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On packing of two copies of a hypergraph

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A *2-packing* of a hypergraph \mathcal{H} is a permutation σ on $V(\mathcal{H})$ such that if an edge e belongs to $\mathcal{E}(\mathcal{H})$, then $\sigma(e)$ does not belong to $\mathcal{E}(\mathcal{H})$. Let \mathcal{H} be a hypergraph of order n which contains edges of cardinality at least 2 and at most $n - 2$. We prove that if \mathcal{H} has at most $n - 2$ edges then it is 2-packable.

Keywords: packing, hypergraphs

1 Introduction

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, where V is the *vertex set* and $\mathcal{E} \subset 2^V$ is the *edge set*. We allow empty edges for technical reasons, hence a complete simple hypergraph of order n has 2^n edges. We consider only finite hypergraphs. The edge of cardinality t is called *t-edge*, and 1-edge is called a *singleton*. A vertex is *isolated* if it does not belong to any edge. The number $d(v)$ of edges containing a vertex v is called the *degree of $v \in V$* . A hypergraph is *t-uniform* if $|e| = t$ for all $e \in \mathcal{E}$. Let \mathcal{H} be a hypergraph of order n . A *packing* of two copies of \mathcal{H} (*2-packing of \mathcal{H}*) is a permutation σ on $V(\mathcal{H})$ such that, if an edge $e = \{x_1, \dots, x_k\}$ belongs to $\mathcal{E}(\mathcal{H})$, then the edge $\sigma(e) = \{\sigma(x_1), \dots, \sigma(x_k)\}$ does not belong to $\mathcal{E}(\mathcal{H})$. Such a permutation (a *packing permutation*) is also called an *embedding* of \mathcal{H} into its complement. Consider a hypergraph \mathcal{H} and a permutation σ on V . We have $\sigma(V) = V$ and $\sigma(\emptyset) = \emptyset$. So, if $V \in \mathcal{E}$ or $\emptyset \in \mathcal{E}$, then \mathcal{H} cannot be packable.

We proved the following result in [4].

Theorem 1 *If a hypergraph \mathcal{H} of order n and size at most $\frac{1}{2}n$ has neither the empty edge nor its complement, then \mathcal{H} is 2-packable.*

Observe that this bound is sharp. Namely, if \mathcal{H} is a hypergraph of order n , and it has more than $\frac{1}{2}n$ edges, and each edge is a singleton, then evidently \mathcal{H} is not packable.

The aim of this paper is to show that if empty edges and singletons (and their complements, i.e. n -edges and $(n - 1)$ -edges) are excluded, then the bound on the size can be improved. We call a hypergraph \mathcal{H} of order n *admissible* if $2 \leq |H| \leq n - 2$ holds for all edges $H \in \mathcal{H}$.

We shall prove the following theorem.

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Theorem 2 *An admissible hypergraph \mathcal{H} of order n and size at most $n - 2$ is 2-packable.*

Recall that a 2-uniform hypergraph is called a graph. The packing problems for graphs have been studied for about thirty years (see for instance chapters in the books by B. Bollobás or H. P. Yap ([2],[8]), or survey papers by H. P. Yap or M. Woźniak ([9], [6], [7] and [5])). One of the first results in this area was the following theorem (see [3]).

Theorem 3 *A graph G of order n and size at most $n - 2$ is 2-packable.*

This bound is tight. Namely, if G is a star (of order n and size $n - 1$), then G is not packable.

Let \mathcal{H} be an admissible hypergraph of order n . First, denote by \mathcal{H}_k a k -uniform hypergraph of order n , which is induced by all k -edges in \mathcal{H} , and let m_k be the size of \mathcal{H}_k . Let m be the size of \mathcal{H} . Thus

$$n - 2 \geq m = m_2 + m_3 + \dots + m_{n-2}.$$

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Consider the hypergraph $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$ with the same vertex set V and the edge set $\tilde{\mathcal{E}}$, obtained from \mathcal{E} in the following way: if $e \in \mathcal{E}$ has at most $\frac{n}{2}$ vertices then e belongs to $\tilde{\mathcal{E}}$ and if e has more than $\frac{n}{2}$ vertices, then e is replaced by $V \setminus e$, with the convention that a double edge conceivably created in this way is replaced by a single one.

Remark 4 *Let \mathcal{H} be an admissible hypergraph of order n . If the hypergraph $\tilde{\mathcal{H}}$ is 2-packable, then also \mathcal{H} is 2-packable. Therefore, we shall assume that \mathcal{H} of order n is restricted to have edges of size at most $n/2$ only.*

Let $\mathcal{H} = (V, \mathcal{E})$ be an admissible hypergraph, and let x be a vertex of \mathcal{H} . We define the hypergraph $\mathcal{H}' = (V', \mathcal{E}')$:= $\mathcal{H} - x$ as follows: $V' = V \setminus \{x\}$, and the set of edges is obtained from \mathcal{E} by deleting 2-edges containing x , and replacing all remaining edges containing x by new edges with x deleted. It should be noted that it may happen that the assumption of Remark 4 does not apply to the hypergraph $\tilde{\mathcal{H}}$. So, if necessary, we use $\tilde{\mathcal{H}}'$ instead of \mathcal{H}' .

2 Lemmas

In the proof of Theorem 2, we shall use the following lemmas.

Lemma 5 *Let \mathcal{H} be an admissible hypergraph of order $n \geq 7$. Let x be an isolated vertex in \mathcal{H}_2 , and let y be a vertex of degree at least two in \mathcal{H}_2 . Suppose that there does not exist any 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. If $\mathcal{H}' = \mathcal{H} - x - y$ is 2-packable, then \mathcal{H} is also 2-packable. Moreover, \mathcal{H}' is an admissible hypergraph.*

Proof: Let x and y be two vertices satisfying the assumptions. It is easy to see that \mathcal{H}' is an admissible hypergraph, since, by assumptions, there is no singleton in \mathcal{H}' , because there is no 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. On the other hand, since $n \geq 7$, there is no $(n' - 1)$ -edge in \mathcal{H}' (where $n' = n - 2$).

Let σ' be a packing permutation of \mathcal{H}' . By the choice of x and y and the property of σ' , it is easy to see that the permutation $\sigma = \sigma' \circ (xy)$, where (xy) denotes a transposition, is a packing permutation of \mathcal{H} . □

The proof of Lemma 6 is analogous to that of Lemma 5.

Lemma 6 Let \mathcal{H} be an admissible hypergraph of order $n \geq 7$. Let x and y be two not adjacent vertices of degree one in \mathcal{H}_2 such that the neighbors x' of x and y' of y are distinct. Suppose that there does not exist any 3-edge $e \in \mathcal{H}$ such that $x \in e$ and $y \in e$. If $\mathcal{H}' = \mathcal{H} - x - y$ is 2-packable, then \mathcal{H} is also 2-packable. Moreover, \mathcal{H}' is an admissible hypergraph.

Lemma 7 Let \mathcal{H} be an admissible hypergraph of order n and size at most $n - 2$. If $m_2 \leq \frac{n}{2}$, then \mathcal{H} is 2-packable.

Proof: Using a probabilistic argument we shall show that a packing permutation exists for \mathcal{H} .

Let e and f be two edges of \mathcal{H} of the same cardinality and let σ be a random permutation on V . We say that edge e covers edge f (with respect to σ), if $\sigma(e) = f$. We denote this fact by $(e \curvearrowright f)$.

Let e and f be two k -edges. The event A such that e covers f (denoted by $A(e \curvearrowright f)$) has probability equal to

$$Pr(A(e \curvearrowright f)) = \frac{k!(n-k)!}{n!} = \binom{n}{k}^{-1}.$$

Observe, that there are m_k^2 ways to choose a pair e, f of k -edges such that e covers f . So, we have

$$Pr\left(\bigcup_{e,f \in \mathcal{H}} A(e \curvearrowright f)\right) \leq \sum_{e,f \in \mathcal{H}} Pr(A(e \curvearrowright f)) = m_2^2 \binom{n}{2}^{-1} + m_3^2 \binom{n}{3}^{-1} + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2 \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1}.$$

Since $k \leq \frac{n}{2}$, the sequence $\left(\binom{n}{2}^{-1}, \binom{n}{3}^{-1}, \dots\right)$ is decreasing, and we have

$$\begin{aligned} m_2^2 \binom{n}{2}^{-1} + m_3^2 \binom{n}{3}^{-1} + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2 \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1} &\leq m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (m_3^2 + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2) \leq \\ &\leq m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2. \end{aligned}$$

If $m_2 = 0$, then $n \geq 5$, and

$$m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2 = \binom{n}{3}^{-1} (n-2)^2.$$

If $m_2 = 1$, then $n \geq 3$, and

$$m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2 = \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-3)^2.$$

If $m_2 \geq 2$, then $n \geq 4$, and

$$m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2 \leq \frac{2n^2}{4n(n-1)} + \frac{6(n-4)^2}{n(n-1)(n-2)}.$$

It is easy to check that in each case

$$\Pr \left(\bigcup_{e, f \in \mathcal{H}} A(e \cap f) \right) < 1.$$

Consequently, a 2-packing of an admissible hypergraph \mathcal{H} of order n and size at most $n - 2$ exists, if $m_2 \leq \frac{n}{2}$. □

3 Proof of Theorem 2

By Remark 4, we consider only hypergraphs with edges of cardinality at most $\frac{n}{2}$. It is easy to see that for $n \leq 6$, either \mathcal{H} has only 2-edges, and we can apply Theorem 3, or the number of 2-edges is less than or equal to $n/2$, and we can apply Lemma 7. So, let $n \geq 7$.

Observe that, by Lemma 7, our claim holds if \mathcal{H}_2 is empty. Therefore, the proof will be divided into two main cases corresponding to the structure of \mathcal{H}_2 which is supposed to be non-empty.

The proof goes by induction on n . Let x, y be two vertices satisfying the assumptions of Lemma 5 or of Lemma 6. A 3-edge containing both of them will be called a *blocking edge*. Observe that if there is no blocking edge in \mathcal{H} , then the induction hypothesis can be applied. Below, we shall very often estimate the number of blocking edges in order to get a contradiction with the size of \mathcal{H} .

Case 1. There is no vertex of degree one in \mathcal{H}_2 .

The hypergraph \mathcal{H}_2 has at most $n - 2$ edges, so it has at least two isolated vertices. Denote by w the number of non-isolated vertices in \mathcal{H}_2 . Observe that $w \geq 3$ and $w \leq m_2$. Let y be a vertex of degree at least 2 in \mathcal{H}_2 . If we can choose an isolated vertex x in \mathcal{H}_2 such that there is no 3-edge containing both x and y , then we are done. So, suppose that a 3-edge containing both x and y exists in \mathcal{H} for every isolated vertex x in \mathcal{H}_2 and for any y . Observe that one 3-edge can cover at most two pairs of vertices x, y satisfying the assumptions of Lemma 5. Hence,

$$m_3 \geq \frac{1}{2}w(n - w) \geq \frac{1}{2}w(n - m_2),$$

$$2m_3 + wm_2 \geq wn.$$

Hence,

$$w(m_2 + m_3) \geq wn,$$

but $m_2 + m_3 \leq n - 2$, a contradiction.

Case 2. There is a vertex of degree one in \mathcal{H}_2 .

Let $b = m_3 + \dots + m_{\lfloor \frac{n}{2} \rfloor}$. If $b = 0$, then \mathcal{H} is a graph, and the claim is true. Hence, let $b > 0$. Then $m_2 = n - 2 - b$. Denote by t the number of tree components in \mathcal{H}_2 . So, $t \geq b + 2$. Next, denote by i the number of isolated vertices in \mathcal{H}_2 , by j the number of isolated edges, by k the number of stars with at least two leaves, and by l the number of trees with diameter greater than two. Thus, $t = i + j + k + l$. We shall consider four subcases.

Case 2A. There are at least two vertices of degree at least two in \mathcal{H}_2 , and $j + k + l \geq 2$.

As above, we shall count, how many blocking edges have to be in \mathcal{H} . Denote by n_2 the number of vertices of degree at least two in \mathcal{H}_2 . By assumption, $n_2 \geq 2$. So, if we are not able to apply Lemma 5, we should have at least $(\frac{1}{2}in_2)$ 3-edges in \mathcal{H} . Similarly, if we are not able to apply Lemma 6, we should have at least $[\frac{1}{3} \cdot 4 \cdot \binom{j+k+l}{2}]$ 3-edges in \mathcal{H} . Observe that one 3-edge can cover at most three pairs of vertices x, y which satisfy the assumptions of Lemma 6. Moreover, between every two tree components with at least two leaves, there are at least four such pairs. There are $\binom{j+k+l}{2}$ such pairs. Observe that all 3-edges mentioned above have to be distinct. Hence, we have

$$\frac{1}{2}in_2 + \frac{4}{3} \cdot \binom{j+k+l}{2} \leq b \leq t-2 = i+j+k+l-2.$$

Observe that

$$\frac{1}{2}in_2 \geq i,$$

and

$$\frac{4}{3} \cdot \frac{1}{2} \cdot (j+k+l)(j+k+l-1) \geq 1 \cdot 1 \cdot (j+k+l-1).$$

Again, we obtain a contradiction.

Case 2B. There are at least two vertices of degree at least two in \mathcal{H}_2 , and $j + k + l < 2$.

Thus, we have $l \leq 1$ and $n_2 \geq 2$. Analogously as in Case 2A, we consider blocking edges in \mathcal{H} . If $l = 0$, we obtain two cases:

1) if $j + k = 0$, then

$$i \leq \frac{1}{2}in_2 \leq b \leq t-2 = i-2;$$

2) if $j + k = 1$, then

$$i \leq \frac{1}{2}in_2 \leq b \leq t-2 = i-1.$$

If $l = 1$ we have at least one blocking edge more. Then,

$$i+1 \leq \frac{1}{2}in_2 + 1 \leq b \leq t-2 = i+l-2 = i-1.$$

In all cases we get a contradiction.

Case 2C. There is at most one vertex of degree at least two in \mathcal{H}_2 , and $j + k + l < 2$.

By definition, $l = 0$. Therefore, we have three subcases to consider. If $k = j = 0$ or $k = 0$ and $j = 1$, then by Lemma 7, our claim is true. Thus, let $j = 0$ and $k = 1$. So, \mathcal{H}_2 consists of a star $K_{1,p}$ and i isolated vertices. Observe that if $p \leq \frac{n}{2}$, then we are done by Lemma 7.

Hence, let $p > \frac{n}{2}$. Then, $n = i + p + 1$. Let y be the center of the star, and let x be an isolated vertex in \mathcal{H}_2 . If for any vertex z , the set $\{x, y, z\}$ is not an edge of \mathcal{H} , then we are done by Lemma 5.

If the vertex y belongs to two edges of the form $\{x, y, z\} \in \mathcal{E}(\mathcal{H})$ for any isolated vertex x , then we have the inequality

$$p + 2 \cdot \frac{i}{2} \leq n - 2.$$

Since $p + 2 \cdot \frac{i}{2} = n - 1$, we obtain a contradiction.

Therefore, there exists an isolated vertex x such that \mathcal{H} contains exactly one 3-edge $\{x, y, z\}$. Now, we construct a hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ such that $V' = V - \{x, y\}$ and the set of edges is obtained from \mathcal{E} as follows: we delete all 2-edges as well as the edge $\{x, y, z\}$, and we replace all remaining edges containing x or y (or x and y) by new edges with these vertices deleted. Then \mathcal{H}' has two vertices less, and at least $p + 1$ edges less than \mathcal{H} .

We shall show that there exists a packing permutation σ' of \mathcal{H}' without fixed points.

By the choice of x and y and the property of σ' , it is easy to see that the permutation $\sigma = \sigma' \circ (xy)$, where (xy) denotes a transposition, will be a packing permutation of \mathcal{H} .

An edge of the form $\{x, s, t\} \in \mathcal{H}$ (where $s \neq y$ and $t \neq y$) will be called an x -edge. Analogously, an edge of the form $\{y, s, t\} \in \mathcal{H}$ (where $s \neq x$ and $t \neq x$) will be called a y -edge.

First, we consider the case where \mathcal{H} has either x -edges or y -edges. We construct the hypergraph $\mathcal{H}'' = (V'', \mathcal{E}'')$ as follows: $V'' = V'$, and the set of edges is obtained from \mathcal{E}' by deleting all x -edges and y -edges. So $m_2'' = 0$ in \mathcal{H}'' . Now, we use a probabilistic argument as in the proof of Lemma 7.

$$Pr \left(\bigcup_{e, f \in \mathcal{H}''} A(e \cap f) \right) \leq \binom{n}{3}^{-1} (n - 2 - p - 1)^2 \leq \frac{6(n - 6)^2}{4(n - 2)(n - 3)(n - 4)} < \frac{1}{e} - \frac{1}{n!}.$$

It is easy to observe that the last inequality holds for $n \geq 6$. (Recall that the probability that a random permutation has no fixed point is greater than or equal to $\frac{1}{e} - \frac{1}{n!}$.)

Now, suppose that there are ξ x -edges and η y -edges in \mathcal{H} . Observe that we have at least $p + 3$ edges in \mathcal{H} (there are p edges of the star, the edge $\{x, y, z\}$, at least one x -edge and at least one y -edge). Then, $p + 3 \leq n - 2$. But $p > \frac{n}{2}$, hence $n \geq 11$. In general, we have at least $(\xi + \eta + 1 + p)$ edges in \mathcal{H} . Therefore $\xi + \eta \leq \frac{n}{2} - 3$. Then a product $\xi\eta$ is maximal if $\xi = \eta = \frac{1}{2}(\frac{n}{2} - 3)$. Analogously as above, we use a probabilistic argument to show that there is a packing permutation σ' without fixed points of \mathcal{H}' . Observe that there are $\xi + \eta$ edges in \mathcal{H}'_2 , and an x -edge cannot be mapped by σ' onto a y -edge (and vice versa). We have

$$\begin{aligned} Pr \left(\bigcup_{e, f \in \mathcal{H}'} A(e \cap f) \right) &\leq \frac{2 \cdot 2\xi\eta \cdot (n - 2)!}{n!} + \binom{n}{3}^{-1} (n - 2 - p - 3)^2 \leq \\ &\leq \frac{(n - 6)^2}{4n(n - 1)} + \frac{3(n - 10)^2}{2(n - 2)(n - 3)(n - 4)} < \frac{1}{e} - \frac{1}{n!}. \end{aligned}$$

It is easy to check that the last inequality is satisfied for $n \geq 11$, and consequently, there exists a packing permutation of \mathcal{H}' without fixed points.

Case 2D. There is at most one vertex of degree at least two in \mathcal{H}_2 , and $j + k + l \geq 2$.

Then, \mathcal{H}_2 has only tree components, $l = 0$ and $k \leq 1$.

If $k = 0$, then $j \geq 2$ and $j \leq \frac{n}{2}$ (because j is the number of isolated edges in \mathcal{H}_2). Then, by Lemma 7, the conclusion holds.

Thus, let $k = 1$ and $j \geq 1$. Denote by $K_{1,p}$ the star in \mathcal{H}_2 . If $p + j \leq \frac{n}{2}$, we are done by Lemma 7.

Hence $p + j > \frac{n}{2}$ and $n = i + 2j + p + 1$. If $j = 1$, then a 3-edge can block at most two possibilities for the choice of two leaves in \mathcal{H}_2 if one leaf is in the star. So, if we are not able to apply Lemma 6, we have to have at least $\frac{2p}{2}$ blocking edges in \mathcal{H} . If we are not able to apply Lemma 5, we have to have at least $\frac{i}{2}$ blocking edges in \mathcal{H} . Observe that in both cases the blocking edges are distinct. Hence, taking into account all 2-edges we get

$$n - 2 \geq |\mathcal{E}| \geq p + 1 + p + \frac{i}{2},$$

and

$$n - 3 \geq 2p + \frac{i}{2}.$$

On the other hand, $n - 3 = i + p$. Therefore, $\frac{i}{2} \geq p$. So, $n - 3 \geq 3p$. It follows that $p < \frac{n}{3}$, a contradiction.

Now, let $j \geq 2$. Observe that the number of 3-edges in \mathcal{H} is at least $\frac{i}{2}$ (because of Lemma 5), and at least $\frac{2pj}{2}$ (because of Lemma 6). (In the latter case, we may assume that one of the leaves comes from the star.)

We have

$$n - 2 \geq |\mathcal{E}| \geq j + p + pj + \frac{i}{2}.$$

But $j + p > \frac{n}{2}$, so

$$n - 2 \geq \frac{n}{2} + pj + \frac{i}{2}.$$

Hence

$$\frac{n}{2} - \frac{i}{2} - 2 \geq pj.$$

We know from a structure of the hypergraph that $n = i + 2j + p + 1$, so it follows from the above inequality that

$$\frac{2j + p + 1 - 4}{2} \geq pj.$$

This inequality together with the fact that $2pj \geq 2p + 2j$ for $p, j \geq 2$, implies

$$2j + p - 3 \geq 2pj \geq 2p + 2j,$$

a contradiction.

This ends the proof of the theorem.

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