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Let G be a simple graph and let us color its edges so that the *multisets* of colors around each vertex are distinct. The smallest number of colors for which such a coloring exists is called the *irregular coloring number* of G and is denoted by $c(G)$. We determine the exact value of the irregular coloring number for almost all 2-regular graphs. The results obtained provide new examples demonstrating that a conjecture by Burris is false. As another consequence, we also determine the value of a graph invariant called the *point distinguishing index* (where *sets*, instead of multisets, are required to be distinct) for the same family of graphs.

Keywords: irregular edge coloring, irregular coloring number, 2-regular graph, point distinguishing index

1 Introduction

Consider a simple (without loops and multiple edges) undirected graph G. Let C be a color set, $w\colon E(G) \to C$ an edge coloring, and let $MS_w(v)$ denote the *multiset* of colors of all the edges incident with a vertex v in G. A coloring w is said to be *irregular* or *vertex distinguishing* if for any two distinct vertices u, v of G the corresponding multisets hold $MS_w(u) \neq MS_w(v)$. We ask for the minimal number of colors needed to obtain an irregular edge coloring and we call it the *irregular coloring number*. Moreover, we denote by $c(G)$ the irregular coloring number of a given graph G. Note that such a coloring does not exist at all if G contains an isolated edge or more than one isolated vertex. In such a case we set $c(G) = \infty$.

The irregular coloring number has also another interesting interpretation, and was introduced by Aigner and Triesch in [1] as a variant of another widely studied (in more than 40 papers, e.g. [9, 12, 14, 15, 16]) parameter, the *irregularity strength* of graphs. Namely, if we consider labelings of edges of a given graph G with positive integers, $c(G)$ can be defined as the minimal cardinality of such a subset of $\mathbb{N}$ (the set of labels) that allows us to distinguish all the vertices of G by the sums of labels of edges incident with them, see [1] for details.

To provide an example, let us recall that in [1] the irregular coloring number was determined to be equal to three for K_n ($n \geq 3$) and $K_{n,n}$ ($n \geq 2$). Other results, mostly for connected graphs, such as paths, cycles (see [1]) or complete multipartite graphs (see [17]), are also known. These are however difficult to generalize in case of their nonconnected equivalents. On the other hand, a general bound

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from [1], $c(G) \leq |G| - 1$ for graphs with $c(G) < \infty$ other than a triangle, is far from being sharp for numerous families of graphs. In particular, Aigner, Triesch and Tuza proved in [2] that for k-regular graphs, $c(G) \leq Cn^{\frac{1}{k}}$, where C is a constant dependent only on k ($k \geq 2$) and n is the order of the graph. Again determining the exact value of the irregular coloring number, or at least the best possible constant C, proved to be complicated, even for 2-regular graphs. A representative of such a family is simply a disjoint union of cycles and can be denoted as $G = C_{m_1} \cup \ldots \cup C_{m_t}$, where C_i is a cycle of length i. The following upper bound was established by Aigner et al..

Theorem 1 ([2]) *Let $G = C_{m_1} \cup \ldots \cup C_{m_t}$ be a simple 2-regular graph of order $n = \sum_{i=1}^{t} m_i$. Then*

$$c(G) \leqslant \frac{9}{\sqrt{2}}\sqrt{n} + O(1).$$

It was then improved by Wittmann.

Theorem 2 ([17]) *Let $G = C_{m_1} \cup \ldots \cup C_{m_t}$ be a simple 2-regular graph of order $n = \sum_{i=1}^{t} m_i$. Then*

$$c(G) \leqslant \sqrt{2n} + O(1).$$

This bound was already best possible except for an additive constant term. On the other hand, the role of this constant is crucial in face of a conjecture by Burris we mention below. Several exact results were then obtained for some special 2-regular graphs. Among others, in [10] and [11] the irregular coloring number was determined for unions of cycles of exclusively *even* lengths. Earlier, a particular, but very important case (we refer to this result at the end of the paper, explaining its significance), was solved by Aigner and Triesch, [1]. Namely, they showed that $c(tC_3) = \min(\min\{r : \lfloor \frac{r}{3} \lfloor \frac{r-1}{2} \rfloor \rfloor \geq t, r \not\equiv 5 (\mathrm{mod}\ 6)\}, \min\{r : \lfloor \frac{r}{3} \lfloor \frac{r-1}{2} \rfloor \rfloor - 1 \geq t, r \equiv 5 (\mathrm{mod}\ 6)\})$, where tC_3 consists of t disjoint triangles.

As usual, let $\delta(G)$ and $\Delta(G)$ stand for the *minimum* and the *maximum*, resp., *degree* of a given graph G. Additionally, let n_d (or $n_d(G)$) denote the number of vertices of degree d in G. Note that if there exists any irregular edge coloring of G with k colors, then, by the standard combinatorial formula for the number of multisets of a given size, we must have that

$$\binom{k+d-1}{d} \geqslant n_d \tag{1}$$

for each $d \geqslant 1$. The following Vizing-type conjecture was posed by Burris in her PhD thesis [7].

Conjecture 3 ([7]) *Let G be a graph without isolated edges, containing at most one isolated vertex, and let k be the minimum integer such that $\binom{k+d-1}{d} \geqslant n_d(G)$ for $1 \leqslant d \leqslant \Delta(G)$. Then $c(G) = k$ or $k + 1$.*

The aim of this paper is to determine the irregular coloring number for almost all (except a finite number) 2-regular graphs, see Corollary 12. Consequently, we will obtain infinitely many counterexamples against Conjecture 3, which will lead to its reformulation in weaker versions, see Conjectures 13 and 14.

As another consequence of our reasoning we also determine the value of the graph invariant called the *point distinguishing chromatic index*, $\chi_0(G)$, for the same family of graphs. This parameter, introduced by Harary and Plantholt [13], differs from $c(G)$ only by the requirement that *sets*, $S_w(v)$, instead of multisets, $MS_w(v)$, are distinct for all the vertices of G, where $w\colon E(G) \to C$ is an edge coloring. One can easily verify that both parameters coincide in the case of 2-regular graphs. For related problems see [3, 5, 6, 8].

2 Equivalent problem

The problem of determining the smallest number of colors for which there exists an irregular edge coloring is in the case of *2-regular graphs* equivalent to a different one. To state this new problem clearly we have to first specify some notations. In our reasoning we will admit (simple) graphs containing possible additional single loops at some of the vertices. Though these are actually pseudographs, we still call them *graphs*. A *loop* at a vertex v of such a graph is then denoted by vv (since it is uniquely determined by v). Analogously, an edge with ends u and v is denoted by uv (or vu).

A *trail* of length n in a graph G is an alternating sequence $v_0e_0v_1e_1 \dots e_{n-1}v_n$ of $n+1$ vertices and n edges (or loops) of G such that the $e_i = v_iv_{i+1}$, $0 \le i < n$, are pairwise distinct. We usually identify such a trail with the graph (subgraph) G' spanned by its edges (and loops), i.e., where $V(G') = \{v_0, \dots, v_n\}$ and $E(G') = \{e_0, \dots, e_{n-1}\}$. In some cases we will write G' simply as a sequence of its "consecutive" vertices $v_0v_1 \dots v_n$ (where we will usually identify all the sequences corresponding to the same graph G'). Such a trail is called *closed* if $v_0 = v_n$, or *open* otherwise. Obviously, some special representatives of trails are *paths* and *cycles*. Let us additionally assume that a single loop does not constitute a graph, hence a single loop is neither a closed trail, nor a cycle. Note that a closed trail may be regarded as an *Eulerian graph*, i.e., a connected graph with all the degrees being even integers (where a loop adds 2 to the degree of a vertex), while an open trail is just a connected graph with exactly two vertices of odd degrees. In general, we will call a graph *even* if the degrees of all its vertices are even integers.

Now given an irregular edge coloring $w : E \to \{1, \dots, k\}$ of a 2-regular graph $G = C_{m_1} \cup \dots C_{m_t}$, we define the *graph H of* such *a coloring* as follows. Let its vertex set contain all the colors, hence $V(H) = \{1, \dots, k\}$, and let us draw an edge in H whenever colors i and j ($i = j$ possibly) meet at some vertex in G. In other words, $E(H) = \{S_w(v) : v \in V(G)\}$. Since our coloring is irregular, we never draw an edge (or loop) twice in H. It is also easy to observe that when we traverse consecutive vertices of a given cycle C_{m_i}, we obtain a closed trail in H as a result. Therefore, H is just a sum of t closed trails of lengths $m_1, \dots, m_t$, hence is an even subgraph of size $\|G\|$ of a graph M_k, where M_k is defined to be *the complete graph K_k with a single loop added at each vertex*. Conversely, if we have an even subgraph H of the graph M_k which can be written as a sum of t edge disjoint closed trails of lengths $m_1, \dots, m_t$, then reversing our process, i.e., traversing each of these closed trails and painting the consecutive edges of the corresponding cycle from G with the colors of the vertices encountered in H, yields an irregular edge coloring of G. Thus we have reduced our problem to the following one.

Observation 4 *For any* 2*-regular graph $G = C_{m_1} \cup \dots C_{m_t}$, the irregular coloring number is equal to the smallest k for which there exists a subgraph of the graph M_k that is an edge disjoint sum of closed trails of lengths $m_1, \dots, m_t$.* □

3 Necessary condition

Consider a problem defined by Observation 4, where we want to determine the smallest possible k complying with the stated requirements for $G = C_{m_1} \cup \dots C_{m_t}$. Note that an obvious necessary condition for such a k is a requirement that there is at least one even subgraph of size $m_1 + \dots + m_t$ in M_k. We must however require something more from such a subgraph. The number of its possible loops might be limited. It is because closed trails of length 3 cannot contain loops, a closed trail of length 4 may have at most one loop, the one of length 5, two loops, etcetera. A general *function L on the maximal possible* *number of loops in a closed trail of length* m ($m \geq 3$) is the following.

$$L(m) = \begin{cases} 0, & \text{for } m = 3, \\ 1, & \text{for } m = 4, \\ \lfloor \frac{m}{2} \rfloor, & \text{for } m \geq 5. \end{cases}$$

We may then denote that in total

$$L(m_1, \ldots, m_t) = L(m_1) + \ldots + L(m_t).$$

On the other hand, we have only k loops available in M_k. Combining these two facts, we obtain the following function L_k on *the maximal possible number of loops in the required even subgraph of* M_k.

$$L_k(m_1, \ldots, m_t) = \min\{k, L(m_1, \ldots, m_t)\}.$$

Finally, we obtain the following necessary condition.

Observation 5 *For any collection of lengths* $m_1, \ldots, m_t$*,* $m_i \geq 3$*, and any* k*, if there exists an even subgraph of size* $m_1 + \ldots + m_t$ *in* M_k *containing at most* $L_k(m_1, \ldots, m_t)$ *loops, then*

(1°) $m_1 + \ldots + m_t \leq \binom{k}{2} - \frac{k}{2} + L_k(m_1, \ldots, m_t)$ *if* k *is even,*

(2°) $m_1 + \ldots + m_t \leq \binom{k}{2} + L_k(m_1, \ldots, m_t)$ *if* k *is odd and either* $k \not\equiv 5 (\mathrm{mod}\ 6)$ *or* $m_i \geq 4$ *for at least one* i*,*

(3°) $m_1 + \ldots + m_t \leq \binom{k}{2}$ *and* $m_1 + \ldots + m_t \neq \binom{k}{2} - 1$ *if* $k \equiv 5 (\mathrm{mod}\ 6)$ *and* $m_j = 3$ *for all* j*.*

Proof: There are $\binom{k}{2}$ edges and k loops in M_k. Since the subgraph whose existence we assumed cannot have more than $L_k(m_1, \ldots, m_t)$ loops, the bound from (2°) is obvious. In (1°) on the other hand, the degree of each vertex in M_k is odd. Since our aim is to find an *even* subgraph of M_k, we must additionally remove at least as many edges (loops do not change the degrees of the vertices) as there are in a perfect matching, that is $\frac{k}{2}$, from M_k. In (3°) in turn, which is a very special case, we have $m_1 = \ldots = m_t = 3$, hence $L_k(m_1, \ldots, m_t) = 0$, and there is no *even* subgraph of K_k (M_k without loops) containing all except *exactly one* (that is $\binom{k}{2} - 1$) of its edges. □

Now, consider for instance the lengths $3, 3, 6, 6$ and $k = 6$. Then $L_6(3, 3, 6, 6) = 6$ and $\binom{6}{2} - \frac{6}{2} + 6 = 18$, hence our necessary condition (see (1°)) is fulfilled. Nevertheless, as one can easily check, it is not enough for the existence of the edge disjoint closed trails of specified lengths in M_6. Surprisingly, it occurs that for k large enough the necessary conditions from Observation 5 are also sufficient. This fact, which also allows us to determine the exact value of the irregular coloring number for almost all 2-regular graphs, is the main result of this paper, see Theorem 11 and Corollary 12.

It is worth noting that for the general case, not only starting from a given (large) n, it is known that $c(G) \leq \lceil \sqrt{2n} \rceil + 1$ for any 2-regular graph, even if we require the coloring to be proper. This bound is off by at most two from the real value, and is a consequence of the following Corollary 1.2 from Balister [3], with two additional simple cases that must be verified separately, i.e., $G = C_8$ and $C_4 \cup C_4$.

Theorem 6 ([3]) *Let* $L = \sum_{i=1}^{t} m_i$*,* $m_i \geq 3$*; then we can write some subgraph of* K_N *as an edge-disjoint union of closed trails of lengths* $m_1, \ldots, m_t$ *if and only if either*

(1) N *is odd,* $L = \binom{N}{2}$ *or* $L \leq \binom{N}{2} - 3$*, or*

(2) N *is even,* $L \leq \binom{N}{2} - \frac{N}{2}$.

4 The idea and lemmas

The main idea of the proof of Theorem 11 is based on the following result concerning *simple* graphs due to Balister.

Theorem 7 ([4]) *There exist absolute constant N and $\varepsilon > 0$ such that for any even simple graph G on n vertices with $n \geq N$ and $\delta(G) \geq (1-\varepsilon)n$, and for any collection of integers $m_1, \ldots, m_t$ with $m_i \geq 3$ and $\sum_{i=1}^{t} m_i = |E(G)|$, one can write G as the edge-disjoint union of closed trails $T_1, \ldots, T_t$ with T_i of length m_i for $i = 1, \ldots, t$. In addition, given any fixed $v \in V(G)$, we can also ensure that T_1 meets v.*

Note that if we want to prove the sufficiency of the conditions (1°)-(3°) from Observation 5, then given the lengths $m_1, \ldots, m_t$, $m_i \geq 3$, of the cycles of a 2-regular graph and the corresponding k complying with (1°)-(3°), our aim is to construct a subgraph of M_k which is an edge disjoint sum of closed trails of the given lengths (compare also with Observation 4). A natural approach is now to first find a ("small") number of necessary closed trails in M_k "covering" all the loops we should use in our construction. If we are able to do it using not too many (bounded by a constant) edges at each vertex, then after removing them (together with all the loops) from M_k, we will be able to use Theorem 7 above for the resulting graph and the remaining set of lengths, provided that k is large enough.

A realization of the first part of this plan is contained in Lemma 10. It is preceded by the following two auxiliary lemmas. First of them will help us to "close" open trails of odd lengths by joining their ends with paths of length two. The second, in turn, will be used later in the construction of closed trails of lengths four taking form of the triangles with single loops.

Lemma 8 *Let G be a simple graph of order $n \geq 51$ and with minimum degree $\delta \geq n-11$, and let U be a set of at most $\frac{n}{2}$ disjoint pairs of its vertices $\{\{x_1, y_1\}, \ldots, \{x_k, y_k\}\}$ ($x_i \neq y_i$). Then there exist vertices $z_1, \ldots, z_k$ such that $x_i z_i y_i$, $i = 1, \ldots, k$, form a set of k edge disjoint paths of length 2 in G, and that G' obtained from G by removal of all the edges of these paths has minimum degree $\delta' \geq n-14$.*

Proof: It is sufficient to choose *pairwise distinct* $z_1, \ldots, z_k$ such that $x_i z_i y_i$, $i = 1, \ldots, k$, are the edge disjoint paths in G. Assume then that U' is a maximal subset of U for which we can find vertices z_i (or equivalently paths of length two) complying with these requirements, and let Z' be the set of the middle vertices of the obtained paths. Suppose that $U' \neq U$ (hence $|U'| = |Z'| \leq \frac{n}{2} - 1$) and let $\{x, y\} \in U \smallsetminus U'$. Now, to complete the proof, it is sufficient to get a contradiction by finding a path complying with our requirements for this pair of vertices. Note that any element $z \in M := N_G(x) \cap N_G(y)$ constitute a middle vertex of some path of length 2 in G with endpoints x and y. A given such z cannot be used in construction of the desired path by one of the two following reasons. First of all we need to exclude $z \in Z'$ (since z_i must be pairwise distinct). Second, we need to eliminate the ends of these (at most two) potential paths from U' in which x or y might be a middle vertex (since the paths are supposed to be edge disjoint), i.e., at most four vertices. Consequently, we are still left with at least $|M| - |Z'| - 4 \geq (2\delta - n) - (\frac{n}{2} - 1) - 4 \geq n - 22 - (\frac{n}{2} - 1) - 4 > 0$ appropriate candidates for z, a contradiction. □

Lemma 9 *Let G be a simple graph of order $n \geq 34$ and with minimum degree $\delta \geq n-5$, and let $W = \{v_1, \ldots, v_l\}$ be a subset of its vertices. Then there exist l edge disjoint triangles, $H_1, \ldots, H_l$, in G such that v_i is a vertex of H_i for each i, and that G' obtained from G by removal of all the edges of these triangles has minimum degree $\delta' \geq n-11$.*

Proof: It is sufficient to show our assertion in the case when $W = V(G)$. By the Dirac condition, the graph G contains a Hamilton cycle C. To prove the theorem, for each vertex $v_i \in W = V(G)$ we will choose a distinct edge $e_i = u_i w_i$ of the cycle C such that $\Delta_i = v_i u_i w_i$ forms a triangle in G (then each vertex will belong to exactly three of these triangles), $v_i u_i, v_i w_i \notin E(C)$, and that the triangles obtained are edge disjoint. Let W' be a maximal *size* subset of W for which we can choose such edges (or equivalently triangles). Denote by E' the set of these edges ($E' \subseteq E(C)$), and let $\mathcal{H}$ be the set of the corresponding triangles. For a given vertex v', let $T_{v'}$ denote the set of the edges of C whose ends span in G triangles together with v', each containing exactly one edge of C. Then $|T_{v'}| \geq n - 2(n-\delta) - 2 = 2\delta - n - 2$. It is because each vertex which is not adjacent with v' meets, hence excludes, at most two of the n edges of C, and we additionally exclude the two edges of C which might span, together with v', triangles containing more than one edge of C in G. Analogously, for a given edge $e' = u'w'$ of C, let $T_{e'}$ denote the set of the vertices spanning in G triangles together with e', each containing exactly one edge of C, hence $|T_{e'}| \geq |N_G(u') \cap N_G(w')| - 2 \geq 2\delta - n - 2$.

Suppose now that the theorem is false, hence $W' \neq W$ ($W = V(G)$), and let $v \in W \smallsetminus W'$. Then at least one edge of C, say $e = uw$, does not belong to any of the triangles from $\mathcal{H}$. Note that by the maximality of W', there cannot exist an edge $xy \in T_v \smallsetminus E'$ such that vxy is a triangle in G which is edge disjoint with all the triangles from $\mathcal{H}$. Denote then by F the set of these edges $xy \in T_v \cap E'$ for which the triangle vxy is edge disjoint with all the triangles from $\mathcal{H}$ except the one containing xy. Observe that if $x'y' \in T_v \smallsetminus F$, then $x'y'$ must be incident with one of at most two triangles of $\mathcal{H}$ containing v, hence there can be at most 4 such edges in T_v. Consequently, $|F| \geq |T_v| - 4 \geq 2\delta - n - 6$. Analogously, by the maximality of W', there cannot exist a vertex $z \in T_e \smallsetminus W'$ such that zuw is a triangle in G which is edge disjoint with all the triangles from $\mathcal{H}$. Denote by U the set of these vertices $z \in T_e \cap W'$ for which the triangle zuw is edge disjoint with all the triangles from $\mathcal{H}$. Observe that if $z' \in T_e \smallsetminus U$, then zu or zw must be an edge of a triangle from $\mathcal{H}$, hence there are at most 6 such vertices in T_e. It is because each of the vertices u, w is incident with at most three edges of the triangles from $\mathcal{H}$ which do not belong to C (contained in at most two triangles from $\mathcal{H}$ for each of them). Consequently, $|U| \geq |T_e| - 6 \geq 2\delta - n - 8$, hence $|F| + |U| \geq 4\delta - 2n - 14 \geq 2n - 34 \geq n = |W| > |W'|$. Therefore, since $F \subseteq E'$ and $U \subseteq W'$, the set $\mathcal{H}$ must contain a triangle $\Delta_j = v_j u_j w_j$ such that $v_j \in U$ and $u_j w_j \in F$. Then, by the definitions of F and U, $\Delta' = v u_j w_j$ and $\Delta'' = v_j uw$ form triangles in G which are edge disjoint with the ones from $\mathcal{H} \smallsetminus \{\Delta_j\}$. Moreover, Δ' and Δ'' are also edge disjoint, since by the definition of U, the edges $e = uw$ and $u_j w_j$ must be independent. The obtained collection of triangles, $(\mathcal{H} \smallsetminus \{\Delta_j\}) \cup \{\Delta', \Delta''\}$, contradicts the maximality of W'. $\square$

Lemma 10 *Let $n \geq 51$ and let $m_1, \ldots, m_t \geq 4$ be integers such that $s = \sum_{i=1}^{t} L(m_i) \leq n$. Then for any $\gamma = 0, 1, 2$ there exists a subgraph G' of M_n of maximum degree at most 12 (not taking loops into account) containing exactly s loops, which is an edge disjoint union of closed trails $T_1, \ldots, T_t$ of lengths $m_1, \ldots, m_{t-1}, m_t + \gamma$, respectively.*

Proof: In other words, we need to find t edge disjoint closed trails $T_1, \ldots, T_t$ of specified lengths in M_n, such that their sum G' is a graph of maximum degree at most 12 (not taking loops into account) which contains s loops.

First partition the sequence $m_1, \ldots, m_{t-1}$ (note that we omit m_t) into three subsequences $e_1, \ldots, e_r$, $f_1, \ldots, f_l$, $o_1, \ldots, o_k$ containing all its even elements of lengths at least 6, all elements equal to four, and all odd elements, respectively (hence $e_i \equiv 0 (\mathrm{mod}\ 2)$, $e_i \geq 6$, $f_i = 4$ and $o_i \equiv 1 (\mathrm{mod}\ 2)$ for all i).

Let $G = M_n$, $G = (V, E)$, and let $V = \{a_1, \ldots, a_n\}$. Then $C = a_1 \ldots a_n a_1$ forms a Hamilton cycle in G. We begin our construction from the first of the subsequences. Note that $\sum_{i=1}^{r} \frac{e_i}{2} = \sum_{i=1}^{r} L(e_i) \leq n - L(m_t)$. Therefore, we can construct the first r closed trails one after another by simply taking consecutive blocks of $\frac{e_i}{2}$, $i = 1, \ldots, r$, vertices on the Hamilton cycle C together with the edges of this cycle joining these vertices, then closing each of the obtained paths up by adding the edge joining its ends, and finally attaching all the loops adjacent with it. This way, the first trail E_1 of length e_1 is of the form $a_1 a_1 a_2 a_2 \ldots a_{L(e_1)} a_{L(e_1)} a_1$ and contains loops at the vertices $a_1, \ldots, a_{L(e_1)}$. Subsequently, the trail E_2 of length e_2 is defined by the sequence $a_{L(e_1)+1}\ a_{L(e_1)+1}\ a_{L(e_1)+2}\ a_{L(e_1)+2} \ldots a_{L(e_1)+L(e_2)} a_{L(e_1)+L(e_2)} a_{L(e_1)+1}$, etcetera.

Now take $L(m_t)$ consecutive (after those already used) vertices on the Hamilton cycle C and join them with a path (in particular we may have a path reduced to one point when $L(m_t) = 1$), adding to this path all the loops encountered. To create a closed trail of length $m_t + \gamma$ from this open trail, we join its ends with a path avoiding the edges of the cycle C (and the edges already used) of length at most $\gamma + 3 \leq 5$. Since we still have a very dense graph (even after removing the edges already used in our construction and all the edges of C), it is easy to find such a path using a greedy algorithm (taking at each step the first "suitable" vertex/edge). This way we have already constructed the closed trails containing $\sum_{i=1}^{r} L(e_i) + L(m_t)$ consecutive loops of the Hamilton cycle. Moreover, each vertex of G meets at most 4 edges (and at most one loop) from the sum of these trails.

Let $v_1, \ldots, v_l$ be the subsequent l consecutive vertices on the Hamilton cycle C (with loops at them unused), and denote $W = \{v_1, \ldots, v_l\}$. Then $v_l = a_q$ for some $q \leq n$. Let us temporarily remove from G the edges $a_{q+1}a_{q+2}, a_{q+2}a_{q+3}, \ldots, a_{n-1}a_n$ (we may need them in the further part of our construction), all the edges of the closed trails already constructed and all the (n) loops. Denote the graph obtained by G_1. Note that $\delta_1 \geq n - 5$, where δ_1 is the minimum degree of G_1. Let $\mathcal{H}$ be the collection of triangles in G_1 guaranteed by Lemma 9. Joining these triangles with the loops at $v_1, \ldots, v_l$, resp., we obtain the desired closed trails of length four ($f_1, \ldots, f_l = 4$) in G.

Remove from G_1 the edges of the triangles from $\mathcal{H}$ and denote the graph obtained by G_2. By Lemma 9, which we have used, its minimum degree is at least $n - 11$. Now add back the edges $a_{q+1}a_{q+2}, a_{q+2}a_{q+3}, \ldots, a_{n-1}a_n$ together with the loops incident with them to this graph. We will use some of them to create open trails of lengths $o_1 - 2, \ldots, o_k - 2$, which will be then completed to form the desired closed trails. Again we do it by simply taking the proper consecutive bits of C together with all the incident loops. For instance, the first of such open trails is of the form $a_{q+1}a_{q+1}a_{q+2}a_{q+2} \ldots a_{L(o_1)}a_{L(o_1)}$ (since o_1 is odd, $L(o_1) = \frac{o_1 - 1}{2}$). Let $\{\{x_1, y_1\}, \ldots, \{x_k, y_k\}\}$ be the pairs of the corresponding ends of these open trails, and let us remove all the edges of these trails and all the remaining loops. Since we have just removed only some part of the edges (and loops) which we had formerly added back to G_2, we still have $\delta_3 \geq n - 11$, where δ_3 is the minimum degree of the obtained graph G_3. By Lemma 8 we can now easily find k edge disjoint paths of length 2 that complete our construction of the trails of lengths $o_1, \ldots, o_k$. Moreover, these paths can be chosen in such a way that if we remove their edges from G_3, then the minimum degree of the graph obtained will be at least $n - 14$. Since initially all the vertices of G had degree $n - 1$ (not taking loops into account), then the graph obtained as a sum of all the closed trails constructed has maximum degree at most 13 (not taking loops into account) and exactly s loops. Finally, since such a graph is even, its maximum degree cannot in fact exceed 12 (not taking loops into account). $\square$

5 Main result

Theorem 11 *There exists an absolute constant K such that for any $k \geq K$ the following conditions are sufficient for existence of a subgraph of M_k which is an edge disjoint union of closed trails of lengths $m_1, \ldots, m_t$, where $m_i \geq 3$.*

(1°) $m_1 + \ldots + m_t \leq \binom{k}{2} - \frac{k}{2} + L_k(m_1, \ldots, m_t)$ *if k is even,*

(2°) $m_1 + \ldots + m_t \leq \binom{k}{2} + L_k(m_1, \ldots, m_t)$ *if k is odd and either $k \not\equiv 5 (\mathrm{mod}\ 6)$ or $m_i \geq 4$ for at least one i,*

(3°) $m_1 + \ldots + m_t \leq \binom{k}{2}$ *and* $m_1 + \ldots + m_t \neq \binom{k}{2} - 1$ *if $k \equiv 5 (\mathrm{mod}\ 6)$ and $m_j = 3$ for all j.*

Proof: Let N and ε be the constants from Theorem 7. Assume that $N_0 \geq N$ is the smallest integer such that $k - 16 \geq (1-\varepsilon)k$ for each $k \geq N_0$, and let $K = \max\{N_0, 51\}$. We shall prove that this K complies with our requirements.

Assume that the lengths $m_1, \ldots, m_t$ are ordered in such a way that all threes are placed at the end of the sequence. Let $n = \sum_{i=1}^t m_i$, $L = L_k(m_1, \ldots, m_t)$, and assume that $k \geq K$ complies with the assumptions of the theorem, hence (1°)-(3°) are fulfilled. We will first use Lemma 10 to find in M_k closed trails "covering" all the loops we want to use in our construction. In most of the cases this will be exactly L loops, but in some special situations we shall limit this number to $L-1$ or $L-2$.

Assume first that k is odd. Below we define the value of a constant L' ($L' \leq L$), dependent on the given cases. This in fact will be the number of loops which we will use while applying Lemma 10. Let $L' = L$ if $n - \binom{k}{2} \leq L - 3$ and $L' = n - \binom{k}{2}$ otherwise. Note that $L' \geq 0$ in all these cases. It is obvious for $L' = L$. For $L' = L - 1$, on the other hand, if we had $L' < 0$, hence $L = 0$, then $n = \binom{k}{2} + 0 - 1$ would have to be divisible by 3 (because if $L = 0$, then $m_i = 3$ for all i). Consequently, we would get a contradiction with (3°), since $\binom{k}{2} - 1$ is divisible by 3 (k odd) only for $k \equiv 5 (\mathrm{mod}\ 6)$. Finally, for $L' = L - 2$, if we had $L' < 0$, then we would analogously as above either obtain $\binom{k}{2} - 2 = 3t$ (for $L = 0$) or $\binom{k}{2} + 1 - 2 = 3t + 1$ (for $L = 1$). In both cases we obtain a contradiction with the fact that $\binom{k}{2} - 2$ is never divisible by three. Let s be the least integer ($s \geq 0$) such that

$$L(m_1) + \ldots + L(m_s) \geq L'.$$

Let $m'_s \leq m_s$ be the least integer so that $\sum_{i=1}^{s-1} L(m_i) + L(m'_s) = L'$, and set $\gamma = m_s - m'_s$ if $m_s - m'_s \leq 2$, or $\gamma = 0$ otherwise. By Lemma 10 there exists a subgraph G' of M_k of maximum degree at most 12 (not taking loops into account) containing exactly L' loops, which is an edge disjoint union of closed trails $T_1, \ldots, T_s$ of lengths $m_1, \ldots, m_{s-1}, m'_s + \gamma$, respectively. Let G'' be a graph obtained from M_k by removing all the edges of G' and all the (k) loops. Since M_k (k odd) is an *even* graph, G'' is an even graph as well. Moreover, $\delta(G'') \geq k - 13$. Note that by our construction (by the choice of L' actually), either $m_s - m'_s - \gamma + \sum_{i=s+1}^t m_i = ||G''||$, or $m_s - m'_s - \gamma + \sum_{i=s+1}^t m_i \leq ||G''|| - 3$, hence there exists m_{t+1} such that $m_s - m'_s - \gamma + \sum_{i=s+1}^{t+1} m_i = ||G''||$ and $m_{t+1} = 0$ or $m_{t+1} \geq 3$. By Theorem 7, G'' can be written as the union of closed trails of lengths $m_s - m'_s - \gamma, m_{s+1}, \ldots, m_{t+1}$ (where we ignore the lengths equal to 0 if there are any), such that the closed trail of length $m_s - m'_s - \gamma$ contains any fixed vertex v of the formerly constructed closed trail T_s. By joining these two trails together, we obtain a one of length m_s, hence complete the construction. (Note that the closed trail of length m_{t+1} plays only a technical role in our reasoning.)

Assume now that k is even. This time we shall use exactly L loops while applying Lemma 10. Let then analogously as above s be the least integer ($s \geq 0$) such that

$$L(m_1) + \ldots + L(m_s) \geq L.$$

Let $m'_s \leq m_s$ be the least integer so that $\sum_{i=1}^{s-1} L(m_i)+L(m'_s) = L$, and let $\gamma = m_s-m'_s$ if $m_s-m'_s \leq 2$, or $\gamma = 0$ otherwise. By Lemma 10 there exists a subgraph G' of M_k of maximum degree at most 12 (not taking loops into account) containing exactly L loops, which is an edge disjoint union of closed trails $T_1, \ldots, T_s$ of lengths $m_1, \ldots, m_{s-1}, m'_s + \gamma$, respectively. Let G'' be a graph obtained from M_k by removing all the edges of G' and all the (k) loops. Note that $m_s-m'_s-\gamma+\sum_{i=s+1}^{t} m_i \leq ||G''||-\frac{k}{2}$, where the degrees of all the vertices in G'' are odd. Let us define $m_{t+1} = ||G''||-\frac{k}{2}-(m_s-m'_s-\gamma+\sum_{i=s+1}^{t} m_i)$ if this number equals at least 3, or set $m_{t+1} = 0$ (for definiteness) otherwise. Consequently,

$$m_s - m'_s - \gamma + \sum_{i=s+1}^{t+1} m_i \in \{||G''|| - \frac{k}{2}, ||G''|| - \frac{k}{2} - 1, ||G''|| - \frac{k}{2} - 2\}. \tag{2}$$

In order to apply now Theorem 7, we need to first modify the graph G'' by removing some edges from it, so that it is even and have a proper size (i.e., $m_s-m'_s-\gamma+\sum_{i=s+1}^{t+1} m_i$). If the sum in (2) equals $||G''||-\frac{k}{2}$, we simply remove a perfect matching, that is $\frac{k}{2}$ independent edges, from G''. Its existence is guaranteed e.g. by the Dirac condition for hamiltonicity (G'' still has “large” minimum degree compared to its order). In the remaining two cases, we first remove a star $K_{1,3}$ (for $||G''|| - \frac{k}{2} - 1$), or two independent stars $K_{1,3}$ (for $||G''|| - \frac{k}{2} - 2$), and then a matching of the remaining (all except the ones in the stars) vertices from G''. Since then we obtain a simple even graph of the desired size and with the minimum degree $\delta \geq k - 16$, we analogously as above finish our proof by first applying Theorem 7 to this graph, and then joining the obtained closed trails of lengths $m_s - m'_s - \gamma$ and $m'_s + \gamma$. □

6 Conclusions

By Observation 4, Observation 5 and Theorem 11 we obtain the following corollary.

Corollary 12 *Let $G = C_{m_1} \cup \ldots \cup C_{m_t}$ be a 2-regular graph and let k be the smallest integer for which (1°)-(3°) hold. If $k \geq K$, then $c(G) = k$ (hence also $\chi_0(G) = k$).* □

Consequently, we have determined the exact value of $c(G)$ for all 2-regular graphs of order $n > \binom{K-1}{2} + (K-1) = \binom{K}{2}$ (where we must have taken the worst case (2°) into account), hence for all except a finite number. One can calculate that then $c(G) \in \{\lceil\sqrt{2n}\rceil - 1, \lceil\sqrt{2n}\rceil, \lceil\sqrt{2n}\rceil + 1\}$, what can be conveniently confronted with the result by Wittmann, see Theorem 2.

We must point out here that K from Corollary 12 above is very large. It is due to its dependency on the size of N, which is very large, and ϵ, which is extremely small, from Theorem 7 of Balister. In particular $\epsilon \leq 10^{-24}$, hence $K \geq 16 \cdot 10^{24}$, see [4] for more details and references.

As we have already mentioned our result also provides many counterexamples disproving Conjecture 3. These include the case already pointed out in the introduction, i.e., $G = tC_3$ with $3t = \binom{r+1}{2} - 1$ for any $r \equiv 4 (\mathrm{mod}\ 6)$, where we should have $c(G) \in \{r, r+1\}$ according to Conjecture 3, while in fact, $c(G) = r+2$, see [1] or Corollary 12. Many other counterexamples can however be derived from Corollary 12 (e.g., $G = tC_3$ with $\binom{r+1}{2} - \frac{r+1}{2} < 3t \leqslant \binom{r+1}{2}$ for $r \equiv 5 (\mathrm{mod}\ 6)$ or $r \equiv 1 (\mathrm{mod}\ 6)$). Nevertheless, we suspect that one of the following conjectures, the first of which implies the other, should hold.

Conjecture 13 *Let G be a graph without isolated edges, containing at most one isolated vertex, and let k be the minimum integer such that $\binom{k+d-1}{d} \geqslant n_d(G)$ for $1 \leqslant d \leqslant \Delta(G)$. Then $c(G) \in \{k, k+1, k+2\}$.*

Conjecture 14 *There exists an absolute constant A such that for each graph G without isolated edges which contains at most one isolated vertex, if k is the minimum integer such that $\binom{k+d-1}{d} \geqslant n_d(G)$ for $1 \leqslant d \leqslant \Delta(G)$, then $k \leq c(G) \leq k + A$.*

It is worth noting at the end that the question whether the conjecture by Burris holds for graphs with $\Delta(G) \geq 3$ remains open. A limited number of (small) examples with $\Delta(G) = 3$ we have verified did not give a negative answer to that question.

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