the  $k$ -clique  $N_G(v)$* .

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<sup>(i)</sup> We consider simple, finite, undirected graphs  $G$  with vertex set  $V(G)$  and edge set  $E(G)$ . We employ standard graph-theoretic terminology; see (Diestel, 2000). For disjoint  $A, B \subseteq V(G)$ , let  $G[A; B]$  denote the bipartite subgraph of  $G$  with vertex set  $A \cup B$  and edge set  $\{vw \in E(G) : v \in A, w \in B\}$ .

What is the maximum book thickness of a  $k$ -tree? Observe that 1-trees are precisely the trees. Bernhart and Kainen (1979) proved that every 1-tree has a 1-page book embedding. In fact, a graph has a 1-page book embedding if and only if it is outerplanar (Bernhart and Kainen, 1979). 2-trees are the edge-maximal series-parallel graphs. Rengarajan and Veni Madhavan (1995) proved that every series parallel graph, and thus every 2-tree, has a 2-page book embedding (also see (Di Giacomo et al., 2006)). This bound is best possible, since  $K_{2,3}$  is series parallel and is not outerplanar. Ganley and Heath (2001) proved that every  $k$ -tree has a  $(k+1)$ -page book embedding; see (Dujmović and Wood, 2007) for an alternative proof. Ganley and Heath (2001) also conjectured that every  $k$ -tree has a  $k$ -page book embedding. This conjecture was refuted by Dujmović and Wood (2007), who constructed a  $k$ -tree with book thickness  $k+1$  for all  $k \geq 3$ . Vandenbussche et al. (2009) independently proved the same result. Therefore the maximum book thickness of a  $k$ -tree is  $k$  for  $k \leq 2$  and is  $k+1$  for  $k \geq 3$ .

Which families of  $k$ -trees have  $k$ -page book embeddings? Togasaki and Yamazaki (2002) proved that every graph with pathwidth  $k$  has a  $k$ -page book embedding (and there are graphs with pathwidth  $k$  and book thickness  $k$ ). This result is equivalent to saying that every  $k$ -tree that has a smooth degree-2 tree decomposition<sup>(ii)</sup> of width  $k$  has a  $k$ -page book embedding. Vandenbussche et al. (2009) extended this result by showing that every  $k$ -tree that has a smooth degree-3 tree decomposition of width  $k$  has a  $k$ -page book embedding. Vandenbussche et al. (2009) then introduced the following natural definition. Let  $m(k)$  be the maximum integer  $d$  such that every  $k$ -tree that has a smooth degree- $d$  tree decomposition of width  $k$  has a  $k$ -page book embedding. Vandenbussche et al. (2009) proved that  $3 \leq m(k) \leq k+1$ , and state that determining  $m(k)$  is an open problem. However, it is easily seen that the  $k$ -tree with book thickness  $k+1$  constructed in (Dujmović and Wood, 2007) has a smooth degree-5 tree decomposition with width  $k$ . Thus  $m(k) \leq 4$  for all  $k \geq 3$ . The main result of this note is to refine the construction in (Dujmović and Wood, 2007) to give a  $k$ -tree with book thickness  $k+1$  that has a smooth degree-4 tree decomposition with width  $k$  for all  $k \geq 4$ . This proves that  $m(k) = 3$  for all  $k \geq 4$ . It is open whether  $m(3) = 3$  or 4. We conjecture that  $m(3) = 3$ .

## 2 Construction

**Theorem 1** *For all  $k \geq 4$  and  $n \geq 11(2k^2+1)+k$ , there is an  $n$ -vertex  $k$ -tree  $Q$ , such that  $\text{bt}(Q) = k+1$  and  $Q$  has a smooth degree-4 tree decomposition of width  $k$ .*

**Proof:** Start with the complete split graph  $K_{k,2k^2+1}^*$ . That is,  $K_{k,2k^2+1}^*$  is the  $k$ -tree obtained by adding a set  $S$  of  $2k^2+1$  vertices onto a  $k$ -clique  $K = \{u_1, u_2, \dots, u_k\}$ , as illustrated in Figure 1. For each vertex  $v \in S$  add a vertex onto the  $k$ -clique  $(K \cup \{v\}) \setminus \{u_1\}$ . Let  $T$  be the set of vertices added in this step. For each  $w \in T$ , if  $v$  is the neighbour of  $w$  in  $S$ , then add a set  $T_2(w)$  of three simplicial vertices onto the  $k$ -clique  $(K \cup \{v, w\}) \setminus \{u_1, u_2\}$ , add a set  $T_3(w)$  of three simplicial vertices onto the  $k$ -clique  $(K \cup \{v, w\}) \setminus \{u_1, u_3\}$ , and add a set  $T_4(w)$  of three simplicial vertices onto the  $k$ -clique  $(K \cup \{v, w\}) \setminus \{u_1, u_4\}$ . This step is well defined since  $k \geq 4$ . For each  $w \in T$ , let  $T(w) := T_2(w) \cup T_3(w) \cup T_4(w)$ . By construction,  $Q$  is a  $k$ -tree, and as illustrated in Figure 2,  $Q$  has a smooth degree-4 tree decomposition of width  $k$ .

<sup>(ii)</sup> See (Diestel, 2000) for the definition of tree decomposition and treewidth. Note that  $k$ -trees are the edge maximal graphs with treewidth  $k$ . A tree decomposition of width  $k$  is *smooth* if every bag has size exactly  $k+1$  and any two adjacent bags have exactly  $k$  vertices in common. Any tree decomposition of a graph  $G$  can be converted into a smooth tree decomposition of  $G$  with the same width. A tree decomposition is *degree- $d$*  if the host tree has maximum degree at most  $d$ .

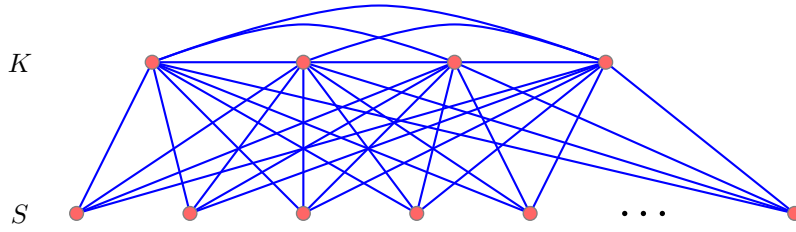


Fig. 1: The complete split graph  $K_{4,|S|}^*$ .

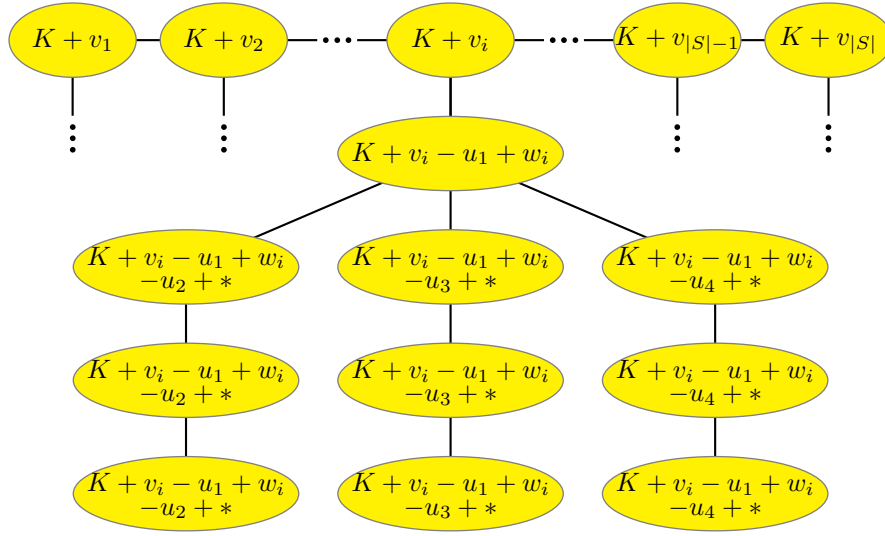


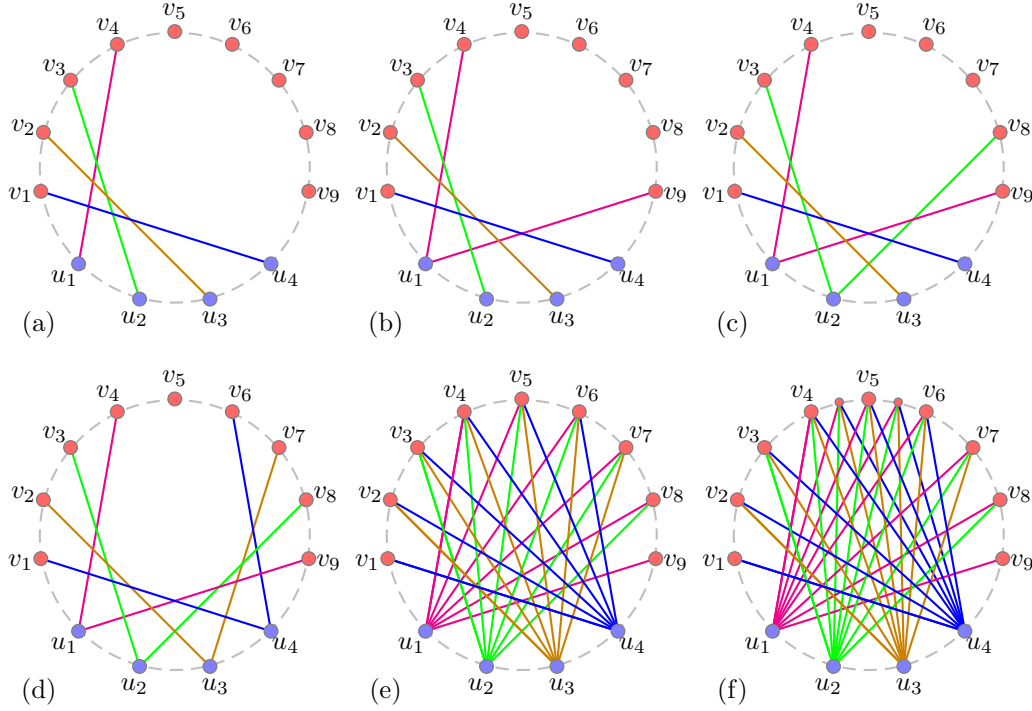
Fig. 2: A smooth degree-4 tree decomposition of  $Q$ .

It remains to prove that  $\text{bt}(Q) \geq k + 1$ . Suppose, for the sake of contradiction, that  $Q$  has a  $k$ -page book embedding. Say the edge colours are  $1, 2, \dots, k$ . For each ordered pair of vertices  $v, w \in V(Q)$ , let  $\widehat{vw}$  be the list of vertices in clockwise order from  $v$  to  $w$  (not including  $v$  and  $w$ ).

Say  $K = (u_1, u_2, \dots, u_k)$  in anticlockwise order. Since there are  $2k^2 + 1$  vertices in  $S$ , by the pigeonhole principle, without loss of generality, there are at least  $2k + 1$  vertices in  $S \cap \widehat{u_1 u_k}$ . Let  $(v_1, v_2, \dots, v_{2k+1})$  be  $2k + 1$  vertices in  $S \cap \widehat{u_1 u_k}$  in clockwise order.

Observe that the  $k$  edges  $\{u_i v_{k-i+1} : 1 \leq i \leq k\}$  are pairwise crossing, and thus receive distinct colours, as illustrated in Figure 3(a). Without loss of generality, each  $u_i v_{k-i+1}$  is coloured  $i$ . As illustrated in Figure 3(b), this implies that  $u_1 v_{2k+1}$  is coloured 1, since  $u_1 v_{2k+1}$  crosses all of  $\{u_i v_{k-i+1} : 2 \leq i \leq k\}$  which are coloured  $2, 3, \dots, k$ . As illustrated in Figure 3(c), this in turn implies that  $u_2 v_{2k}$  is coloured 2, and so on. By an easy induction,  $u_i v_{2k+2-i}$  is coloured  $i$  for each  $i \in \{1, 2, \dots, k\}$ , as illustrated in Figure 3(d). It follows that for all  $i \in \{1, 2, \dots, k\}$  and  $j \in \{k - i + 1, k - i + 2, \dots, 2k + 2 - i\}$ , the edge  $u_i v_j$  is coloured  $i$ , as illustrated in Figure 3(e). Moreover, as illustrated in Figure 3(f):

If  $qu_i \in E(Q)$  and  $q \in \widehat{v_k v_{k+2}}$ , then  $qu_i$  is coloured  $i$ . (★)



**Fig. 3:** Illustration of the proof of Theorem 1 with  $k = 4$ .

Note that the argument up to now is the same as in (Dujmović and Wood, 2007). Let  $w$  be the vertex in  $T$  adjacent to  $v_{k+1}$ . Recall that  $w$  is adjacent to each vertex in  $K \setminus \{u_1\}$ . Vertex  $w$  is in  $\widehat{v_k v_{k+2}}$ , as otherwise the edge  $wv_{k+1}$  crosses  $k$  edges of  $Q[\{v_k, v_{k+2}\}; K]$  that are all coloured differently. Without loss of generality,  $w$  is in  $\widehat{v_k v_{k+1}}$ . Each vertex  $x \in T(w)$  is in  $\widehat{v_k v_{k+1}}$ , as otherwise  $xw$  crosses  $k$  edges in  $Q[\{v_k, v_{k+1}\}; K]$  that are all coloured differently. Therefore, all nine vertices in  $T(w)$  are in  $\widehat{v_k v_{k+1}}$ . By the pigeonhole principle, at least one of  $\widehat{v_k w}$  or  $\widehat{wv_{k+1}}$  contains two vertices from  $T_i(w)$  and two vertices from  $T_j(w)$  for some  $i, j \in \{2, 3, 4\}$  with  $i \neq j$ . Let  $x_1, x_2, x_3, x_4$  be these four vertices in clockwise order in  $\widehat{v_k w}$  or  $\widehat{wv_{k+1}}$ .

*Case 1.*  $x_1, x_2, x_3$  and  $x_4$  are in  $\widehat{v_k w}$ : By (★), the edges in  $Q[\{w\}; K]$  are coloured  $2, 3, \dots, k$ . Thus  $x_2v_{k+1}$ , which crosses all the edges in  $Q[\{w\}; K]$ , is coloured 1. At least one of the vertices in  $\{x_2, x_3, x_4\}$  is adjacent to  $\{K \setminus \{u_1, u_i\}\}$  and at least one to  $\{K \setminus \{u_1, u_j\}\}$ . Thus, by (★), the edges in  $Q[\{x_2, x_3, x_4\}; K]$  are coloured  $2, 3, \dots, k$ . Thus  $x_1w$ , which crosses all the edges of  $Q[\{x_2, x_3, x_4\}; K]$  is coloured 1. Thus  $x_2v_{k+1}$  and  $x_1w$  cross and are both coloured 1, which is the desired contradiction.

*Case 2.*  $x_1, x_2, x_3$  and  $x_4$  are in  $\widehat{wv_{k+1}}$ : As in Case 1, the edges in  $Q[\{x_2, x_3, x_4\}; K]$  are coloured  $2, 3, \dots, k$ . Thus  $x_1v_{k+1}$ , which crosses all the edges in  $Q[\{x_2, x_3, x_4\}; K]$ , is coloured 1. Since the edges in  $Q[\{x_1, x_2, x_3\}; K]$  are coloured  $2, 3, \dots, k$ , the edge  $x_4w$ , which crosses all the edges of

$Q[\{x_1, x_2, x_3\}; K]$ , is coloured 1. Thus  $x_1v_{k+1}$  and  $x_4w$  cross and are both coloured 1, which is the desired contradiction.

Finally, observe that  $|V(Q)| = |K| + |S| + |T| + \sum_{w \in Q} |T(w)| = |K| + 11|S| = k + 11(2k^2 + 1)$ . Adding more  $k$ -simplicial vertices to  $Q$  does not reduce its book thickness. Moreover, it is simple to verify that the graph obtained from  $Q$  by adding simplicial vertices onto  $K$  has a smooth degree-4 tree decomposition of width  $k$ . Thus for all  $n \geq 11(2k^2 + 1) + k$ , there is a  $k$ -tree  $G$  with  $n$  vertices and  $\text{bt}(G) = k + 1$  that has the desired tree decomposition.  $\square$

### 3 Final Thoughts

For  $k \geq 3$ , the minimum book thickness of a  $k$ -tree is  $\lceil \frac{k+1}{2} \rceil$  (since every  $k$ -tree contains  $K_{k+1}$ , and  $\text{bt}(K_{k+1}) = \lceil \frac{k+1}{2} \rceil$ ; see (Bernhart and Kainen, 1979)). However, we now show that the range of book thicknesses of sufficiently large  $k$ -trees is very limited.

**Proposition 1** *Every  $k$ -tree  $G$  with at least  $\frac{1}{2}k(k+1)$  vertices has book thickness  $k-1$ ,  $k$  or  $k+1$ .*

**Proof:** Ganley and Heath (2001) proved that  $\text{bt}(G) \leq k+1$ . It remains to prove that  $\text{bt}(G) \geq k-1$  assuming  $|V(G)| \geq \frac{k(k+1)}{2}$ . Numerous authors (Bernhart and Kainen, 1979; Cottafava and D'Antona, 1984; Keys, 1975) observed that  $|E(G)| < (\text{bt}(G) + 1)|V(G)|$  for every graph  $G$ . Thus

$$(k-1)|V(G)| \leq k|V(G)| - \frac{1}{2}k(k+1) = |E(G)| < (\text{bt}(G) + 1)|V(G)| .$$

Hence  $k-1 < \text{bt}(G) + 1$ . Since  $k$  and  $\text{bt}(G)$  are integers,  $\text{bt}(G) \geq k-1$ .  $\square$

We conclude the paper by discussing some natural open problems regarding the computational complexity of calculating the book thickness for various classes of graphs.

Proposition 1 begs the question: Is there a characterisation of the  $k$ -trees with book thickness  $k-1$ ,  $k$  or  $k+1$ ? And somewhat more generally, is there a polynomial-time algorithm to determine the book thickness of a given  $k$ -tree? Note that the  $k$ -th power of paths are an infinite class of  $k$ -trees with book thickness  $k-1$ ; see (Swaminathan et al., 1995).

$k$ -trees are the edge-maximal chordal graphs with no  $(k+2)$ -clique, and also are the edge-maximal graphs with treewidth  $k$ . Is there a polynomial-time algorithm to determine the book thickness of a given chordal graph? Is there a polynomial-time algorithm to determine the book thickness of a given graph with bounded treewidth?

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