# Colouring the Square of the Cartesian Product of Trees

David R. Wood

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# *Colouring the Square of the Cartesian Product of Trees*

David R. Wood †

*Department of Mathematics and Statistics, The University of Melbourne, Melbourne, Australia*

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We prove upper and lower bounds on the chromatic number of the square of the cartesian product of trees. The bounds are equal if each tree has even maximum degree.

**Keywords:** cartesian product, colouring, square graph

## 1 Introduction

This paper studies colourings of the square of cartesian products of trees. For simplicity we assume that a tree has at least one edge.

For our purposes, a *colouring* of a graph $G$ is a function $c : V(G) \to \mathbb{Z}$ such that $c(v) \neq c(w)$ for every edge $vw$ of $G$. The *square* graph $G^2$ of $G$ has vertex set $V(G)$, where two vertices are adjacent in $G^2$ whenever they are adjacent in $G$ or have a common neighbour in $G$. Thus, a colouring of $G^2$ corresponds to a colouring of $G$, such that in addition, vertices with a common neighbour in $G$ are assigned distinct colours.

Let $[a, b] := \{a, a + 1, \dots, b\}$. The *cartesian product* of graphs $G_1, \dots, G_d$ is the graph $G_1 \square \cdots \square G_d$ with vertex set $\{(v_1, \dots, v_d) : v_i \in V(G_i)\}$, where vertices $v = (v_1, \dots, v_d)$ and $w = (w_1, \dots, w_d)$ are adjacent whenever $v_i w_i \in E(G_i)$ for some $i \in [1, d]$, and $v_j = w_j$ for all $j \neq i$. In this case, $vw$ is in *dimension* $i$. Let $\Delta(G)$ be the maximum degree of $G$.

**Theorem 1** *Let $T_1, \dots, T_d$ be trees. Let $G := T_1 \square T_2 \square \cdots \square T_d$. Then*

$$1 + \sum_{i=1}^{d} \Delta(T_i) \leq \chi(G^2) \leq 1 + 2 \sum_{i=1}^{d} \lceil \tfrac{1}{2} \Delta(T_i) \rceil \ .$$

This upper bound improves upon a similar bound by Jamison et al. (2006), who proved $\chi(G^2) \leq 1 + 2\sum_{i=1}^{d}(\Delta(T_i) - 1)$, assuming that each $\Delta(T_i) \geq 2$. Theorem 1 implies:

† woodd@unimelb.edu.au. Supported by a QEII Research Fellowship from the Australian Research Council.

**Corollary 1** *Let $T_1, \dots, T_d$ be trees, such that $\Delta(T_i)$ is even for all $i \in [1, d]$. Let $G := T_1 \square T_2 \square \cdots \square T_d$. Then*

$$\chi(G^2) = 1 + \sum_{i=1}^{d} \Delta(T_i) \ .$$

This corollary generalises a result of Fertin et al. (2003), who proved it when each $T_i$ is a path, and thus $G$ is a $d$-dimensional grid. See (Sopena and Wu, 2010; Fertin et al., 2004; Pór and Wood, 2009; Jamison et al., 2006; Chiang and Yan, 2008) for more related results.

## 2 The Proof

For a colouring $c$ of a graph $G$, the *span* of an edge $vw$ of $G$ is $|c(v) - c(w)|$. The following lemma is well known; see (Pór and Wood, 2009) for example.

**Lemma 1** *Let $G$ be a graph. If $G^2$ has a colouring in which every edge of $G$ has span at most $s$, then $G^2$ is $(2s + 1)$-colourable.*

**Proof:** Let $c : V(G^2) \to \mathbb{Z}$ be the given colouring of $G^2$. Since every edge of $G$ has span at most $s$, every edge of $G^2$ has span at most $2s$. Let $c'(v) := c(v) \bmod (2s + 1)$ for each vertex $v$. Then $c'(v) \neq c'(w)$ for each edge $vw$ of $G^2$. Thus $G^2$ is $(2s + 1)$-colourable. □

**Lemma 2** *For every tree $T$ and non-negative integer $s$, $T^2$ has a colouring such that every edge of $T$ has span in $[s + 1, s + \lceil \frac{1}{2}\Delta(T)\rceil]$.*

**Proof:** We proceed by induction on $|V(T)|$. If $|V(T)| = 2$ the result is trivial. Now assume that $|V(T)| \geq 3$. Let $v$ be a leaf vertex of $T$. Let $w$ be the neighbour of $v$. By induction, $(T - v)^2$ has a colouring $c$ such that every edge of $T - v$ has span in $[s + 1, s + \lceil \frac{1}{2}\Delta(T)\rceil]$. Let

$$X := \{x \in \mathbb{Z} : |x| \in [s + 1, s + \lceil \tfrac{1}{2}\Delta(T)\rceil]\} \ .$$

Each neighbour of $w$ in $T - v$ is coloured $c(w) + x$ for some $x \in X$. Since $|X| \geq \Delta(T)$ and $w$ has degree less than $\Delta(T)$ in $T - v$, for some $x \in X$, no neighbour of $w$ is coloured $c(w) + x$. Set $c(v) := c(w) + x$. Thus $|c(v) - c(w)| = |x| \in [s + 1, s + \lceil \frac{1}{2}\Delta(T)\rceil]$. No two neighbours of $w$ receive the same colour. Hence $c$ is the desired colouring of $T$. □

**Proof of Theorem 1:** The lower bound is well known (Jamison et al., 2006). In particular, for $i \in [1, d]$, let $v_i$ be a vertex of maximum degree in $T_i$. Then $(v_1, \dots, v_d)$ has degree $\sum_i \Delta(T_i)$ in $G$. This vertex and its neighbours in $G$ receive distinct colours in any colouring of $G^2$. Thus $\chi(G^2) \geq 1 + \sum_i \Delta(T_i)$.

Now we prove the upper bound. Let $s_1 := 0$ and $s_i := \sum_{j=1}^{i-1} \lceil \frac{1}{2}\Delta(T_j)\rceil$. By Lemma 2, $T_i^2$ has a colouring $c_i$ such that every edge of $T_i$ has span in $[s_i + 1, s_i + \lceil \frac{1}{2}\Delta(T_i)\rceil]$. Thus the spans of edges in distinct trees are distinct.

Colour each vertex $v = (v_1, \dots, v_d)$ of $G$ by $c(v) := \sum_{i=1}^{d} c_i(v_i)$.

Suppose on the contrary that $c(v) = c(w)$ for some edge $vw$ of $G$. Say $vw$ is in dimension $i$. Thus $v_j = w_j$ for all $j \neq i$. Hence $c_i(v_i) = c_i(w_i)$, and $c_i$ is not a colouring of $G$. This contradiction proves that $c$ is a colouring of $G$.

Suppose on the contrary that $c(x) = c(y)$ for two vertices $x$ and $y$ with a common neighbour $v$ in $G$. Say $vx$ is in dimension $i$, and $vy$ is in dimension $j$. Thus $v_\ell = x_\ell$ for all $\ell \neq i$, and $v_\ell = y_\ell$ for all $\ell \neq j$. Now $c_i(x_i) - c_i(v_i) = c(x) - c(v) = c(y) - c(v) = c_j(y_j) - c_j(v_j)$. Thus the edges $x_i v_i$ and $y_j v_j$ have the same span. Since the spans of edges in distinct trees are distinct, $i = j$. Hence $c_i(x_i) = c_i(y_i)$. However, $v_i$ is a common neighbour of $x_i$ and $y_i$ in $T_i$, implying $c_i$ is not a colouring of $T_i^2$. This contradiction proves that $c$ is a colouring of $G^2$.

Each edge of $G$ has span at most $\sum_{i=1}^{d} \lceil \frac{1}{2}\Delta(T_i) \rceil$. The result follows from Lemma 1. □

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