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# Colouring the Square of the Cartesian Product of Trees

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We prove upper and lower bounds on the chromatic number of the square of the cartesian product of trees. The bounds are equal if each tree has even maximum degree.

**Keywords:** cartesian product, colouring, square graph

## 1 Introduction

This paper studies colourings of the square of cartesian products of trees. For simplicity we assume that a tree has at least one edge.

For our purposes, a *colouring* of a graph  $G$  is a function  $c : V(G) \rightarrow \mathbb{Z}$  such that  $c(v) \neq c(w)$  for every edge  $vw$  of  $G$ . The *square* graph  $G^2$  of  $G$  has vertex set  $V(G)$ , where two vertices are adjacent in  $G^2$  whenever they are adjacent in  $G$  or have a common neighbour in  $G$ . Thus, a colouring of  $G^2$  corresponds to a colouring of  $G$ , such that in addition, vertices with a common neighbour in  $G$  are assigned distinct colours.

Let  $[a, b] := \{a, a + 1, \dots, b\}$ . The *cartesian product* of graphs  $G_1, \dots, G_d$  is the graph  $G_1 \square \dots \square G_d$  with vertex set  $\{(v_1, \dots, v_d) : v_i \in V(G_i)\}$ , where vertices  $v = (v_1, \dots, v_d)$  and  $w = (w_1, \dots, w_d)$  are adjacent whenever  $v_i w_i \in E(G_i)$  for some  $i \in [1, d]$ , and  $v_j = w_j$  for all  $j \neq i$ . In this case,  $vw$  is in *dimension*  $i$ . Let  $\Delta(G)$  be the maximum degree of  $G$ .

**Theorem 1** *Let  $T_1, \dots, T_d$  be trees. Let  $G := T_1 \square T_2 \square \dots \square T_d$ . Then*

$$1 + \sum_{i=1}^d \Delta(T_i) \leq \chi(G^2) \leq 1 + 2 \sum_{i=1}^d \lceil \frac{1}{2} \Delta(T_i) \rceil .$$

This upper bound improves upon a similar bound by Jamison et al. (2006), who proved  $\chi(G^2) \leq 1 + 2 \sum_{i=1}^d (\Delta(T_i) - 1)$ , assuming that each  $\Delta(T_i) \geq 2$ . Theorem 1 implies:

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**Corollary 1** *Let  $T_1, \dots, T_d$  be trees, such that  $\Delta(T_i)$  is even for all  $i \in [1, d]$ . Let  $G := T_1 \square T_2 \square \dots \square T_d$ . Then*

$$\chi(G^2) = 1 + \sum_{i=1}^d \Delta(T_i) .$$

This corollary generalises a result of Fertin et al. (2003), who proved it when each  $T_i$  is a path, and thus  $G$  is a  $d$ -dimensional grid. See (Sopena and Wu, 2010; Fertin et al., 2004; Pór and Wood, 2009; Jamison et al., 2006; Chiang and Yan, 2008) for more related results.

## 2 The Proof

For a colouring  $c$  of a graph  $G$ , the *span* of an edge  $vw$  of  $G$  is  $|c(v) - c(w)|$ . The following lemma is well known; see (Pór and Wood, 2009) for example.

**Lemma 1** *Let  $G$  be a graph. If  $G^2$  has a colouring in which every edge of  $G$  has span at most  $s$ , then  $G^2$  is  $(2s + 1)$ -colourable.*

**Proof:** Let  $c : V(G^2) \rightarrow \mathbb{Z}$  be the given colouring of  $G^2$ . Since every edge of  $G$  has span at most  $s$ , every edge of  $G^2$  has span at most  $2s$ . Let  $c'(v) := c(v) \bmod (2s + 1)$  for each vertex  $v$ . Then  $c'(v) \neq c'(w)$  for each edge  $vw$  of  $G^2$ . Thus  $G^2$  is  $(2s + 1)$ -colourable.  $\square$

**Lemma 2** *For every tree  $T$  and non-negative integer  $s$ ,  $T^2$  has a colouring such that every edge of  $T$  has span in  $[s + 1, s + \lceil \frac{1}{2}\Delta(T) \rceil]$ .*

**Proof:** We proceed by induction on  $|V(T)|$ . If  $|V(T)| = 2$  the result is trivial. Now assume that  $|V(T)| \geq 3$ . Let  $v$  be a leaf vertex of  $T$ . Let  $w$  be the neighbour of  $v$ . By induction,  $(T - v)^2$  has a colouring  $c$  such that every edge of  $T - v$  has span in  $[s + 1, s + \lceil \frac{1}{2}\Delta(T) \rceil]$ . Let

$$X := \{x \in \mathbb{Z} : |x| \in [s + 1, s + \lceil \frac{1}{2}\Delta(T) \rceil]\} .$$

Each neighbour of  $w$  in  $T - v$  is coloured  $c(w) + x$  for some  $x \in X$ . Since  $|X| \geq \Delta(T)$  and  $w$  has degree less than  $\Delta(T)$  in  $T - v$ , for some  $x \in X$ , no neighbour of  $w$  is coloured  $c(w) + x$ . Set  $c(v) := c(w) + x$ . Thus  $|c(v) - c(w)| = |x| \in [s + 1, s + \lceil \frac{1}{2}\Delta(T) \rceil]$ . No two neighbours of  $w$  receive the same colour. Hence  $c$  is the desired colouring of  $T$ .  $\square$

**Proof of Theorem 1:** The lower bound is well known (Jamison et al., 2006). In particular, for  $i \in [1, d]$ , let  $v_i$  be a vertex of maximum degree in  $T_i$ . Then  $(v_1, \dots, v_d)$  has degree  $\sum_i \Delta(T_i)$  in  $G$ . This vertex and its neighbours in  $G$  receive distinct colours in any colouring of  $G^2$ . Thus  $\chi(G^2) \geq 1 + \sum_i \Delta(T_i)$ .

Now we prove the upper bound. Let  $s_1 := 0$  and  $s_i := \sum_{j=1}^{i-1} \lceil \frac{1}{2}\Delta(T_j) \rceil$ . By Lemma 2,  $T_i^2$  has a colouring  $c_i$  such that every edge of  $T_i$  has span in  $[s_i + 1, s_i + \lceil \frac{1}{2}\Delta(T_i) \rceil]$ . Thus the spans of edges in distinct trees are distinct.

Colour each vertex  $v = (v_1, \dots, v_d)$  of  $G$  by  $c(v) := \sum_{i=1}^d c_i(v_i)$ .

Suppose on the contrary that  $c(v) = c(w)$  for some edge  $vw$  of  $G$ . Say  $vw$  is in dimension  $i$ . Thus  $v_j = w_j$  for all  $j \neq i$ . Hence  $c_i(v_i) = c_i(w_i)$ , and  $c_i$  is not a colouring of  $G$ . This contradiction proves that  $c$  is a colouring of  $G$ .

Suppose on the contrary that  $c(x) = c(y)$  for two vertices  $x$  and  $y$  with a common neighbour  $v$  in  $G$ . Say  $vx$  is in dimension  $i$ , and  $vy$  is in dimension  $j$ . Thus  $v_\ell = x_\ell$  for all  $\ell \neq i$ , and  $v_\ell = y_\ell$  for all  $\ell \neq j$ . Now  $c_i(x_i) - c_i(v_i) = c(x) - c(v) = c(y) - c(v) = c_j(y_j) - c_j(v_j)$ . Thus the edges  $x_i v_i$  and  $y_j v_j$  have the same span. Since the spans of edges in distinct trees are distinct,  $i = j$ . Hence  $c_i(x_i) = c_i(y_i)$ . However,  $v_i$  is a common neighbour of  $x_i$  and  $y_i$  in  $T_i$ , implying  $c_i$  is not a colouring of  $T_i^2$ . This contradiction proves that  $c$  is a colouring of  $G^2$ .

Each edge of  $G$  has span at most  $\sum_{i=1}^d \lceil \frac{1}{2} \Delta(T_i) \rceil$ . The result follows from Lemma 1.  $\square$

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