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# On the complexity of vertex-coloring edge-weightings

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Given a graph  $G = (V, E)$  and a weight function  $w : E \rightarrow \mathbb{R}$ , a coloring of vertices of  $G$ , induced by  $w$ , is defined by  $\chi_w(v) = \sum_{e \ni v} w(e)$  for all  $v \in V$ . In this paper, we show that determining whether a particular graph has a weighting of the edges from  $\{1, 2\}$  that induces a proper vertex coloring is NP-complete.

**Keywords:** vertex-coloring, 1-2-3 conjecture, NP-completeness

## 1 Introduction

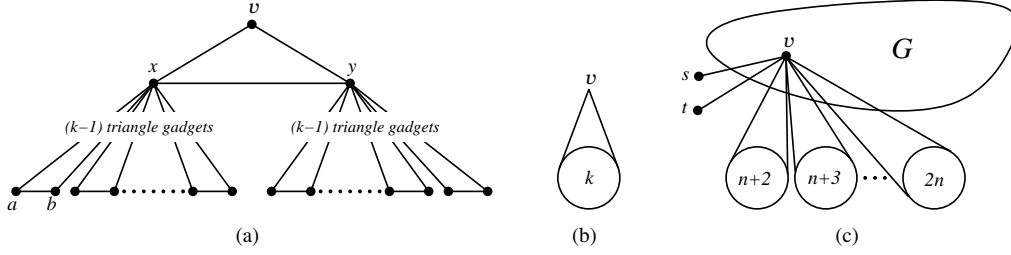
For a given graph  $G = (V, E)$ , let  $w : E \rightarrow \mathbb{R}$  be a weight function. We say that  $w$  is *proper* if the coloring of the vertices  $\chi_w(v) = \sum_{e \ni v} w(e)$ ,  $v \in V$ , is proper. In 2004, Karoński, Łuczak, and Thomason (2004) showed that any graph with no components isomorphic to  $K_2$  has a proper weighting from a finite set of reals. Furthermore, they conjectured that every graph with no components isomorphic to  $K_2$  has a proper weighting from  $W = \{1, 2, 3\}$ . Addario-Berry, Dalal, McDiarmid, Reed, and Thomason (2007) showed that the above holds if  $W = \{1, \dots, 30\}$ . This result was improved by Addario-Berry, Dalal, and Reed (2008), who showed that one can take  $W = \{1, \dots, 16\}$ . Subsequently, Wang and Yu (2008) proved that  $W = \{1, \dots, 13\}$  suffices. A recent breakthrough by Kalkowski, Karoński, and Pfender (2010) showed that the set of weights can be as small as  $W = \{1, 2, 3, 4, 5\}$ .

On the other hand, Addario-Berry, Dalal, and Reed (2008) showed that almost all graphs have a proper weighting from  $\{1, 2\}$ . In this paper, we show that determining whether a particular graph has a proper weighting of the edges from  $\{1, 2\}$  is NP-complete. Consequently, there is no simple characterization of graphs with proper weightings from  $\{1, 2\}$ , unless  $P=NP$ . Formally, let

$$1\text{-}2\text{WEIGHT} = \{G : G \text{ is a graph having a proper weighting from } \{1, 2\}\}.$$

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**Fig. 1:** A  $k$ -disallowing gadget (a) and its symbolic representation (b); a construction of  $h(G)$  (c).

**Theorem 1.1** *1-2WEIGHT is NP-complete.*

Before we prove this statement, we consider a similar theorem with a somewhat simpler proof, which we use as a template to prove Theorem 1.1. By analogy to 1-2WEIGHT we denote by 0-1WEIGHT the family of graphs with a proper weighting from  $\{0, 1\}$  and show the following:

**Theorem 1.2** *0-1WEIGHT is NP-complete.*

## 2 0-1WEIGHT is NP-complete

Here we prove Theorem 1.2.

First note that 0-1WEIGHT is clearly in NP, since one can verify in polynomial time for a given graph whether a weighting of its edges from  $\{0, 1\}$  is proper.

Next we consider the well-known NP-hard problem

$$3\text{-COLOR} = \{G : G \text{ is a graph having a proper 3-vertex-coloring}\}.$$

In order to prove that 0-1WEIGHT is NP-hard (and hence NP-complete), we show a reduction from 3-COLOR to 0-1WEIGHT. To this end, we define a polynomial time reduction  $h$ , such that  $G \in 3\text{-COLOR}$  if and only if  $h(G) \in 0\text{-1WEIGHT}$ . To achieve this, we need two auxiliary gadgets.

We refer to the first gadget as a **triangle gadget**. This consists of a triangle  $xab$ , with  $x$  referred to as the *top* and with  $a$  and  $b$  each having no other coinciding edges. Note that any proper weighting  $w$  from  $\{0, 1\}$  of a graph with such a triangle must hold  $w(xa) \neq w(xb)$ ; otherwise  $\chi_w(a) = w(ab) + w(ax) = w(ba) + w(bx) = \chi_w(b)$ . Hence,  $\{w(xa), w(xb)\} = \{0, 1\}$  and so every such triangle gadget contributes exactly 1 to  $\chi_w(x)$ .

The second gadget, called a  **$k$ -disallowing gadget**, consists of a *main triangle*  $vxy$  with  $v$  referred to as the *root* and with  $x$  and  $y$  each constituting the top of  $k - 1$  distinct triangle gadgets (see Figure 1(a)). Note that in any proper weighting  $w$  from  $\{0, 1\}$ ,  $w(vx) \neq w(vy)$ ; otherwise, as both  $\chi_w(x)$  and  $\chi_w(y)$  have  $k - 1$  contributed by  $x$  and  $y$ 's triangles,  $\chi_w(x) = w(xv) + w(xy) + k - 1 = w(yv) + w(yx) + k - 1 = \chi_w(y)$ . Therefore, if  $w(xy) = 0$  then  $\{\chi_w(x), \chi_w(y)\} = \{k - 1, k\}$  and, if  $w(xy) = 1$  then  $\{\chi_w(x), \chi_w(y)\} = \{k, k + 1\}$ . In either case,  $v$  has one neighbor  $z \in \{x, y\}$  with  $\chi_w(z) = k$ , and consequently,  $\chi_w(v) \neq k$  in any proper weighting from  $\{0, 1\}$ . Also  $\{w(vx), w(vy)\} = \{0, 1\}$  and hence this gadget contributes exactly 1 to  $\chi_w(v)$ .

Now we are ready to show a reduction from 3-COLOR to 0-1WEIGHT,  $h$ , such that  $G \in 3\text{-COLOR}$  if and only if  $h(G) \in 0\text{-1WEIGHT}$ . Let  $G = (V, E)$  be a graph of order  $n$ . We may assume that  $n \geq 3$ . Otherwise,  $n \leq 2$  and  $G$  is in 3-COLOR and so it suffices to take as  $h(G)$  an empty graph which is trivially in 0-1WEIGHT. For  $n \geq 3$  we construct the graph  $h(G) = (W, F)$  as follows (see Figure 1(c)). We start with  $G = (V, E)$ . For each  $v \in V$ :

- (i) connect  $v$  to two new vertices,  $s$  and  $t$  (distinct for each  $v$ );
- (ii) add  $n - 1$  new  $k$ -disallowing gadgets for all  $k \in \{n + 2, n + 3, \dots, 2n\}$  with  $v$  as their root.

Clearly,  $h(G)$  can be calculated in time polynomial in the size of  $G$ .

**Fact 2.1** *In  $h(G)$  the following holds: any proper weighting  $w$  from  $\{0, 1\}$  satisfies  $\chi_w(v) \in \{n - 1, n, n + 1\}$  for every  $v \in V$ .*

**Proof:** Fix  $v \in V$ . Since  $w(vs) + w(vt) \in \{0, 1, 2\}$ ,  $v$  is the endpoint of  $\deg(v) \leq n - 1$  edges in  $V$ , and  $v$  is the root of  $(n - 1)$   $k$ -disallowing gadgets (each contributing 1 to  $\chi_w(v)$ ), we have:

$$\chi_w(v) \in \{0, 1, 2\} + \{0, 1, \dots, \deg(v)\} + \{n - 1\} \subseteq \{n - 1, n, \dots, 2n\},$$

where by  $A + B$  we mean the set of all sums of an element from  $A$  with an element from  $B$ . Observing the above and the fact that  $v$  is the root of  $k$ -disallowing gadgets for all  $k \in \{n + 2, \dots, 2n\}$ , we find that any proper weighting  $w$  from  $\{0, 1\}$  satisfies  $\chi_w(v) \in \{n - 1, n, n + 1\}$ , as claimed.  $\square$

It remains to show that  $G \in 3\text{-COLOR}$  if and only if  $h(G) \in 0\text{-1WEIGHT}$ .

First let us assume that  $G \in 3\text{-COLOR}$ . That means there exists a proper 3-coloring of  $G$ , say  $\chi : V \rightarrow \{n - 1, n, n + 1\}$ . We define a weighting of the edges of  $h(G)$ ,  $w : F \rightarrow \{0, 1\}$  as follows. For all  $e \in E$  let  $w(e) = 0$ . For all  $v \in V$ , if  $\chi(v) = n - 1$  then  $w(vs) = w(vt) = 0$ ; otherwise, if  $\chi(v) = n$  then  $w(vs) = 1$  and  $w(vt) = 0$ ; and finally, if  $\chi(v) = n + 1$  then  $w(vs) = w(vt) = 1$ . All other edges (parts of gadgets) are weighted as follows: For a triangle gadget  $xab$  with root  $x$ ,  $w(xa) = 1, w(xb) = w(ab) = 0$ . For a  $k$ -disallowing gadget with root  $v$ , and main triangle  $vxy$ ,  $w(vx) = w(xy) = 1, w(vy) = 0$ , and the weighting of all other triangle gadgets as described above. Note that  $w$  is a proper weighting of  $h(G)$  (satisfying  $\chi_w(v) = \chi(v)$  for all  $v \in V$ ), as required.

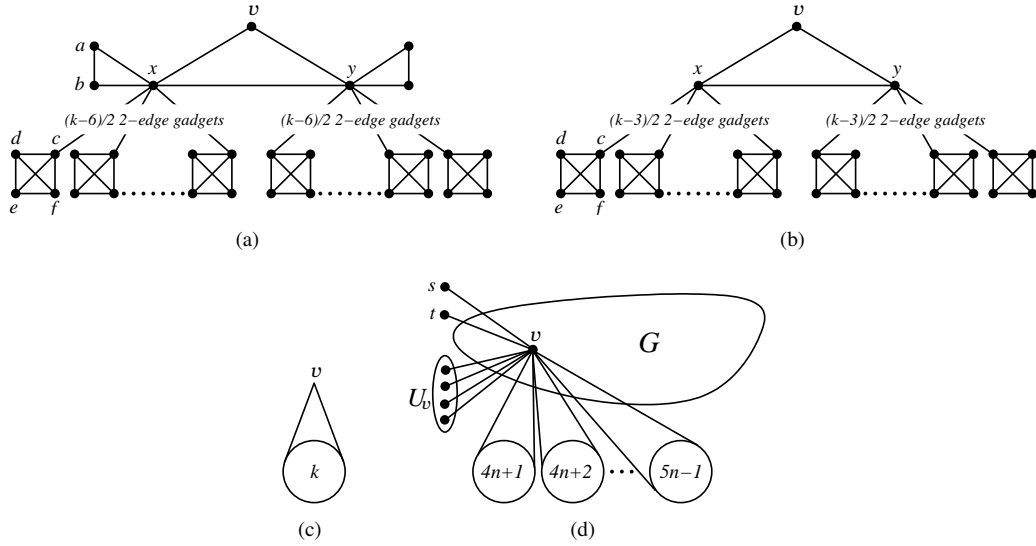
Now let us assume that  $G \notin 3\text{-COLOR}$ . Therefore, for all  $\chi : V \rightarrow \{n - 1, n, n + 1\}$ ,  $\chi$  is not proper. But, from Fact 2.1, any proper weighting from  $\{0, 1\}$  of  $h(G)$  satisfies  $\chi_w(v) \in \{n - 1, n, n + 1\}$  for all  $v \in V$ . Thus, there is no such proper weighting and hence  $h(G) \notin 0\text{-1WEIGHT}$ .

This completes the proof of Theorem 1.2.

### 3 1-2WEIGHT is NP-complete

The proof of Theorem 1.1 extends the ideas introduced in the proof of Theorem 1.2. Since clearly 1-2WEIGHT is in NP, it remains to show that 1-2WEIGHT is NP-hard. As before, we show a reduction from 3-COLOR to 1-2WEIGHT. To this end, we define a polynomial time reduction  $f$ , such that  $G \in 3\text{-COLOR}$  if and only if  $f(G) \in 1\text{-2WEIGHT}$ . Below we define auxiliary gadgets.

As in Section 2, we will use a **triangle gadget**. Now note that every triangle  $xab$ , with only  $x$  having other adjacent edges ( $x$  is referred to as the *top*), contributes exactly 3 to  $\chi_w(x)$  in any proper weighting  $w$  from  $\{1, 2\}$ .



**Fig. 2:** A  $k$ -disallowing gadget (a) for even  $k$ , (b) for odd  $k$  and their symbolic representation (c); a construction of  $f(G)$ , (d).

Now we define a **2-edge gadget** consisting of the set of vertices  $\{x, c, d, e, f\}$  with  $x$  and  $c$  adjacent and  $\{c, d, e, f\}$  spanning a complete graph  $K_4$ . One can check that every proper weighting  $w$  from  $\{1, 2\}$  of a graph adjacent to such a gadget only at  $x$  requires  $w(xc) = 2$ . We refer to  $x$  as the *endpoint*.

We use the above gadgets to construct another gadget, called a  **$k$ -disallowing gadget**. As we will see, this gadget has similar properties as its namesake in Section 2. We therefore allow ourselves the re-use of the name for this new, slightly different, gadget. We assume that  $k \geq 8$ . The  $k$ -disallowing gadget contains a *main triangle*  $vxy$  with  $v$  referred to as the *root*. Moreover, if  $k$  is even,  $x$  and  $y$  each form the endpoint of  $(k-6)/2$  edge disjoint 2-edge gadgets and  $x$  and  $y$  are each tops of distinct triangle gadgets (see Figure 2(a)). If  $k$  is odd,  $x$  and  $y$  each form the endpoint of  $(k-3)/2$  edge disjoint 2-edge gadgets (see Figure 2(b)). Note that in any proper weighting  $w$  from  $\{1, 2\}$ ,  $w(vx) \neq w(vy)$ ; otherwise, since the weight contributed by gadgets to  $\chi_w(x)$  and  $\chi_w(y)$  is  $k-3$ , then  $\chi_w(x) = w(xv) + w(xy) + k-3 = w(yv) + w(yx) + k-3 = \chi_w(y)$ . Therefore, for any  $k$ , if  $w(xy) = 1$  then  $\{\chi_w(x), \chi_w(y)\} = \{k-1, k\}$  and, if  $w(xy) = 2$  then  $\{\chi_w(x), \chi_w(y)\} = \{k, k+1\}$ . In either case,  $v$  has one neighbor  $z \in \{x, y\}$  with  $\chi_w(z) = k$ , and consequently,  $\chi_w(v) \neq k$  in any proper weighting from  $\{1, 2\}$ . Also  $\{w(vx), w(vy)\} = \{1, 2\}$ , and hence this gadget contributes exactly 3 to  $\chi_w(v)$ .

Now we are ready to show a polynomial time reduction from 3-COLOR to 1-2WEIGHT,  $f$ , such that  $G \in 3\text{-COLOR}$  if and only if  $f(G) \in 1\text{-2WEIGHT}$ . Let  $G = (V, E)$  be a graph of order  $n$ . As in Section 2, we may assume that  $n \geq 3$ . We construct the graph  $f(G) = (W, F)$  as follows (see Figure 2(d)). We start with  $G = (V, E)$ . For each  $v \in V$ :

- (i) connect  $v$  to two new vertices  $s$  and  $t$  (distinct for each  $v$ );
- (ii) connect  $v$  to all vertices from a new set  $U_v$  (distinct for each  $v$ ) with  $|U_v| = n - 1 - \deg(v)$ ;

(iii) add  $n - 1$  new  $k$ -disallowing gadgets for all  $k \in \{4n + 1, 4n + 2, \dots, 5n - 1\}$  with  $v$  as their root.

Clearly,  $f(G)$  can be calculated in time polynomial in the size of  $G$ .

**Fact 3.1** *In  $f(G)$  the following holds: any proper weighting  $w$  from  $\{1, 2\}$  satisfies  $\chi_w(v) \in \{4n - 2, 4n - 1, 4n\}$  for every  $v \in V$ .*

**Proof:** Fix  $v \in V$ . Since  $w(vs) + w(vt) \in \{2, 3, 4\}$ ,  $v$  is the endpoint of  $n - 1$  edges with endpoints in  $V \cup U_v$  and  $v$  is the root of  $(n - 1)$   $k$ -disallowing gadgets (each contributing 3 to  $\chi_w(v)$ ), we have:

$$\chi_w(v) \in \{2, 3, 4\} + \{n - 1, \dots, 2n - 2\} + \{3n - 3\} = \{4n - 2, \dots, 5n - 1\}.$$

Observing the above and the fact that  $v$  is the root of  $k$ -disallowing gadgets for all  $k \in \{4n + 1, 4n + 2, \dots, 5n - 1\}$ , we find that any proper weighting  $w$  from  $\{1, 2\}$  satisfies  $\chi_w(v) \in \{4n - 2, 4n - 1, 4n\}$ , as claimed.  $\square$

Now we show that  $G \in 3\text{-COLOR}$  if and only if  $f(G) \in 1\text{-2WEIGHT}$ .

First let us assume that  $G \in 3\text{-COLOR}$ . That means there exists a proper 3-coloring of  $G$ , say  $\chi : V \rightarrow \{4n - 2, 4n - 1, 4n\}$ . We define a weighting of the edges of  $f(G)$ ,  $w : F \rightarrow \{1, 2\}$  as follows. For all  $e \in E$  let  $w(e) = 1$ . For all edges  $e = vu$  with  $v \in V$  and  $u \in U_v$  we set  $w(e) = 1$ . For all  $v \in V$ , if  $\chi(v) = 4n - 2$  then  $w(vs) = w(vt) = 1$ ; otherwise, if  $\chi(v) = 4n - 1$  then  $w(vs) = 1$  and  $w(vt) = 2$ ; finally, if  $\chi(v) = 4n$  then  $w(vs) = w(vt) = 2$ . All other edges (parts of gadgets) are weighted as follows: For a triangle gadget  $xab$  with root  $x$ ,  $w(xa) = 2, w(xb) = w(ab) = 1$ . For a 2-gadget defined by  $\{x, c, d, e, f\}$  with  $x$  adjacent to  $c$ , we have  $w(xc) = w(cd) = w(ce) = w(de) = w(df) = 2$  and  $w(cf) = w(ef) = 1$ . For a  $k$ -disallowing gadget with root  $v$  and main triangle  $vxy$ ,  $w(vx) = w(xy) = 2, w(vy) = 1$ , and the weighting of all other gadgets as described above. Note that  $w$  is a proper weighting of  $f(G)$  (satisfying  $\chi_w(v) = \chi(v)$  for all  $v \in V$ ), as required.

Next let us assume that  $G \notin 3\text{-COLOR}$ . Therefore, for all  $\chi : V \rightarrow \{4n - 2, 4n - 1, 4n\}$ ,  $\chi$  is not a proper vertex coloring. But, from Fact 3.1, any proper weighting from  $\{1, 2\}$  of  $f(G)$  satisfies  $\chi_w(v) \in \{4n - 2, 4n - 1, 4n\}$  for all  $v \in V$ . Thus, there is no such proper weighting and hence  $f(G) \notin 1\text{-2WEIGHT}$ .

This concludes the proof of Theorem 1.1.

## 4 Concluding remarks

In this paper we showed that determining whether a graph has a proper weighting from either  $\{0, 1\}$  or  $\{1, 2\}$  is NP-complete. As a matter of fact, these two problems are not the same, in the sense that the corresponding families of graphs 0-1WEIGHT and 1-2WEIGHT are not equal. For example, the graph consisting only of one 2-edge gadget is in 1-2WEIGHT, as seen before, but it is easy to check that it is not in 0-1WEIGHT. Furthermore, we believe that our approach can be generalized to show that determining whether a graph has a proper weighting from  $\{a, b\}$  is NP-complete for any different rational numbers  $a$  and  $b$ . It is not clear if the same would hold for any two distinct irrational numbers.

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