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# On packing of two copies of a hypergraph

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A *2-packing* of a hypergraph  $\mathcal{H}$  is a permutation  $\sigma$  on  $V(\mathcal{H})$  such that if an edge  $e$  belongs to  $\mathcal{E}(\mathcal{H})$ , then  $\sigma(e)$  does not belong to  $\mathcal{E}(\mathcal{H})$ . Let  $\mathcal{H}$  be a hypergraph of order  $n$  which contains edges of cardinality at least 2 and at most  $n - 2$ . We prove that if  $\mathcal{H}$  has at most  $n - 2$  edges then it is 2-packable.

**Keywords:** packing, hypergraphs

## 1 Introduction

Let  $\mathcal{H} = (V, \mathcal{E})$  be a hypergraph, where  $V$  is the *vertex set* and  $\mathcal{E} \subset 2^V$  is the *edge set*. We allow empty edges for technical reasons, hence a complete simple hypergraph of order  $n$  has  $2^n$  edges. We consider only finite hypergraphs. The edge of cardinality  $t$  is called *t-edge*, and 1-edge is called a *singleton*. A vertex is *isolated* if it does not belong to any edge. The number  $d(v)$  of edges containing a vertex  $v$  is called the *degree of*  $v \in V$ . A hypergraph is *t-uniform* if  $|e| = t$  for all  $e \in \mathcal{E}$ . Let  $\mathcal{H}$  be a hypergraph of order  $n$ . A *packing* of two copies of  $\mathcal{H}$  (*2-packing* of  $\mathcal{H}$ ) is a permutation  $\sigma$  on  $V(\mathcal{H})$  such that, if an edge  $e = \{x_1, \dots, x_k\}$  belongs to  $\mathcal{E}(\mathcal{H})$ , then the edge  $\sigma(e) = \{\sigma(x_1), \dots, \sigma(x_k)\}$  does not belong to  $\mathcal{E}(\mathcal{H})$ . Such a permutation (a *packing permutation*) is also called an *embedding* of  $\mathcal{H}$  into its complement. Consider a hypergraph  $\mathcal{H}$  and a permutation  $\sigma$  on  $V$ . We have  $\sigma(V) = V$  and  $\sigma(\emptyset) = \emptyset$ . So, if  $V \in \mathcal{E}$  or  $\emptyset \in \mathcal{E}$ , then  $\mathcal{H}$  cannot be packable.

We proved the following result in [4].

**Theorem 1** *If a hypergraph  $\mathcal{H}$  of order  $n$  and size at most  $\frac{1}{2}n$  has neither the empty edge nor its complement, then  $\mathcal{H}$  is 2-packable.*

Observe that this bound is sharp. Namely, if  $\mathcal{H}$  is a hypergraph of order  $n$ , and it has more than  $\frac{1}{2}n$  edges, and each edge is a singleton, then evidently  $\mathcal{H}$  is not packable.

The aim of this paper is to show that if empty edges and singletons (and their complements, i.e.  $n$ -edges and  $(n - 1)$ -edges) are excluded, then the bound on the size can be improved. We call a hypergraph  $\mathcal{H}$  of order  $n$  *admissible* if  $2 \leq |H| \leq n - 2$  holds for all edges  $H \in \mathcal{H}$ .

We shall prove the following theorem.

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**Theorem 2** *An admissible hypergraph  $\mathcal{H}$  of order  $n$  and size at most  $n - 2$  is 2-packable.*

Recall that a 2-uniform hypergraph is called a graph. The packing problems for graphs have been studied for about thirty years (see for instance chapters in the books by B. Bollobás or H. P. Yap ([2],[8]), or survey papers by H. P. Yap or M. Woźniak ([9], [6], [7] and [5])). One of the first results in this area was the following theorem (see [3]).

**Theorem 3** *A graph  $G$  of order  $n$  and size at most  $n - 2$  is 2-packable.*

This bound is tight. Namely, if  $G$  is a star (of order  $n$  and size  $n - 1$ ), then  $G$  is not packable.

Let  $\mathcal{H}$  be an admissible hypergraph of order  $n$ . First, denote by  $\mathcal{H}_k$  a  $k$ -uniform hypergraph of order  $n$ , which is induced by all  $k$ -edges in  $\mathcal{H}$ , and let  $m_k$  be the size of  $\mathcal{H}_k$ . Let  $m$  be the size of  $\mathcal{H}$ . Thus

$$n - 2 \geq m = m_2 + m_3 + \dots + m_{n-2}.$$

Let  $\mathcal{H} = (V, \mathcal{E})$  be a hypergraph. Consider the hypergraph  $\tilde{\mathcal{H}} = (V, \tilde{\mathcal{E}})$  with the same vertex set  $V$  and the edge set  $\tilde{\mathcal{E}}$ , obtained from  $\mathcal{E}$  in the following way: if  $e \in \mathcal{E}$  has at most  $\frac{n}{2}$  vertices then  $e$  belongs to  $\tilde{\mathcal{E}}$  and if  $e$  has more than  $\frac{n}{2}$  vertices, then  $e$  is replaced by  $V \setminus e$ , with the convention that a double edge conceivably created in this way is replaced by a single one.

**Remark 4** *Let  $\mathcal{H}$  be an admissible hypergraph of order  $n$ . If the hypergraph  $\tilde{\mathcal{H}}$  is 2-packable, then also  $\mathcal{H}$  is 2-packable. Therefore, we shall assume that  $\mathcal{H}$  of order  $n$  is restricted to have edges of size at most  $n/2$  only.*

Let  $\mathcal{H} = (V, \mathcal{E})$  be an admissible hypergraph, and let  $x$  be a vertex of  $\mathcal{H}$ . We define the hypergraph  $\mathcal{H}' = (V', \mathcal{E}')$  :=  $\mathcal{H} - x$  as follows:  $V' = V \setminus \{x\}$ , and the set of edges is obtained from  $\mathcal{E}$  by deleting 2-edges containing  $x$ , and replacing all remaining edges containing  $x$  by new edges with  $x$  deleted. It should be noted that it may happen that the assumption of Remark 4 does not apply to the hypergraph  $\tilde{\mathcal{H}}$ . So, if necessary, we use  $\tilde{\mathcal{H}}'$  instead of  $\mathcal{H}'$ .

## 2 Lemmas

In the proof of Theorem 2, we shall use the following lemmas.

**Lemma 5** *Let  $\mathcal{H}$  be an admissible hypergraph of order  $n \geq 7$ . Let  $x$  be an isolated vertex in  $\mathcal{H}_2$ , and let  $y$  be a vertex of degree at least two in  $\mathcal{H}_2$ . Suppose that there does not exist any 3-edge  $e \in \mathcal{H}$  such that  $x \in e$  and  $y \in e$ . If  $\mathcal{H}' = \mathcal{H} - x - y$  is 2-packable, then  $\mathcal{H}$  is also 2-packable. Moreover,  $\mathcal{H}'$  is an admissible hypergraph.*

**Proof:** Let  $x$  and  $y$  be two vertices satisfying the assumptions. It is easy to see that  $\mathcal{H}'$  is an admissible hypergraph, since, by assumptions, there is no singleton in  $\mathcal{H}'$ , because there is no 3-edge  $e \in \mathcal{H}$  such that  $x \in e$  and  $y \in e$ . On the other hand, since  $n \geq 7$ , there is no  $(n' - 1)$ -edge in  $\mathcal{H}'$  (where  $n' = n - 2$ ).

Let  $\sigma'$  be a packing permutation of  $\mathcal{H}'$ . By the choice of  $x$  and  $y$  and the property of  $\sigma'$ , it is easy to see that the permutation  $\sigma = \sigma' \circ (xy)$ , where  $(xy)$  denotes a transposition, is a packing permutation of  $\mathcal{H}$ . □

The proof of Lemma 6 is analogous to that of Lemma 5.

**Lemma 6** Let  $\mathcal{H}$  be an admissible hypergraph of order  $n \geq 7$ . Let  $x$  and  $y$  be two not adjacent vertices of degree one in  $\mathcal{H}_2$  such that the neighbors  $x'$  of  $x$  and  $y'$  of  $y$  are distinct. Suppose that there does not exist any 3-edge  $e \in \mathcal{H}$  such that  $x \in e$  and  $y \in e$ . If  $\mathcal{H}' = \mathcal{H} - x - y$  is 2-packable, then  $\mathcal{H}$  is also 2-packable. Moreover,  $\mathcal{H}'$  is an admissible hypergraph.

**Lemma 7** Let  $\mathcal{H}$  be an admissible hypergraph of order  $n$  and size at most  $n - 2$ . If  $m_2 \leq \frac{n}{2}$ , then  $\mathcal{H}$  is 2-packable.

**Proof:** Using a probabilistic argument we shall show that a packing permutation exists for  $\mathcal{H}$ .

Let  $e$  and  $f$  be two edges of  $\mathcal{H}$  of the same cardinality and let  $\sigma$  be a random permutation on  $V$ . We say that edge  $e$  covers edge  $f$  (with respect to  $\sigma$ ), if  $\sigma(e) = f$ . We denote this fact by  $(e \curvearrowright f)$ .

Let  $e$  and  $f$  be two  $k$ -edges. The event  $A$  such that  $e$  covers  $f$  (denoted by  $A(e \curvearrowright f)$ ) has probability equal to

$$Pr(A(e \curvearrowright f)) = \frac{k!(n-k)!}{n!} = \binom{n}{k}^{-1}.$$

Observe, that there are  $m_k^2$  ways to choose a pair  $e, f$  of  $k$ -edges such that  $e$  covers  $f$ . So, we have

$$Pr\left(\bigcup_{e,f \in \mathcal{H}} A(e \curvearrowright f)\right) \leq \sum_{e,f \in \mathcal{H}} Pr(A(e \curvearrowright f)) = m_2^2 \binom{n}{2}^{-1} + m_3^2 \binom{n}{3}^{-1} + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2 \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1}.$$

Since  $k \leq \frac{n}{2}$ , the sequence  $\left(\binom{n}{2}^{-1}, \binom{n}{3}^{-1}, \dots\right)$  is decreasing, and we have

$$\begin{aligned} m_2^2 \binom{n}{2}^{-1} + m_3^2 \binom{n}{3}^{-1} + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2 \binom{n}{\lfloor \frac{n}{2} \rfloor}^{-1} &\leq m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (m_3^2 + \dots + m_{\lfloor \frac{n}{2} \rfloor}^2) \leq \\ &\leq m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2. \end{aligned}$$

If  $m_2 = 0$ , then  $n \geq 5$ , and

$$m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2 = \binom{n}{3}^{-1} (n-2)^2.$$

If  $m_2 = 1$ , then  $n \geq 3$ , and

$$m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2 = \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-3)^2.$$

If  $m_2 \geq 2$ , then  $n \geq 4$ , and

$$m_2^2 \binom{n}{2}^{-1} + \binom{n}{3}^{-1} (n-2-m_2)^2 \leq \frac{2n^2}{4n(n-1)} + \frac{6(n-4)^2}{n(n-1)(n-2)}.$$

It is easy to check that in each case

$$\Pr \left( \bigcup_{e, f \in \mathcal{H}} A(e \cap f) \right) < 1.$$

Consequently, a 2-packing of an admissible hypergraph  $\mathcal{H}$  of order  $n$  and size at most  $n - 2$  exists, if  $m_2 \leq \frac{n}{2}$ . □

### 3 Proof of Theorem 2

By Remark 4, we consider only hypergraphs with edges of cardinality at most  $\frac{n}{2}$ . It is easy to see that for  $n \leq 6$ , either  $\mathcal{H}$  has only 2-edges, and we can apply Theorem 3, or the number of 2-edges is less than or equal to  $n/2$ , and we can apply Lemma 7. So, let  $n \geq 7$ .

Observe that, by Lemma 7, our claim holds if  $\mathcal{H}_2$  is empty. Therefore, the proof will be divided into two main cases corresponding to the structure of  $\mathcal{H}_2$  which is supposed to be non-empty.

The proof goes by induction on  $n$ . Let  $x, y$  be two vertices satisfying the assumptions of Lemma 5 or of Lemma 6. A 3-edge containing both of them will be called a *blocking edge*. Observe that if there is no blocking edge in  $\mathcal{H}$ , then the induction hypothesis can be applied. Below, we shall very often estimate the number of blocking edges in order to get a contradiction with the size of  $\mathcal{H}$ .

**Case 1.** There is no vertex of degree one in  $\mathcal{H}_2$ .

The hypergraph  $\mathcal{H}_2$  has at most  $n - 2$  edges, so it has at least two isolated vertices. Denote by  $w$  the number of non-isolated vertices in  $\mathcal{H}_2$ . Observe that  $w \geq 3$  and  $w \leq m_2$ . Let  $y$  be a vertex of degree at least 2 in  $\mathcal{H}_2$ . If we can choose an isolated vertex  $x$  in  $\mathcal{H}_2$  such that there is no 3-edge containing both  $x$  and  $y$ , then we are done. So, suppose that a 3-edge containing both  $x$  and  $y$  exists in  $\mathcal{H}$  for every isolated vertex  $x$  in  $\mathcal{H}_2$  and for any  $y$ . Observe that one 3-edge can cover at most two pairs of vertices  $x, y$  satisfying the assumptions of Lemma 5. Hence,

$$m_3 \geq \frac{1}{2}w(n - w) \geq \frac{1}{2}w(n - m_2),$$

$$2m_3 + wm_2 \geq wn.$$

Hence,

$$w(m_2 + m_3) \geq wn,$$

but  $m_2 + m_3 \leq n - 2$ , a contradiction.

**Case 2.** There is a vertex of degree one in  $\mathcal{H}_2$ .

Let  $b = m_3 + \dots + m_{\lfloor \frac{n}{2} \rfloor}$ . If  $b = 0$ , then  $\mathcal{H}$  is a graph, and the claim is true. Hence, let  $b > 0$ . Then  $m_2 = n - 2 - b$ . Denote by  $t$  the number of tree components in  $\mathcal{H}_2$ . So,  $t \geq b + 2$ . Next, denote by  $i$  the number of isolated vertices in  $\mathcal{H}_2$ , by  $j$  the number of isolated edges, by  $k$  the number of stars with at least two leaves, and by  $l$  the number of trees with diameter greater than two. Thus,  $t = i + j + k + l$ . We shall consider four subcases.

**Case 2A.** There are at least two vertices of degree at least two in  $\mathcal{H}_2$ , and  $j + k + l \geq 2$ .

As above, we shall count, how many blocking edges have to be in  $\mathcal{H}$ . Denote by  $n_2$  the number of vertices of degree at least two in  $\mathcal{H}_2$ . By assumption,  $n_2 \geq 2$ . So, if we are not able to apply Lemma 5, we should have at least  $(\frac{1}{2}in_2)$  3-edges in  $\mathcal{H}$ . Similarly, if we are not able to apply Lemma 6, we should have at least  $[\frac{1}{3} \cdot 4 \cdot \binom{j+k+l}{2}]$  3-edges in  $\mathcal{H}$ . Observe that one 3-edge can cover at most three pairs of vertices  $x, y$  which satisfy the assumptions of Lemma 6. Moreover, between every two tree components with at least two leaves, there are at least four such pairs. There are  $\binom{j+k+l}{2}$  such pairs. Observe that all 3-edges mentioned above have to be distinct. Hence, we have

$$\frac{1}{2}in_2 + \frac{4}{3} \cdot \binom{j+k+l}{2} \leq b \leq t-2 = i+j+k+l-2.$$

Observe that

$$\frac{1}{2}in_2 \geq i,$$

and

$$\frac{4}{3} \cdot \frac{1}{2} \cdot (j+k+l)(j+k+l-1) \geq 1 \cdot 1 \cdot (j+k+l-1).$$

Again, we obtain a contradiction.

**Case 2B.** There are at least two vertices of degree at least two in  $\mathcal{H}_2$ , and  $j + k + l < 2$ .

Thus, we have  $l \leq 1$  and  $n_2 \geq 2$ . Analogously as in Case 2A, we consider blocking edges in  $\mathcal{H}$ . If  $l = 0$ , we obtain two cases:

1) if  $j + k = 0$ , then

$$i \leq \frac{1}{2}in_2 \leq b \leq t-2 = i-2;$$

2) if  $j + k = 1$ , then

$$i \leq \frac{1}{2}in_2 \leq b \leq t-2 = i-1.$$

If  $l = 1$  we have at least one blocking edge more. Then,

$$i+1 \leq \frac{1}{2}in_2 + 1 \leq b \leq t-2 = i+l-2 = i-1.$$

In all cases we get a contradiction.

**Case 2C.** There is at most one vertex of degree at least two in  $\mathcal{H}_2$ , and  $j + k + l < 2$ .

By definition,  $l = 0$ . Therefore, we have three subcases to consider. If  $k = j = 0$  or  $k = 0$  and  $j = 1$ , then by Lemma 7, our claim is true. Thus, let  $j = 0$  and  $k = 1$ . So,  $\mathcal{H}_2$  consists of a star  $K_{1,p}$  and  $i$  isolated vertices. Observe that if  $p \leq \frac{n}{2}$ , then we are done by Lemma 7.

Hence, let  $p > \frac{n}{2}$ . Then,  $n = i + p + 1$ . Let  $y$  be the center of the star, and let  $x$  be an isolated vertex in  $\mathcal{H}_2$ . If for any vertex  $z$ , the set  $\{x, y, z\}$  is not an edge of  $\mathcal{H}$ , then we are done by Lemma 5.

If the vertex  $y$  belongs to two edges of the form  $\{x, y, z\} \in \mathcal{E}(\mathcal{H})$  for any isolated vertex  $x$ , then we have the inequality

$$p + 2 \cdot \frac{i}{2} \leq n - 2.$$

Since  $p + 2 \cdot \frac{i}{2} = n - 1$ , we obtain a contradiction.

Therefore, there exists an isolated vertex  $x$  such that  $\mathcal{H}$  contains exactly one 3-edge  $\{x, y, z\}$ . Now, we construct a hypergraph  $\mathcal{H}' = (V', \mathcal{E}')$  such that  $V' = V - \{x, y\}$  and the set of edges is obtained from  $\mathcal{E}$  as follows: we delete all 2-edges as well as the edge  $\{x, y, z\}$ , and we replace all remaining edges containing  $x$  or  $y$  (or  $x$  and  $y$ ) by new edges with these vertices deleted. Then  $\mathcal{H}'$  has two vertices less, and at least  $p + 1$  edges less than  $\mathcal{H}$ .

We shall show that there exists a packing permutation  $\sigma'$  of  $\mathcal{H}'$  without fixed points.

By the choice of  $x$  and  $y$  and the property of  $\sigma'$ , it is easy to see that the permutation  $\sigma = \sigma' \circ (xy)$ , where  $(xy)$  denotes a transposition, will be a packing permutation of  $\mathcal{H}$ .

An edge of the form  $\{x, s, t\} \in \mathcal{H}$  (where  $s \neq y$  and  $t \neq y$ ) will be called an  $x$ -edge. Analogously, an edge of the form  $\{y, s, t\} \in \mathcal{H}$  (where  $s \neq x$  and  $t \neq x$ ) will be called a  $y$ -edge.

First, we consider the case where  $\mathcal{H}$  has either  $x$ -edges or  $y$ -edges. We construct the hypergraph  $\mathcal{H}'' = (V'', \mathcal{E}'')$  as follows:  $V'' = V'$ , and the set of edges is obtained from  $\mathcal{E}'$  by deleting all  $x$ -edges and  $y$ -edges. So  $m_2'' = 0$  in  $\mathcal{H}''$ . Now, we use a probabilistic argument as in the proof of Lemma 7.

$$Pr \left( \bigcup_{e, f \in \mathcal{H}''} A(e \cap f) \right) \leq \binom{n}{3}^{-1} (n - 2 - p - 1)^2 \leq \frac{6(n - 6)^2}{4(n - 2)(n - 3)(n - 4)} < \frac{1}{e} - \frac{1}{n!}.$$

It is easy to observe that the last inequality holds for  $n \geq 6$ . (Recall that the probability that a random permutation has no fixed point is greater than or equal to  $\frac{1}{e} - \frac{1}{n!}$ .)

Now, suppose that there are  $\xi$   $x$ -edges and  $\eta$   $y$ -edges in  $\mathcal{H}$ . Observe that we have at least  $p + 3$  edges in  $\mathcal{H}$  (there are  $p$  edges of the star, the edge  $\{x, y, z\}$ , at least one  $x$ -edge and at least one  $y$ -edge). Then,  $p + 3 \leq n - 2$ . But  $p > \frac{n}{2}$ , hence  $n \geq 11$ . In general, we have at least  $(\xi + \eta + 1 + p)$  edges in  $\mathcal{H}$ . Therefore  $\xi + \eta \leq \frac{n}{2} - 3$ . Then a product  $\xi\eta$  is maximal if  $\xi = \eta = \frac{1}{2}(\frac{n}{2} - 3)$ . Analogously as above, we use a probabilistic argument to show that there is a packing permutation  $\sigma'$  without fixed points of  $\mathcal{H}'$ . Observe that there are  $\xi + \eta$  edges in  $\mathcal{H}'_2$ , and an  $x$ -edge cannot be mapped by  $\sigma'$  onto a  $y$ -edge (and vice versa). We have

$$\begin{aligned} Pr \left( \bigcup_{e, f \in \mathcal{H}'} A(e \cap f) \right) &\leq \frac{2 \cdot 2\xi\eta \cdot (n - 2)!}{n!} + \binom{n}{3}^{-1} (n - 2 - p - 3)^2 \leq \\ &\leq \frac{(n - 6)^2}{4n(n - 1)} + \frac{3(n - 10)^2}{2(n - 2)(n - 3)(n - 4)} < \frac{1}{e} - \frac{1}{n!}. \end{aligned}$$

It is easy to check that the last inequality is satisfied for  $n \geq 11$ , and consequently, there exists a packing permutation of  $\mathcal{H}'$  without fixed points.

**Case 2D.** There is at most one vertex of degree at least two in  $\mathcal{H}_2$ , and  $j + k + l \geq 2$ .

Then,  $\mathcal{H}_2$  has only tree components,  $l = 0$  and  $k \leq 1$ .

If  $k = 0$ , then  $j \geq 2$  and  $j \leq \frac{n}{2}$  (because  $j$  is the number of isolated edges in  $\mathcal{H}_2$ ). Then, by Lemma 7, the conclusion holds.

Thus, let  $k = 1$  and  $j \geq 1$ . Denote by  $K_{1,p}$  the star in  $\mathcal{H}_2$ . If  $p + j \leq \frac{n}{2}$ , we are done by Lemma 7.

Hence  $p + j > \frac{n}{2}$  and  $n = i + 2j + p + 1$ . If  $j = 1$ , then a 3-edge can block at most two possibilities for the choice of two leaves in  $\mathcal{H}_2$  if one leaf is in the star. So, if we are not able to apply Lemma 6, we have to have at least  $\frac{2p}{2}$  blocking edges in  $\mathcal{H}$ . If we are not able to apply Lemma 5, we have to have at least  $\frac{i}{2}$  blocking edges in  $\mathcal{H}$ . Observe that in both cases the blocking edges are distinct. Hence, taking into account all 2-edges we get

$$n - 2 \geq |\mathcal{E}| \geq p + 1 + p + \frac{i}{2},$$

and

$$n - 3 \geq 2p + \frac{i}{2}.$$

On the other hand,  $n - 3 = i + p$ . Therefore,  $\frac{i}{2} \geq p$ . So,  $n - 3 \geq 3p$ . It follows that  $p < \frac{n}{3}$ , a contradiction.

Now, let  $j \geq 2$ . Observe that the number of 3-edges in  $\mathcal{H}$  is at least  $\frac{i}{2}$  (because of Lemma 5), and at least  $\frac{2pj}{2}$  (because of Lemma 6). (In the latter case, we may assume that one of the leaves comes from the star.)

We have

$$n - 2 \geq |\mathcal{E}| \geq j + p + pj + \frac{i}{2}.$$

But  $j + p > \frac{n}{2}$ , so

$$n - 2 \geq \frac{n}{2} + pj + \frac{i}{2}.$$

Hence

$$\frac{n}{2} - \frac{i}{2} - 2 \geq pj.$$

We know from a structure of the hypergraph that  $n = i + 2j + p + 1$ , so it follows from the above inequality that

$$\frac{2j + p + 1 - 4}{2} \geq pj.$$

This inequality together with the fact that  $2pj \geq 2p + 2j$  for  $p, j \geq 2$ , implies

$$2j + p - 3 \geq 2pj \geq 2p + 2j,$$

a contradiction.

This ends the proof of the theorem.



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