

Vertex-colouring edge-weightings with two edge weights

Mahdad Khatirinejad, Reza Naserasr, Mike Newman, Ben Seamone, Brett Stevens

► **To cite this version:**

Mahdad Khatirinejad, Reza Naserasr, Mike Newman, Ben Seamone, Brett Stevens. Vertex-colouring edge-weightings with two edge weights. *Discrete Mathematics and Theoretical Computer Science*, DMTCS, 2012, Vol. 14 no. 1 (1), pp.1-20. <hal-00990567>

HAL Id: hal-00990567

<https://hal.inria.fr/hal-00990567>

Submitted on 13 May 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Vertex-colouring edge-weightings with two edge weights

Mahdad Khatirinejad^{1*} Reza Naserasr² Mike Newman^{3*}
Ben Seamone^{2†} Brett Stevens^{2*}

¹ Department of Communications and Networking, Aalto University, Finland

² School of Mathematics and Statistics, Carleton University, Canada

³ Department of Mathematics, University of Ottawa, Canada

received 24th July 2010, accepted 24th November 2011.

An edge-weighting vertex colouring of a graph is an edge-weight assignment such that the accumulated weights at the vertices yields a proper vertex colouring. If such an assignment from a set S exists, we say the graph is S -weight colourable. It is conjectured that every graph with no isolated edge is $\{1, 2, 3\}$ -weight colourable.

We explore the problem of classifying those graphs which are $\{1, 2\}$ -weight colourable. We establish that a number of classes of graphs are S -weight colourable for much more general sets S of size 2. In particular, we show that any graph having only cycles of length $0 \pmod 4$ is S -weight colourable for most sets S of size 2. As a consequence, we classify the minimal graphs which are not $\{1, 2\}$ -weight colourable with respect to subgraph containment. We also demonstrate techniques for constructing graphs which are not $\{1, 2\}$ -weight colourable.

Keywords: edge weighting, graph colouring

1 Introduction

Let G be a simple graph and S be a set of real numbers. An S -edge-weighting of G is an assignment $w : E(G) \rightarrow S$. Given an S -edge-weighting, the *weighted degree* of a vertex v , denoted $w(v)$, is the sum of weights of the edges incident with v . An S -edge-weighting gives a *vertex colouring* if the weighted degrees of adjacent vertices are different. If an S -edge-weighting vertex colouring w exists, we also call w an S -weight colouring and we say G is S -weight colourable. For a positive integer k , we say G has a k -weight colouring or G is k -weight colourable if it is S -weight colourable for every set S of size k . The most commonly studied sets S are those of the form $\{1, \dots, k\}$.

Problem 1 Given a graph G with no isolated edges, find the minimum k such that G is $\{1, \dots, k\}$ -weight colourable.

* Partially supported by NSERC Canada.

† Partially supported by *Fonds québécois de la recherche sur la nature et les technologies*.

‡ Emails: mahdad.khatirinejad@tkk.fi, {bseamone, brett}@math.carleton.ca, mnewman@uottawa.ca

It is not hard to verify that K_4 with a single leaf attached is $\{1, 2\}$ -weight colourable but is not $\{0, 1\}$ -weight colourable. It follows that the S -weight colourability of a graph is not only dependent on the size of S but also on the particular elements of S . However, if a graph G is S -weight colourable then there exists an $i_0 = i_0(G, S)$ such that for all $i > i_0$ the graph is also $\{s + i : s \in S\}$ -weight colourable. One such value for i_0 , though not necessarily the smallest, is $i_0 = |S| \cdot \Delta(G) \cdot \max\{|s| : s \in S\}$, where $\Delta(G)$ is the maximum degree of G .

Let us start by considering the 2-weight colourability of a simple class of graphs – paths. If a and b are non-zero real numbers, then every path of length at least 2 has an $\{a, b\}$ -weight colouring. Assigning the edge weights $a, a, b, b, a, a, b, b, \dots$, beginning with one leaf of the path, gives such a colouring. However, a path has a $\{0, a\}$ -weight colouring if and only if it is not of length $1 \pmod 4$. The reader can easily check that paths of length 2, 3 and 4 have a $\{0, a\}$ -weight colouring. However, if we let $P = e_1, e_2, e_3, e_4, e_5$ be a path of length 5 (we omit vertex labels) then if $w(e_2) = 0$ (or $w(e_4) = 0$) then the ends of e_1 (e_5) will have equal weight. Thus the only way to achieve a $\{0, a\}$ -weight colouring of P is if $w(e_2) = w(e_4) = a$. However, this implies that the ends of e_3 will have the same weight, and hence a $\{0, a\}$ -weight colouring cannot exist. These examples easily extend to longer paths; the details are left to the reader.

In general, it is unknown how difficult it is to decide if a given graph admits a $\{1, 2\}$ -weight colouring, or more generally an $\{a, b\}$ -weight colouring. As such, we present the following question:

Problem 2 *Is it NP-complete to decide whether a given graph is 2-weight colourable?*

Returning to Problem 1, we state the following conjecture, due to Karoński, Łuczak, and Thomason [KŁT04], which motivates most of the known results on the $\{1, \dots, k\}$ -weight colourability of graphs.

Conjecture 1.1 *Every graph with no isolated edge is $\{1, 2, 3\}$ -weight colourable.*

Karoński et al. [KŁT04] showed that the Conjecture 1.1 is true for 3-colourable graphs. They also proved that if S is any set of at least 183 real numbers which are linearly independent over the rational numbers then every graph with no isolated edge is S -weight colourable. Recently, Kalkowski et al. [KKP09] showed that every graph with no isolated edge is $\{1, \dots, k\}$ -weight colourable for $k = 5$. This result is an improvement on the previous bounds on k established by Addario-Berry et al. [ABDM⁺07], Addario-Berry et al. [ABDR08], and Wang et al. [WY08], who obtained the bounds $k = 30$, $k = 16$, and $k = 13$, respectively.

Our work in this paper is similarly motivated by Conjecture 1.1. However, where most others have attempted to lower the best known value of k as described above, our focus is on establishing which graphs are $\{1, 2\}$ -weight colourable. Addario-Berry, Dalal and Reed [ABDR08] showed that asymptotically almost every graph is $\{1, 2\}$ -weight colourable, however it is not known which ones are not. Chang et al and Lu et al ([CLWY10], [LYZ10]) have made some progress in determining which classes of graphs are $\{1, 2\}$ -weight colourable, notably having shown that 3-connected bipartite graphs are one such class. A complete classification of such graphs would determine those graphs for which $k = 3$ is the smallest possible solution in Problem 1, and would reduce Conjecture 1.1 to just those graphs.

The results that follow are, for the most part, concerned with a more general problem than that of finding $\{1, 2\}$ -weight colourings, namely that of finding $\{a, b\}$ -weight colourings for more general values of a and b . In such cases, the existence of a $\{1, 2\}$ -weight colouring follows as an unstated corollary. In Section 2, we establish a wide range of basic graphs which admit $\{a, b\}$ -weight colourings. We also establish classes of graphs which do not admit $\{a, b\}$ -weight colourings, but which do admit an $\{a, b\}$ -edge weighting which is almost a proper colouring. These results provide building blocks for our results on the weight

colourability of bipartite graphs in Section 3 and of other general classes of graphs, particularly direct products of graphs, in Section 4. Of note, we show in Section 3 that if every cycle of G is of length $0 \pmod 4$, then G is $\{1, 2\}$ -weight colourable.

2 Building blocks: Weight colourings of basic graphs

We will use standard graph theory terminology; the reader may refer to [BM08] for clarification of any terms which are not specifically defined here.

The *length* of a path (walk) is defined to be the number of edges of the path (walk). A *thread* in a graph G is a walk connecting two vertices x and y , not necessarily distinct, such that the internal vertices are distinct from all others on the walk, all internal vertices have degree 2 in G , and $\deg(x), \deg(y) \geq 3$. If x and y are distinct, then the walk is in fact a path and in this case we may refer to the thread as an *ear*. If the condition that $\deg(x), \deg(y) \geq 3$ is changed to $\deg(x), \deg(y) \geq 2$ in either case, we have a *subthread* or *subear* respectively.

A *cut vertex* of a graph is one whose removal disconnects the graph. A graph is 2-connected if it has no cut vertex. A graph (not necessarily simple) is called *separable* if it can be decomposed into two nonempty subgraphs with exactly one vertex in common. A simple graph is separable if and only if it is not 2-connected. A maximal nonseparable subgraph of G is a *block* of G . Note that a block is isomorphic either to K_2 or to a 2-connected graph. An *end block* of G is a block which contains at most one cut vertex of G .

A graph is *c-colourable* if the vertices can be coloured with c colours so that adjacent vertices get different colours.

K_n and C_n , respectively, denote the complete graph and the cycle on n vertices. The *Cartesian product* of two graphs G and H , denoted by $G \square H$, is defined as the graph having vertex set $V(G) \times V(H)$ where two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H or $u' = v'$ and u is adjacent to v in G .

We present a few simple observations.

Proposition 2.1 *Let a, b, t be nonzero real numbers and G a graph. Then*

- (i) G is $\{a, b\}$ -weight colourable if and only if G is $\{at, bt\}$ -weight colourable, and
- (ii) if G is $\{a, b\}$ -weight colourable then G is $\{p, q\}$ -weight colourable for any nonzero $p, q \in \mathbb{R}$ which are linearly independent over \mathbb{Q} .

Proof: (i) This follows from the fact that $w(u) \neq w(v)$ if and only if $t \cdot w(u) \neq t \cdot w(v)$. (ii) Note that if two adjacent vertices receive distinct linear combinations of a and b as weights, then the coefficients of these linear combinations will suffice for any two linearly independent nonzero reals. \square

From Proposition 2.1 we deduce the following, adopting the convention that 0 and 1 are relatively prime integers:

Corollary 2.2 *A graph G is 2-weight colourable if and only if G is $\{a, b\}$ -weight colourable for every pair of relatively prime integers a and b .*

Proposition 2.1 allows us to reduce our proofs of positive results on the existence of $\{a, b\}$ -weight colourings of a graph to relatively prime integers. Results in which we show that G does not admit an $\{a, b\}$ -weight colouring will not rely on such assumptions – we will prove them for all real a, b .

Proposition 2.3 *If G is d -regular and $\{a, b\}$ -weight colourable for a fixed choice of a and b then (i) it is d -colourable, and (ii) it is 2-weight colourable.*

Proof: (i) The weighted degree of each vertex must be a number of the form $ta + (d - t)b$ for some $0 \leq t \leq d$, and a vertex of weighted degree da cannot be adjacent to a vertex of weighted degree db . Thus putting the vertices of weighted degree da or db in the same colour class gives a d -colouring.

(ii) In an $\{a, b\}$ -edge weighting of a d -regular graph, the accumulated weight at any vertex is in a one-to-one correspondence with the number of incident edges of weight a . Thus if one choice of a and b gives a vertex colouring, then any other choice of a and b will as well. \square

Corollary 2.4 *If $\chi(G) = \Delta(G) + 1$ or, equivalently (by Brooks theorem), if G is an odd cycle or a complete graph then G is not S -weight colourable for any set S of size 2.*

Even though the complete graph is not S -weight colourable for any set of size 2, it has an S -edge-weighting that is very close to being an S -weight colouring. This specific weighting will be useful in constructing families of 2-weight colourable graphs and non-2-weight colourable graphs in Section 4.

Lemma 2.5 *Given $n \geq 2$ and $a \neq b \in \mathbb{R}$, there is an $\{a, b\}$ -edge-weighting of K_n such that the weighted degrees of all the vertices are distinct except for 2 of them. Furthermore, in any such $\{a, b\}$ -edge-weighting, the degree sequence of the subgraph induced by the edges of weight a (as well as the subgraph induced by the edges of weight b) is either*

$$(1, 2, \dots, \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \dots, n - 2, n - 1),$$

or

$$(0, 1, \dots, \lceil \frac{n}{2} \rceil - 2, \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil, \dots, n - 3, n - 2).$$

Proof: We prove the first part with an explicit construction. Choose any two vertices and assign weight a to the edge joining them. Choose a new vertex and assign weight b to all the edges joining this vertex to the previous two vertices. Choose another vertex and assign weight a to all the edges joining this vertex to the previous three vertices. By repeating this process until all vertices are exhausted, we achieve the desired edge-weighting since the two vertices chosen first will have the same weight while the remainder of the graph is properly coloured. Note that we achieve the same result by swapping a and b in this argument.

We prove the second part of the lemma by induction on n . Suppose w is such an edge-weighting of K_n and let $w(u) = w(v)$. It is easy to verify the claim for $n = 2$ and $n = 3$. If $w(x) \notin \{(n - 1)a, (n - 1)b\}$ for every vertex x then $w(x)$ can only take $n - 2$ values, a contradiction to the choice of w . If $w(u) = w(v) \in \{(n - 1)a, (n - 1)b\}$ then by removing u and v , w induces an $\{a, b\}$ -weight colouring of K_{n-2} , a contradiction to Corollary 2.4. Thus there exists a vertex $x \neq u, v$ such that $w(x) \in \{(n - 1)a, (n - 1)b\}$. The claim follows by induction on $K_n - x$. \square

The following technical lemmata will be useful for the rest of the paper, since they establish useful tools for finding edge-weighting vertex colourings of graphs with specific structural properties.

Lemma 2.6 *Suppose G has a vertex v with a set of leaf neighbours L where $|L| \geq \lceil \deg(v)/2 \rceil$. Let $a \neq b$ be real numbers with $ab > 0$. If $G \setminus L$ is $\{a, b\}$ -weight colourable, then so is G .*

Proof: As mentioned, Proposition 2.1 allows us to only consider $a, b \in \mathbb{Z}^+$. Suppose w is an $\{a, b\}$ -weight colouring of $G \setminus L$. The possible extensions of w to G give exactly $|L| + 1$ possible weights for v . Since v has at most $|L|$ neighbours in $G \setminus L$, in at least one of the extensions, the weighted degree of v is different from the weighted degrees of the neighbours of v in $G \setminus L$. The weighted degree of v is also different from the weighted degrees of the neighbours of v in L , since $ab > 0$. \square

Corollary 2.7 *Every tree with at least 3 vertices is $\{a, b\}$ -weight colourable, where $a \neq b$ are real numbers with $ab > 0$.*

Proof: The statement holds for any star, $K_{1, n-1}$, since the assignment of a to all edges achieves the desired result. As such the result holds for $n = 3$ since the unique tree on 3 vertices is a star. Let T be a tree on n vertices which is not a star and assume the result holds for any tree with fewer than n vertices. Every tree has a vertex v that has at least $\lceil \deg(v)/2 \rceil$ leaf neighbours. Since T is not a star, removing the leaf neighbours of v gives a subtree T' on at least 3 vertices. By the induction hypothesis T' has an $\{a, b\}$ -weight colouring. By Lemma 2.6, T does as well. \square

The following lemma establishes that we may contract long threads in a way that maintains weight colourability.

Lemma 2.8 *Let G be a graph, $P = v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4, e_5, v_5$ be a subthread of G , and $a \neq b$ be any two real numbers. Let $G' = G/\{e_1, e_2, e_3, e_4\}$ Then,*

- (i) *If w is an $\{a, b\}$ -weight colouring of G , then $w(e_1) = w(e_5) \neq w(e_3)$.*
- (ii) *If G' is $\{a, b\}$ -weight colourable, then so is G .*
- (iii) *If $\deg(v_0) = 2$ or $\deg(v_5) = 2$, then G is $\{a, b\}$ -weight colourable if and only if G' is $\{a, b\}$ -weight colourable.*

Proof: (i) If $w(e_1) \neq w(e_5)$ then either one of the two choices for $w(e_3)$ results in an improper colouring at e_2 or e_4 . Hence $w(e_1) = w(e_5)$ and $w(e_3)$ must be distinct.

(ii) For convenience, we still denote the vertex obtained from the contraction by v_0 . Suppose w' is an $\{a, b\}$ -weight colouring of G' . Then $w'(v_0) \neq w'(v_5)$. Without loss of generality assume $w'(v_0v_5) = a$. Let $w(e) = w'(e)$ for each $e \notin \{e_1, e_2, e_3, e_4, e_5\}$, $w(e_1) = w(e_5) = a$ and $w(e_3) = b$. There are two possibilities for the weights of e_2 and e_4 . Assigning $w(e_2) = a$ and $w(e_4) = b$ does not yield a proper vertex colouring of G if and only if either $w(v_0) = 2a$ or $w(v_5) = a + b$. Similarly, defining $w(e_2) = b$ and $w(e_4) = a$ does not yield a proper vertex colouring of G if and only if either $w(v_0) = a + b$ or $w(v_5) = 2a$. Suppose that neither weighting works. If the first possibility gives $w(v_0) = 2a$, then the second must give $w(v_5) = 2a$. If the first possibility gives $w(v_5) = a + b$, then the second gives $w(v_0) = a + b$. In either case $w(v_0) = w(v_5)$, a contradiction.

(iii) Assume $\deg(v_0) = 2$ and let e_0 be the other edge incident with v_0 . Suppose w is an $\{a, b\}$ -weight colouring of G . By (i) we have $w(e_0) = w(e_4)$ and $w(e_1) = w(e_5)$. Hence $w(v_0) = w(v_4) \neq w(v_5)$. Thus, by assigning the common weight of e_1 and e_5 to the edge v_0v_5 , we get an $\{a, b\}$ -weight colouring of G' . \square

The degree condition on the ends of P in Lemma 2.8 (iii) cannot be dropped. For example, by taking G to be the path of length 5, $a = 1$, and $b = 2$, Lemma 2.8 (iii) fails.

From this lemma we may deduce necessary and sufficient conditions for the existence of $\{a, b\}$ -weight colourings of cycles.

Proposition 2.9 *Let a and b be any distinct real numbers. Then C_n is $\{a, b\}$ -weight colourable if and only if $n \equiv 0 \pmod{4}$.*

In lieu of a proof, we simply note that, by Lemma 2.8 (iii), the proof of this proposition may be reduced to the cases C_3 , C_4 , C_5 and C_6 . The details are left to the reader. There are $\{a, b\}$ -edge weightings of other cycles of length $4k + 1$, $4k + 2$ and $4k + 3$ which give vertex colourings with as few conflicts as possible. These results are largely technical, though not difficult to prove.

Proposition 2.10 *Let a and b be any distinct real numbers. Then C_{2k+1} has an $\{a, b\}$ -edge weighting w such that only one edge $e = uv$ has the property that $w(u) = w(v)$.*

Proposition 2.11 *Let a and b be any distinct real numbers. Then C_{4k+2} has an $\{a, b\}$ -edge weighting w such that precisely two edges $e = uv$ and $e' = u'v'$ have the property that $w(u) = w(v)$ and $w(u') = w(v')$. Furthermore,*

- *the distance between e and e' is even,*
- *e and e' may be chosen to be any two edges at an even distance, and*
- *if f_1 and f_2 are the edges incident to e , then their weights are equal and can be chosen to be either a or b (similar for e').*

We present a specific consequence of Proposition 2.11 which we will find useful.

Proposition 2.12 *Let k be an integer, $k \geq 1$. Then C_{4k+2} has an $\{a, b\}$ -edge-weighting such that three consecutive vertices have equal weight and the rest of the cycle is properly coloured. Furthermore, the edge-weighting can be chosen so that the weights of the four edges which contribute to the weights of those three vertices will all be a , all b , or alternate between a and b .*

Let $\Theta_{(m_1, \dots, m_d)}$, $d \geq 3$, be the graph constructed from d internally disjoint paths between distinct vertices x and y , where the i -th path has of length m_i . For simplicity, we assume $m_1 \leq m_2 \leq \dots \leq m_d$. Such graphs will be referred to as *theta graphs*. We present necessary and sufficient conditions for theta graphs to be 2-weight colourable.

Theorem 2.13 *Let $d \geq 3$ and let a, b be real numbers. The graph $\Theta_{(m_1, m_2, \dots, m_d)}$ is 2-weight colourable if and only if it is not of the form $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$.*

Proof: Let x and y be the two vertices of degree greater than two, and let $\{P_i | 1 \leq i \leq d\}$ be the d internally disjoint paths between x and y .

Suppose w is an $\{a, b\}$ -weight colouring of $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$. By applying Lemma 2.8 (i) to each P_i , we observe that on any of the d disjoint paths between x and y the first and last edges must receive same weight. Thus $w(x) = w(y)$, a contradiction since x and y are adjacent. Hence $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$ is not $\{a, b\}$ -weight colourable for any a, b .

Consider $\Theta_{(m_1, m_2, \dots, m_d)} \not\cong \Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$. We can assume that $|a| \geq |b|$. Let n_j be the number of paths that have length equivalent to $j \pmod{4}$. Note that $n_0 + n_1 + n_2 + n_3 = d = \deg(x) = \deg(y)$. For each path P_i , weight each edge according to Lemma 2.8 so that the edges incident with x are weighted

a . Then $w(x) = da$ so it has no conflicts with its neighbours since $d \geq 3$ and the condition on the magnitudes of a and b gives $da \notin \{2a, a + b\}$. Note that, if $|P_i| \geq 2$, there are two choices for the next edge's weight on P_i which determines the rest of the weights. Given one such weighting of a path P_i , the effects of switching to the alternate weighting where the edge incident to x receives weight a depend on the parity of the length of the path. If $|P_i|$ is even, the weight of the edge incident to y and the vertex weights of the neighbours of x and y on P_i all change. If $|P_i|$ odd, the weight of the edge incident to y remains unchanged, but the vertex weights of the neighbours of x and y on P_i do change. In all cases the only possible weights on path-neighbours of x or y are $2b, a + b$ and $2b$. We prove, by cases, that there is an appropriate set of choices which make $w(y)$ distinct from its neighbours.

$\mathbf{n}_0 + \mathbf{n}_2 \geq 4$: Our choice of weightings for even P_i 's give at least 5 possible values for $w(y)$, so there is a choice such that $w(y) \notin \{2b, a + b, 2a, da\}$.

$\mathbf{n}_0 + \mathbf{n}_2 = 3$: If no P_i has length 1, da is not a forbidden weight for y . Also, if $n_3 \geq 1$ then there is an edge incident to y with weight b , and $w(y) \neq da$. In either case there is a choice of weightings so that $w(y) \notin \{2b, a + b, 2a\}$.

So, assume that $m_1 = 1$ and $n_3 = 0$. If the initial weighting fails then we must have

$$\{2b, a + b, 2a, da\} = \{(d - 3)a + 3b, (d - 2)a + 2b, (d - 1)a + b, da\}$$

which implies that $b = -(d - 3)a$ and $d \geq 4$. The fact that $|a| \geq |b|$ gives that $d = 4$, implying $n_1 = 1$ and $b = -a$. We weight all edges explicitly. The single edge on the path of length 1 receives weight a . If $n_0 = 3$ then weight the edges of one even path $a, a, \dots, -a, -a$ and the other two $a, -a, \dots, -a, a$. If $n_0 = 2$ and $n_2 = 1$ weight the edges of the paths of length $0 \pmod 4$ with $a, a, \dots, -a, -a$ and the other even path with a, a, \dots, a, a . If $n_0 = 1$ and $n_2 = 2$ weight the edges of the path of length $0 \pmod 4$ with $a, -a, \dots, -a, a$ and the two other even paths with $a, -a, \dots, a, -a$. Finally if $n_0 = 0$ and $n_2 = 3$ weight the edges of all even paths with $a, -a, \dots, a, -a$. Each weighting gives a vertex-colouring for its respective case.

$\mathbf{n}_0 + \mathbf{n}_2 = 2$: If $n_3 = 0$ and $n_0 > 0$ then assign weights to the edges of one path which is length $0 \pmod 4$ so that the weights of the first and last edges are both a . Weight the edges of the other even path so that the edge incident to x is weighted a and the edge incident to y is weighted b . If $n_3 = n_0 = 0$ but either $d > 3$ or $b \neq 0$ then assign weights to the edges of both even paths so that their edges incident with x are weighted a , one of the edges incident with y is weighted a and the other is weighted b . In both cases weight the edges of the paths of length $1 \pmod 4$ so the weights are, in order beginning with the edge incident with x , a, a, \dots, b, a (if the path is a single edge, give it weight a). In the case when $n_3 = n_0 = 0$, $d = 3$ and $b = 0$ weight the edges of the two even paths $a, 0, 0, \dots, a, 0$ and the single odd path with $0, 0, a, a, \dots, a, 0$ (beginning with the edge incident with x in each case). The weighting given in each case gives a proper vertex colouring.

Assume $n_3 \geq 1$. If $n_0 \neq n_2$ then choose weightings for each P_i so that $w(x) = da$ and each remaining neighbour of y has accumulated weight $a + b$. Then $w(y) = an_0 + an_1 + bn_2 + bn_3$. Since $n_3 \geq 1$ we have $w(x) \neq w(y)$, so the only possible conflict is if $w(y) = a + b$. In this case change both even P_i 's to

their alternate weighting, maintaining $w(x) = da$ and producing a new weight at y :

$$\begin{aligned} w'(y) &= bn_0 + an_1 + an_2 + bn_3 \\ &= w(y) + (a - b)(n_2 - n_0) \\ &= a + b + (a - b)(n_2 - n_0) \\ &= \begin{cases} 3a - b & \text{if } n_0 = 0, n_2 = 2 \\ 3b - a & \text{if } n_0 = 2, n_2 = 0 \end{cases} \end{aligned}$$

In either case $w'(y) \neq a + b$. If $n_0 = 2$ then y has neighbours with weights $2b$, and $3b - a \neq 2b$. Similarly if $n_2 = 2$ then the weight at y avoids conflict with its neighbours with weight $2a$.

If $n_0 = n_2 = 1$ we start again with choices from the basic strategy that leave all path-neighbours with weight $a + b$. We have $w(x) = da \neq (n_1 + 1)a + (n_3 + 1)b = w(y)$. Thus the only conflict can again be if $w(y) = a + b$ or equivalently, $an_1 + bn_3 = 0$. In this case we weight the edges of P_i 's of lengths equivalent to 0, 1, 2, and 3 mod 4 with $\{a, a, \dots, b, b\}$, $\{a, b, \dots, a, a\}$, $\{a, a, \dots, a, a\}$ and $\{a, b, \dots, b, b\}$ respectively. We still have that $w(y) = a + b \neq da = w(x)$ and no neighbour of y has weight $a + b$.

$n_0 + n_2 = 1$: If $n_3 = 0$ then weight the edges of the even path so that the edge incident with x receives weight a and the edge incident with y receives weight b . Weight the edges of the paths of length 1 mod 4 so the weights are, in order beginning with the edge incident with x , $a, a \dots b, a$ (if the path is a single edge, give it weight a). This weighting gives a proper vertex colouring. Assume $n_3 \geq 1$. Again, weight the edges of each P_i so that $w(x) = da$ and each neighbor of y (distinct from x) has accumulated weight $a + b$. Since $n_3 \geq 1$ we have that $w(x) = da \neq w(y)$. If $w(y) \neq a + b$, then w is an $\{a, b\}$ -weight colouring. Suppose $w(y) = a + b$. Equivalently

$$(n_0 + n_1 - 1)a + (n_2 + n_3 - 1)b = 0. \quad (1)$$

Change the edge weights of the even length path to begin with b, a . Call this weighting w' . We now have $w'(x) = (d - 1)a + b$ and $w'(y) \neq a + b$. All neighbours of y still have weight $a + b$, so the only possible conflicts are between x and its neighbours. We reduce all potential conflicts to one of four cases, which are solved explicitly.

If $w'(x) = w'(y)$ then since $w'(x) = (d - 1)a + b$, y is incident with precisely one edge with weight b . Since $n_3 \geq 1$, the edge with weight b comes from a path of length 3 mod 4. This gives $n_0 = 0$, $n_2 = 1$ and $n_3 = 1$ and then Equation 1 and $|a| \geq |b|$ gives either

- $n_0 = 0$, $n_1 = 1$, $n_2 = 1$, $n_3 = 1$ and $b = 0$ (**case iii.** below).
- $n_0 = 0$, $n_1 = 2$, $n_2 = 1$, $n_3 = 1$ and $b = -a$ (**case iv.** below).

The neighbours of x have accumulated weights either $a + b$ or $2a$. If $w(x) = (d - 1)a + b = a + b$ then this implies that $d = 2$ but the hypotheses of the theorem include $d \geq 3$. If $w(x) = (d - 1)a + b = 2a$ then $b = -(d - 3)a$. The fact that $d \geq 3$ and $|a| \geq |b|$ now give either

- $n_0 = 1$, $n_1 = 0$, $n_2 = 0$, $n_3 = 2$ and $b = 0$ which is dealt with in **case i.** below.
- $n_0 = 1$, $n_1 = 1$, $n_2 = 0$, $n_3 = 2$ and $b = -a$ which is dealt with in **case ii.** below.
- $n_0 = 0$, $n_1 = 1$, $n_2 = 1$, $n_3 = 1$ and $b = 0$ which is dealt with in **case iii.** below.

- $n_0 = 0, n_1 = 2, n_2 = 1, n_3 = 1$ and $b = -a$ which is dealt with in **case iv.** below.

case i. In this case x and y are not adjacent. Weight the edges of the path of length equivalent to $0 \pmod 4$ with $0, 0, \dots, a, a$ and the two odd paths with $a, 0, \dots, 0, 0$.

case ii. In this case x and y are not adjacent. Weight the edges of the paths of lengths equivalent to $0 \pmod 4, 1 \pmod 4$ and $3 \pmod 4$ with $-a, -a, \dots, a, a, a, -a, \dots, a, a$ and $a, -a, \dots, -a, -a$ respectively.

case iii. In this case x and y may be adjacent. Weight the edges of the paths of lengths equivalent to $1 \pmod 4, 2 \pmod 4$ and $3 \pmod 4$ with $a, a, \dots, 0, a, 0, 0, \dots, 0, 0$ and $0, 0, \dots, 0, a$ respectively.

case iv. In this case x and y may be adjacent. Weight the edges of the paths of lengths equivalent to $1 \pmod 4, 2 \pmod 4$ and $3 \pmod 4$ with $a, -a, \dots, a, a, -a, -a, \dots, -a, -a$ and $a, -a, \dots, -a, -a$ respectively.

Each of these edge-weightings gives a proper vertex colouring.

$\mathbf{n}_0 + \mathbf{n}_2 = 0$: Every weighting of the paths P_i which gives $w(x) = da$ must give $w(y) = an_1 + bn_3$. If $m_1 = 1$ then, since our graph is not $\Theta_{(1,4k_2+1, \dots, 4k_d+1)}$, we have $n_3 \geq 1$ and thus $w(x) \neq w(y)$. Suppose $m \neq 1$. For each P_i we have two choices for y 's neighbour. Each choice leaves $w(y)$ constant. Thus there is a choice for each path which gives an edge-weighting vertex-colouring. \square

3 Bipartite graphs

We begin the section by noting that the property of being $\{a, b\}$ -weight colourable is not one that is inherited by subgraphs, nor is the property of being non- $\{a, b\}$ -weight colourable. For example, the graph consisting of K_4 with a leaf attached is $\{1, 2\}$ -weight colourable, however K_4 is not $\{a, b\}$ -colourable for any choice of a and b . Similarly K_4 contains the subgraph C_4 which is 2-weight colourable.

We can, however, characterize the minimal graphs with respect to subgraph containment in the class of graph which are not $\{a, b\}$ -weight colourable for many pairs $\{a, b\}$ (in particular, $\{1, 2\}$). In Theorem 3.9 we establish that any graph which is not $\{a, b\}$ -weight colourable must contain C_{2k+1} or C_{4k+2} as a subgraph for some positive integer k .

Definition 3.1 A graph G is round if every cycle of G has length $0 \pmod 4$.

The class of round graphs is much richer than merely those obtained by taking a graph and subdividing each edge into a path of length 4. For example, $\Theta_{(2,2,2)} \cong K_{2,3}$ is a round graph which is not obtained in this way.

The following lemma establishes a useful subgraph condition of round graphs which we will use in our study of the $\{a, b\}$ -weight colourability of round graphs.

Proposition 3.2 If G is a round graph and $\Theta_{(i,j,k)}$ is a subgraph of G , then i, j and k are even and $i \equiv j \equiv k \pmod 4$.

Proof: Let $\Theta_{(i,j,k)}$ be a subgraph of G and let P_i, P_j and P_k be the corresponding paths of length i, j and k respectively. Since G is round, $i + j \equiv i + k \equiv j + k \equiv 0 \pmod 4$. The result follows. \square

Before proceeding with our results on bipartite graphs we present the following definition which we adopt throughout this section (and this section only). In any $\{a, b\}$ -edge-weighting of a graph, the weighted degree of every vertex is of the form $ra + sb$ for some nonnegative integers r, s . We will call a weighted vertex *even* (*odd*) if its weighted degree is $ra + sb$ with r even (odd). Note that the parity

of a weighted vertex does not necessarily refer to the parity of its weight. However, by Proposition 2.1, if a and b are not independent over \mathbb{Q} then we will assume that they are relatively prime integers, and so we will assume that a is an odd integer in this case. If b is even, which will be the case in a number of the following results, then the parity of a weighted vertex does coincide with the parity of its weight.

Since a number of our arguments rely on this notion of parity, we often exclude those pairs of numbers whose ratio may be reduced to a ratio of odd integers. We define the sets:

$$\mathcal{E} = \left\{ \{a, b\} \mid \frac{a}{b} = \frac{p}{q}, p, q \text{ odd integers} \right\}$$

$$\mathcal{N} = \left\{ \{a, b\} \mid \frac{a}{b} = \frac{p}{q}, p, q \in \mathbb{Z}, pq \leq 0 \right\}$$

We have already seen examples of bipartite graphs which are 2-weight colourable (C_{4k} for any $k \geq 1$, bipartite theta graphs except $\Theta_{(1, 4k_2+1, \dots, 4k_d+1)}$) and some which are not (C_{4k+2} for any $k \geq 1$). From these examples, we note that a bipartite graph G with both parts of odd size is not necessarily $\{a, b\}$ -weight colourable. However, if G has one part of even size, we are able to prove G is $\{a, b\}$ -weight colourable for particular values of a and b .

Theorem 3.3 *Let $a, b \in \mathbb{R}$ be such that $\{a, b\} \notin \mathcal{E}$. If G is a connected bipartite graph with at least one part being of even size, then G is $\{a, b\}$ -weight colourable.*

Proof: Let $V(G) = X \cup Y$ be a bipartition of the vertices of G with $|X|$ even. By Corollary 2.2 and since $\{a, b\} \notin \mathcal{E}$, we may assume that a is an odd integer and b is an even integer. We assign the weight b to each edge of G . Clearly v is even for each $v \in V(G)$. Let $V(X) = \{x_1, x_2, \dots, x_{2k}\}$ and let P_i be an $x_{2i-1}x_{2i}$ -path in G . By changing every edge weight along P_1 we only change the parity of x_1 and x_2 . By repeating this process for each P_i we have that every vertex of X has odd parity and every vertex of Y has even parity.

Call the resulting edge-weighting w . Suppose that w is not an $\{a, b\}$ -weight colouring. Then there are adjacent vertices x and y such that $w(x) = w(y)$. Thus there exist integers r, r', s, s' such $w(x) = ra + sb$ where r is odd, $w(y) = r'a + s'b$ where r' is even, and $ra + sb = r'a + s'b$. If a and b are linearly independent over \mathbb{Q} , we must have $r = r'$, a contradiction. Hence $b = (p/q)a$ for some $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Thus $rq + sp = r'q + s'p$. Since r is odd and r' is even, p even implies q must be even, a contradiction. Hence p is odd. Similarly, q is odd. Therefore, $b/a = p/q$ with p, q odd, contradicting our choice of a and b . Thus w is an $\{a, b\}$ -weight colouring of G . \square

Corollary 3.4 *Let $a, b \in \mathbb{R}$ be such that $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$. Let $G \neq K_2$ be a connected bipartite graph with a vertex of degree 1. Then G is $\{a, b\}$ -weight colourable. In particular, trees are $\{a, b\}$ -weight colourable.*

Proof: Let $V(G) = X \cup Y$ be a bipartition of the vertices of G . Let $x \in X$ be a vertex of degree 1 and let $y \in Y$ be its neighbour. If $|X|$ or $|Y|$ is even, then G is $\{a, b\}$ -weight colourable by Theorem 3.3. If $|X|$ is odd, then $G - x$ has an $\{a, b\}$ -weight colouring by Theorem 3.3, say w' , such that vertices in $X \setminus \{x\}$ are odd and vertices in Y are even. By assigning b to the edge xy we maintain the parity of all the vertices. Also, since $\{a, b\} \notin \mathcal{N}$ we have $w'(y) \neq 0$ and so x and y will receive different weights, thus giving an $\{a, b\}$ -weight colouring of G . \square

Theorem 3.5 *Let $a, b \in \mathbb{R}$ be such that $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$. Let G be a connected bipartite graph with a thread of even length P and let U be the internal vertices of P . If $G - U$ is connected then G is $\{a, b\}$ -weight colourable.*

Proof: We may assume that a is a positive odd integer and b is a positive even integer. If $X \cup Y$ is the bipartition of $V(G)$ and either $|X|$ or $|Y|$ is even, then G is $\{a, b\}$ -weight colourable by Theorem 3.3. Assume both parts of G are of odd size. Let x and y be the ends of P . We first assume that x and y are distinct. By Lemma 2.8, we may assume that P is a path of length either 2 or 4.

Consider the case that P is of length 2, say $P = xvy$. Let G' be the bipartite graph obtained from G by deleting v and adding two leaves, v_1 adjacent to x and v_2 adjacent to y . Now G' is connected and bipartite with an even side, where v_1 and v_2 both belong to the even side. Theorem 3.3 gives an $\{a, b\}$ -weight colouring of G' , say w' , so that v_1 and v_2 are both odd vertices. Hence xv_1 and yv_2 must both receive a as their weight. Let w be an $\{a, b\}$ -edge-weighting of G , where $w(xv) = w'(xv_1) = a$, $w(yv) = w'(yv_2) = a$ and $w(e) = w'(e)$ for all other edges $e \in E(G)$. If w is not an $\{a, b\}$ -weight colouring of G , then either $w(x) = 2a$ or $w(y) = 2a$. Without loss of generality, suppose $w(x) = ra + sb = 2a$ (a similar argument will hold for y). Since $w(xv) = a$ and r even, we have $r \geq 2$. If $r = 2$, then $sb = 0$ which implies $s = 0$ or equivalently $\deg_G(x) = 2$, a contradiction. If $r \geq 3$, then $sb < 0$ which gives $b < 0$, a contradiction. Thus w is an $\{a, b\}$ -weight colouring of G .

Suppose $|P| = 4$. Let $P = xv_1v_2v_3y$ and let $G' = G - v_2$. Now G' is bipartite with an even side X' , and $x, y \in X'$. Theorem 3.3 gives an $\{a, b\}$ -weight colouring of G' , say w' , so that v_1 and v_3 are both even vertices. Hence xv_1 and yv_3 must both receive b as their weight. Let w be an $\{a, b\}$ -edge-weighting of G , where $w(v_1v_2) = w(v_2v_3) = a$ and $w(e) = w'(e)$ for all other edges $e \in E(G)$. If w is not an $\{a, b\}$ -weight colouring of G , then either $w(x) = a + b$ or $w(y) = a + b$. Suppose $w(x) = ra + sb = a + b$. Then $(r - 1)a = -(s - 1)b$, and thus r is odd. Again, we have that a and b are positive integers. Thus either $r - 1 < 0$ or $s - 1 < 0$. However, since $w(xv_1) = b$, we have $s \neq 0$, and since r is odd, $r \neq 0$. Thus w is an $\{a, b\}$ -weight colouring of G .

Now, suppose x and y are not distinct (call this vertex x). Then P is a cycle which is an end block of G and x is a cut vertex of G . Let z_1 and z_2 be the neighbours of x in P . Since $G' = G - U$ is a connected bipartite graph with one part having even size, then by Theorem 3.3 there is an $\{a, b\}$ -weight colouring of G' , say w' . We give an edge weighting w'' of P as follows:

- if P has length 2 (mod 4), then by Proposition 2.12 we may define an $\{a, b\}$ -weight colouring of P , w'' , so that $w''(z_1) = w''(x) = w''(z_2) = 2a$ and P is properly coloured elsewhere;
- if P has length 0 (mod 4), then by Proposition 2.9 we may define an $\{a, b\}$ -weight colouring of P , w'' , so that $w''(x)$ is the larger of $2a$ and $2b$ and P is properly coloured.

Let w be the weighting obtained by combining w' and w'' . Then $w(x) > w(z_1), w(z_2)$ and x has the same parity under w as under w' . Hence the weight of x is distinct from its neighbours in G . Since all other vertices are properly coloured by w' or w'' , w gives an $\{a, b\}$ -weight colouring of G . \square

Theorem 3.6 *If G is a 2-connected round graph which is not a cycle then G contains at least 2 even ears.*

Proof:

We first claim that G contains no proper 2-connected subgraph which contains all even ears of G . Toward a contradiction suppose H is a 2-connected maximal proper subgraph of G that contains all even ears of G . There exist two vertices of H , say x and y , which are connected by a path P such that $H \cap P = \{x, y\}$. Since H is 2-connected, there are also 2 edge disjoint paths P' and P'' in H between x and y . Thus $P \cup P' \cup P''$ is a theta graph, and by Lemma 3.2 P must be of even length. Since H already contains all even ears of G , $H' = H \cup P$ must be a proper subgraph of G but H' is also 2-connected which contradicts the maximality of H .

Now, if G has no even ear, then any cycle of G is a 2-connected subgraph containing all the even ears and this is a contradiction as G is not a cycle. If G has only one ear, let T be the ear and let x and y be the two ends of T . Then there are 2 edge disjoint paths connecting x and y , one of which must be edge disjoint from T . This path together with T forms a cycle that contains all the even ears of G , a contradiction. \square

Corollary 3.7 *If G is a round graph and all threads of G are odd, then G has at least two leaves.*

We are now able to prove that round graphs can be edge-weight vertex-coloured with most sets of size 2.

Theorem 3.8 *Every round graph is $\{a, b\}$ -weight colourable for $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$.*

Proof: Let G be a round graph. Let B be an end block with vertex of attachment v . If B is isomorphic to K_2 , then G is a bipartite graph with a leaf and thus is $\{a, b\}$ -weight colourable by Corollary 3.4. If B is a cycle, then B is an even thread and G is $\{a, b\}$ -weight colourable by Theorem 3.5. Otherwise, if B is a 2-connected graph which is not a cycle, then by Theorem 3.6, B has at least two even ears and thus B has at least one even ear, say P , which does not contain v as an internal vertex. Let U be the internal vertices of P . Since $G - U$ is connected, G is $\{a, b\}$ -weight colourable by Theorem 3.5. \square

Theorem 3.8, together with Proposition 2.9, gives a class of minimal subgraphs with respect to containment which cannot be $\{a, b\}$ -weight coloured for the pairs $\{a, b\}$ on which we have focused.

Corollary 3.9 *Let a and b be real numbers such that $\{a, b\} \notin \mathcal{E} \cup \mathcal{N}$. Any graph which is not $\{a, b\}$ -weight colourable must contain a cycle of length $1, 2$ or $3 \pmod{4}$.*

We end this section with the following problem.

Problem 3 *Is it true that all bipartite graphs except C_{4k+2} and $\Theta_{(1,4k_1+1,4k_2+1,\dots,4k_d+1)}$ are 2-weight colourable?*

4 More families of graphs with determined 2-weight colourability

We have given a number of examples of $\{a, b\}$ -weight colourable graphs for values of a and b subject to particular restrictions. However we have seen few examples of graphs for which a and b can be any distinct real numbers. We note that the Petersen graph provides such an example of a 2-weight colourable graph. One such edge-weighting is given in Figure 1. By Proposition 2.3, note that any 2-weight colouring of the Petersen graph gives a 3-colouring of it, which is also an optimal proper vertex colouring.

In the rest of this section we describe more families of 2-weight colourable graphs as well as a class of nonbipartite graphs which are $\{a, b\}$ -weight colourable when $ab > 0$. In particular we show that all

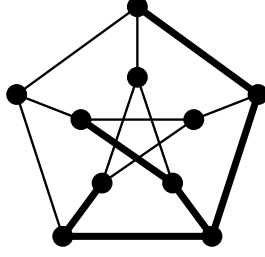


Fig. 1: An $\{a, b\}$ -weight colouring of the Petersen graph. Bold edges are to receive weight b .

unicyclic graphs except cycles of length $1, 2, 3 \pmod{4}$ are 2-weight colourable. We also provide a number of results on Cartesian products of graphs. Finally, we explore techniques for constructing graphs which do not admit $\{a, b\}$ -weight colourings for any choice of a and b .

We begin with our result on unicyclic graphs.

Theorem 4.1 *Every connected unicyclic graph except C_{2k+1} and C_{4m+2} is $\{a, b\}$ -weight colourable, where a and b are real numbers with $ab > 0$.*

Proof: We may assume that $0 < a < b$. By contradiction, let G be the smallest counterexample to our claim. Let C be the only cycle of G . We first note that by Lemma 2.6, we may assume that every vertex of G is either on C or is adjacent to a vertex of C . We may also assume that every vertex of G has degree at most 3. Next, we claim that there are at least two vertices of degree at least 3 on C . If not, let v be the only vertex of degree at least 3 on C . Let x and y be the neighbours of v on C . It is easy to find an edge-weighting w of C which yields a proper colouring on $C - v$ and $w(v) \geq w(x), w(y)$. By assigning b to the other edge incident with v , we get an $\{a, b\}$ -weight colouring of G , a contradiction.

Next, we claim that G has at most one ear of length at least 2. If not, then we choose some maximal path of degree 3 vertices on C , x_1, \dots, x_k , and remove all leaves of G adjacent to those vertices. Call this subgraph G' . By minimality of our choice of G , we can assign an $\{a, b\}$ -weight colouring w' to G' . Let w be the weighting of $E(G)$ given by $w(e) = w'(e)$ if $e \in E(G')$ and $w(e) = b$ otherwise. The only possible conflicts are between x_1 and its neighbour on C which is not x_2 , say y (or, similarly, between x_k and its neighbour on C which is not x_{k-1}). However, since $w(x_1) \geq a + b + w(x_1y)$ and $w(y) \leq w(x_1y) + b$ (similar for x_k), w is an $\{a, b\}$ -weight colouring of G which contradicts our choice of G .

If G has exactly one ear of length at least 2, let $e = rs$ and $e' = r's'$ be the two edges that have exactly one endpoint of degree 2. Specifically, let $\deg(r) = \deg(r') = 2$, $\deg(s) = \deg(s') = 3$. Note that r and r' need not be distinct, but, since there are at least 2 vertices on C of degree 3, s and s' are distinct. We construct an $\{a, b\}$ -weight colouring of G based on the length of $C \pmod{4}$.

- Suppose $|C|$ is odd. By Proposition 2.10, C has an $\{a, b\}$ -edge weighting w' which gives a proper vertex colouring except across rs . Let $w(e) = w'(e)$ if $e \in E(C)$. If $w'(r') - w'(s') = a$, let $w(e) = b$ for all $e \in E(G) \setminus E(C)$. Otherwise, let $w(e) = a$ for all $e \in E(G) \setminus E(C)$. Clearly each leaf's neighbour has a weight strictly greater than its own. Since w' gives a proper colouring of C except for r and s , the only adjacent vertices of G which might not be properly coloured are r

and s or r' and s' . However, our choice of weights for the leaves of G guarantees that r, s, r', s' are properly coloured as well. Thus w is an $\{a, b\}$ -weight colouring of G .

- Suppose $|C| \equiv 0 \pmod{4}$. By Proposition 2.9, C has an $\{a, b\}$ -weight colouring w' such that $w(r) = 2a$ and $w(s) = a + b$. Let $w(e) = w'(e)$ if $e \in E(C)$. If $w'(r') - w'(s') = a$, let $w(e) = b$ for all $e \in E(G) \setminus E(C)$. Otherwise, let $w(e) = a$ for all $e \in E(G) \setminus E(C)$. By the same argument as above, w is an $\{a, b\}$ -weight colouring of G .
- Suppose $|C| \equiv 2 \pmod{4}$. Let t be the other neighbour of s on C and let t' be the other neighbour of r' on C . By Proposition 2.11, there is an $\{a, b\}$ -edge weighting such that all vertices are properly coloured except r, s and t , and such that $w(t'r') = w(r's') = a$. Let $w(e) = w'(e)$ for all $e \in E(C)$. Let f be the edge between s and its leaf, and let $w(f) = a$. For each $e \in E(G) \setminus E(C) \setminus \{f\}$, let $w(e) = b$. The only possible improperly coloured pairs of vertices are r and s , s and t or r' and s' . However,

$$\begin{aligned} w(r) &= w'(r) = w'(s) < w(s) \\ w(s) &= w'(s) + a = w'(t) + a < w'(t) + b = w(t) \\ w(r') &= 2a < a + 2b = w(s') \end{aligned}$$

and so w is an $\{a, b\}$ -weight colouring of G .

The only remaining case is that every vertex of C has degree 3. If $|C|$ is even, assign the same weight to all the edges on the cycle and alternating weights to the leaf edges. The reader can verify that a solution for the cases when $|C| = 3$ or $|C| = 5$ exists. Each of these cases can be extended to larger odd cycle by making the replacement indicated in Figure 2. Note that the variables $\bar{\ell}$ and \bar{n} refer to the weights different from ℓ and n , respectively.

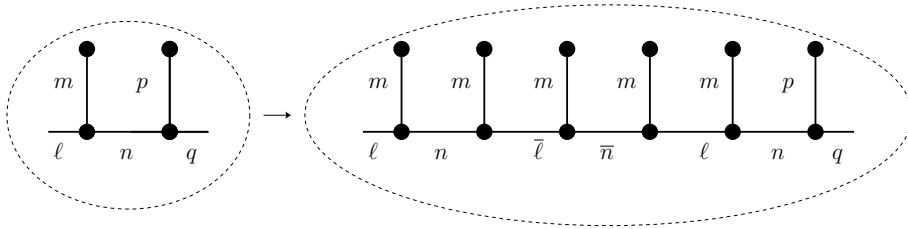


Fig. 2: Replacement operation to expand 2-weight colourings to larger cycles.

Thus, no minimal counterexample G exists. □

Proposition 4.2 For $n \geq 4$, the graph $K_2 \square K_n$ is 2-weight colourable.

Proof: Let K_n and K'_n be the two copies of the complete graph. Denote the vertices of K_n and K'_n , respectively by

$$\begin{aligned} &\{u_1, u_2, \dots, u_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor}, v_{\lfloor n/2 \rfloor + 1}, \dots, v_{n-2}, v_{n-1}\}, \\ &\{u'_1, u'_2, \dots, u'_{\lfloor n/2 \rfloor}, v'_{\lfloor n/2 \rfloor}, v'_{\lfloor n/2 \rfloor + 1}, \dots, v'_{n-2}, v'_{n-1}\}. \end{aligned}$$

Let p be a derangement (permutation with no fixed points) of $\{1, \lfloor n/2 \rfloor\}$ and π be a derangement of $\{\lfloor n/2 \rfloor, n-1\}$. Let u_i be adjacent to $u'_{p(i)}$ for all $1 \leq i \leq \lfloor n/2 \rfloor$ and v_i be adjacent to $v'_{\pi(i)}$ for $\lfloor n/2 \rfloor \leq i \leq n-1$.

Since the graph is n -regular, if adjacent vertices have distinct weights then they have distinct numbers of incident edges having weight b . Using Lemma 2.5, we may weight the edges of K_n and K'_n so that the subscript of the vertex is precisely equal to the number of edges weighted b incident to that edge in K_n . Label $u_i u'_{p(i)}$ with a for all $1 \leq i \leq \lfloor n/2 \rfloor$ and weight $v_i v'_{\pi(i)}$ with b for $\lfloor n/2 \rfloor \leq i \leq n-1$. Then any two vertices that are adjacent have a distinct number of incident edges weighted b and thus $K_2 \square K_n$ is 2-weight colourable. \square

Figure 3 gives an illustration of this construction.

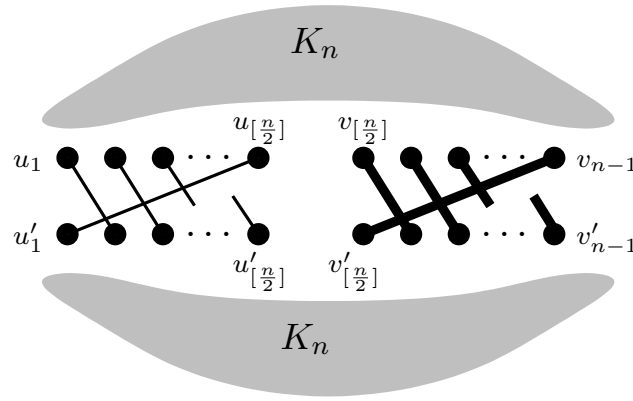


Fig. 3: An $\{a, b\}$ -weight colouring of $K_2 \square K_n$. Bold edges are to receive weight b .

Proposition 4.3 *The graph $K_2 \square C_n$ is 2-weight colourable if and only if $n \geq 4$ and $n \neq 5$.*

Proof: If n is even, then give every edge of one copy of C_n weight a and every edge of the other copy weight b . By alternating the weights of the images of K_2 between a and b along the cycles, we have the desired $\{a, b\}$ -weight colouring.

An example of an $\{a, b\}$ -weight colouring of $K_2 \square C_7$ is given in Figure 4. It can be extended to an $\{a, b\}$ -weight colouring of $K_2 \square C_9$ by replacing the left subgraph in Figure 5 with the right graph. Note that the right subgraph contains the left one, and thus this operation may be repeated as many times as needed to give an $\{a, b\}$ -weight colouring for any $K_2 \square C_{2k+1}$ ($k \geq 3$). The reader may verify that no $\{a, b\}$ -weight colouring of $K_2 \square C_3$ or $K_2 \square C_5$ exists. \square

Theorem 4.4 *Let G be a graph and H be a regular bipartite graph. If $G \square K_2$ is 2-weight colourable, then $G \square H$ is 2-weight colourable.*

Proof: Let w be an $\{a, b\}$ -weight colouring of $G \square K_2$. Denote the two copies of G by G_1 and G_2 and denote the vertices of K_2 by t_1 and t_2 . Since H is regular (say d -regular) and bipartite, Hall's Theorem guarantees a perfect matching M of H . Let X and Y be the parts of $V(H)$.

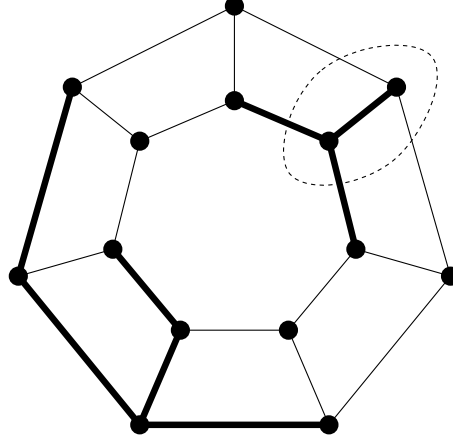


Fig. 4: An $\{a, b\}$ -weight colouring of $K_2 \square C_7$. Bold edges are to receive weight b .

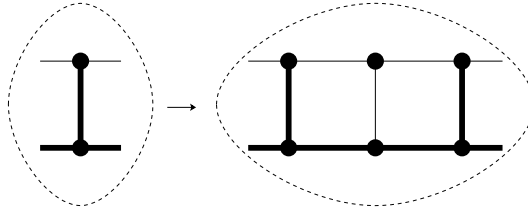


Fig. 5: Replacement operation for obtaining an $\{a, b\}$ -weight colouring of $K_2 \square C_{2k+1}$ for $k \geq 4$.

Define an edge-weighting of $G \square H$ as follows. For each edge $e = xy \in M$ where $x \in X$ and $y \in Y$, weight the edges of the subgraph $G \square e$ by w so that each vertex $(u_G, x) \in V(G \square H)$ has weight $w(u_G, t_1)$ and $(u_G, y) \in V(G \square H)$ has weight $w(u_G, t_2)$. Assign every other edge of $G \square H$ weight a . Call this weighting ϕ .

We have that $\phi(u) = w(u_G, t_1) + (d-1)a$ if $u_H \in X$ and $\phi(u) = w(u_G, t_2) + (d-1)a$ if $u_H \in Y$. Two vertices are adjacent if either their H -coordinates agree and they are adjacent in a copy of G or if their G -coordinates agree and they are adjacent in a copy of H . In the former case, their weights are distinct under ϕ since they are distinct under w . In the latter, consider two adjacent vertices $u = (u_G, u_H)$ and $u' = (u_G, u'_H)$ where $u_H \in X, u'_H \in Y$. Then, $w(u_G, t_1) \neq w(u_G, t_2)$ by choice of w , which implies that $\phi(u) \neq \phi(u')$. Thus ϕ is an $\{a, b\}$ -weight colouring of $G \square H$. \square

Corollary 4.5 *If G and H are regular bipartite graphs, then the following graphs are 2-weight colourable:*

- (i) $K_n \square H$, if $n \geq 4$
- (ii) $C_n \square H$ if $n \geq 4, n \neq 5$
- (iii) $G \square H$

Proof: Applying Theorem 4.4 to Propositions 4.2 and 4.3 immediately gives results (i) and (ii) respectively. For (iii), since $K_2 \square K_2 \cong C_4$, $K_2 \square K_2$ is 2-weight colourable by Proposition 2.9. By Theorem 4.4, $K_2 \square H$ is 2-weight colourable; applying Theorem 4.4 again gives us that $G \square H$ is 2-weight colourable. \square

In order to construct non-2-weight colourable graphs below, we make use of a class of “gadget” graphs. These gadgets are themselves 2-weight colourable, but they have the property that in any of their 2-weight colourings, certain edges receive a predetermined weight.

Define the graph \widehat{K}_n to be the graph obtained from K_n by subdividing one edge exactly once.

Proposition 4.6 *For $n \geq 4$, the graph \widehat{K}_n is 2-weight colourable. Moreover, in any 2-weight colouring of \widehat{K}_n , the edges incident to its degree 2 vertex must receive the same colour.*

Proof: Let x be the vertex of \widehat{K}_n of degree 2 and let u, v be its neighbours. An $\{a, b\}$ -weight colouring of \widehat{K}_4 is given in Figure 6. So assume $n \geq 5$. Let K_n be obtained by adding the edge uv to $\widehat{K}_n - x$. By Lemma 2.5, there exists an edge-weighting w of K_n such that all the vertices have distinct weighted degrees except for u and v . Moreover, $w(u) = w(v) = ra + (n-1-r)b$, where $r \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil - 1\}$. Assign the weight $w(uv)$ from K_n to the edges xu and xv in \widehat{K}_n . Note that w is an $\{a, b\}$ -weight colouring as long as $w(u) = w(v) \neq w(x)$. We have $w(x) \in \{2a, 2b\}$. Since a and b may be swapped in Lemma 2.5, we assume that $w(x) = 2a$. If $w(u) \neq 2a$, we are done. Suppose $w(u) = 2a$. We consider two cases:

- If n is odd, then the edge weighting w' given by swapping every edge's weight gives $w'(u) = w(u) = 2a \neq 2b = w'(x)$.
- If n is even then, by the construction of the weighting in Lemma 2.5, $w(u) = \frac{n}{2}a + (\frac{n}{2} - 1)b$. So, $2a = \frac{n}{2}(a+b) - b$. If the edge weighting w' given by swapping every edge's weight gives a conflict between u and x , then $2b = \frac{n}{2}(a+b) - a$. Together, these imply that $a = b$, a contradiction.

Thus \widehat{K}_n admits an $\{a, b\}$ -weight colouring.

To prove the second part, toward a contradiction, suppose \widehat{K}_n is the smallest counterexample for which there exists an $\{a, b\}$ -weight colouring w such that $w(xu) \neq w(xv)$. By inspection, we may check that \widehat{K}_4 does not admit such edge-weighting. So assume $n \geq 5$. Note that there exists no vertex $y \neq u, v$ such that $w(y) \in \{(n-1)a, (n-1)b\}$, otherwise $\widehat{K}_n - y$ would be a smaller counterexample. Therefore, since w induces a vertex colouring and all the weighted degrees (except for x) are of the form $ra + (n-1-r)b$ for some $0 \leq r \leq n-1$, we must have $w(u), w(v) \in \{(n-1)a, (n-1)b\}$. But then by removing u, v , and x we get an $\{a, b\}$ -weight colouring of K_{n-2} , a contradiction to Corollary 2.4. \square

Corollary 4.7 *Given a graph G , let G' be obtained from identifying a vertex of G with the degree 2 vertex of \widehat{K}_n . Then in any 2-weight colouring of G' , edges in \widehat{K}_n incident to its degree 2 vertex must receive same colour.*

Proof: Since the proof of Proposition 4.6 did not depend in any way on the accumulated weight at vertex x , then regardless of graph joined to \widehat{K}_n at x , the two edges incident with x in \widehat{K}_n must still receive the same weight. \square

An example is given on the left of Figure 6. In the case $G = K_2$ and $n = 4$, the weight of the leaf's edge is forced to be equal to that of its incident edges; this is another useful gadget. It is shown on the right of Figure 6.

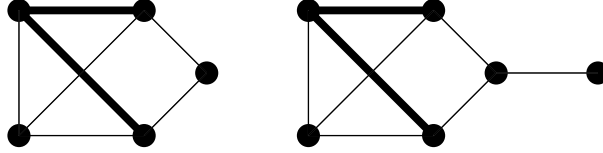


Fig. 6: The graphs \hat{K}_4 and \hat{K}_4 with a leaf are 2-weight colourable. Bold edges represent one weight-class.

We use Proposition 4.6, which established the weight colourability of \hat{K}_n , to construct the following examples of non 2-weight colourable graphs.

Example 4.8 *The following graphs cannot be 2-weight coloured:*

- (i) *Join two copies of \hat{K}_4 by an edge attached at their vertices of degree 2.*
- (ii) *Join $2n + 1$ copies of \hat{K}_4 to a C_{2n+1} by an edge attaching the degree 2 vertex in each copy of \hat{K}_4 to a distinct cycle vertex.*

To see why the graph defined in (ii), which we denote H , cannot be S -weight coloured for any set S of size 2, consider the accumulated weight at one of the cycle vertices, say v . Since H is 3-regular graph, $w(v) \in \{3a, 2a+b, a+2b, 3b\}$. If $w(v) = 3a$, then the noncycle edge, e , incident with v must have weight a and, as shown in Figure 6, so must the two edges in the copy of \hat{K}_4 joined to v by an edge. Thus both endpoints of e would have weight $3a$. A similar argument shows that $w(v) \neq 3b$. Thus the only possible accumulated weights on cycle vertices are $2a + b$ and $2b + a$. Since an odd cycle cannot be properly 2-coloured, we see that H cannot be 2-weight coloured.

Our next family of gadget graphs are described below.

Proposition 4.9 *Let $0 \leq a \in \mathbb{Z}$ and $d \mid a$. Let H be a graph and G be a graph obtained from identifying a vertex u of H with a vertex of a K_n (all other vertices of H and K_n being disjoint). If*

$$\deg_H(u) < \left(\frac{d}{a+d} \right) \left\lfloor \frac{n-1}{2} \right\rfloor,$$

then G is not $\{a, a+d\}$ -weight colourable. Furthermore, if

$$\deg_H(u) = \left(\frac{d}{a+d} \right) \left\lfloor \frac{n-1}{2} \right\rfloor,$$

then in any $\{a, a+d\}$ -weight colouring of G , all edges in H incident to u must receive weight $a+d$.

Proof: We first prove the statement for $d = 1$. Toward a contradiction, suppose w is an $\{a, a+1\}$ -weighting colouring of G . Every vertex of $K_n - u$ has weighted degree $ra + (n-1-r)(a+1) = (n-1)(a+1) - r$ for some $0 \leq r \leq n-1$ and both of the weights $(n-1)a$ and $(n-1)(a+1)$ cannot

appear simultaneously on $K_n - u$. If $w(u) < (n-1)(a+1)$ then there are only $n-1$ colours available for the vertices of K_n , a contradiction. So $w(u) \geq (n-1)(a+1)$.

Let $w|_{K_n}$ be the edge-weighting of K_n induced by w . By Corollary 2.4, K_n is not $\{a, a+1\}$ -edge-weight colourable. Thus, there must be exactly two vertices of K_n with the same weight given by $w|_{K_n}$ and u must be one such vertex. By Lemma 2.5, we get $w|_{K_n}(u) = ra + (n-1-r)(a+1) = (n-1)(a+1) - r$, where $r \in \{\lfloor n/2 \rfloor, \lceil n/2 \rceil - 1\}$. Note that $r \geq \lfloor (n-1)/2 \rfloor$. If u is incident with s edges of weight a in H , then we have

$$(n-1)(a+1) \leq w(u) = ra + (n-1-r)(a+1) + sa + (\deg_H(u) - s)(a+1),$$

which simplifies to $\deg_H(u) \geq \left(\frac{1}{a+1}\right)(r+s)$. Hence

$$\deg_H(u) \geq \left(\frac{1}{a+1}\right) \left(\left\lfloor \frac{n-1}{2} \right\rfloor + s \right).$$

This is a contradiction since $\deg_H(u) < \left(\frac{1}{a+1}\right) \lfloor \frac{n-1}{2} \rfloor$. Also, if $\deg_H(u) = \left(\frac{1}{a+1}\right) \lfloor \frac{n-1}{2} \rfloor$ then we must have $s = 0$, proving the second claim.

Now, let d be any positive divisor of a . By Proposition 2.1, G has an $\{a, a+d\}$ -weight colouring if and only if it has a $\{\frac{a}{d}, \frac{a}{d}+1\}$ -weight colouring. If $\deg_H(u) < \left(\frac{1}{a/d+1}\right) \lfloor \frac{n-1}{2} \rfloor$ then G has no $\{\frac{a}{d}, \frac{a}{d}+1\}$ -weight colouring by the above argument. Hence, if $\deg_H(u) < \left(\frac{d}{a+d}\right) \lfloor \frac{n-1}{2} \rfloor$ then G has no $\{a, a+d\}$ -weight colouring. The second result follows similarly. \square

We may use Proposition 4.9 to construct many graphs which are not $\{a, a+1\}$ -weight colourable and so, in particular, are not $\{1, 2\}$ -weight colourable. In fact, if H is any graph and u any vertex of H , then there is an n large enough so that attaching K_n to u (and only to u) gives a graph which is not $\{a, a+1\}$ -weight colourable. We can also use the equality condition to construct graphs with no $\{a, a+1\}$ -weight colouring (for a specific a). For example, the graph obtained by joining two copies of K_n , $n \geq 5$, with a path of length 3, say e_1, e_2, e_3 , is not $\{a, a+1\}$ -weight colourable for $a = \lfloor \frac{n-1}{2} \rfloor - 1$ since the weights of e_1 and e_3 are forced to be $a+1$, and thus the ends of e_2 will receive the same weight.

We finish this section with a problem whose solution would be useful in the study of $\{1, 2, 3\}$ -weight colourable graphs.

Problem 4 *Does there exist a graph which is either*

- *uniquely $\{1, 2, 3\}$ -weight colourable (up to isomorphism), or*
- *$\{1, 2, 3\}$ -weight colourable where any such colouring forces certain edges of G to receive a particular weight?*

Moreover, does there exist a graph with either of these properties which maintains that property when it is attached in some way to another graph?

References

- [ABDM⁺07] Louigi Addario-Berry, Ketan Dalal, Colin McDiarmid, Bruce A. Reed, and Andrew Thomason. Vertex-colouring edge-weightings. *Combinatorica*, 27(1):1–12, 2007.
- [ABDR08] L. Addario-Berry, K. Dalal, and B. A. Reed. Degree constrained subgraphs. *Discrete Appl. Math.*, 156(7):1168–1174, 2008.
- [BM08] Adrian Bondy and U.S.R. Murty. “Graph Theory” Springer, 2008.
- [CLWY10] Gerard Chang, Changhong Lu, Jiaojiao Wu, and Qinglin Yu. Vertex-coloring edge-weightings of graphs. *Taiwanese J. Math.*, 15(4):1807–1813, 2011
- [KKP09] Maciej Kalkowski, Michal Karonski, and Florian Pfender. Vertex-coloring edge-weightings: Towards the 1-2-3-conjecture. *J. Combin. Theory Ser. B*, 100(3):347–349, 2010.
- [KŁT04] Michał Karoński, Tomasz Łuczak, and Andrew Thomason. Edge weights and vertex colours. *J. Combin. Theory Ser. B*, 91(1):151–157, 2004.
- [LYZ10] Hongliang Lu, Qinglin Yu and Cun-Quan Zhang. Vertex-Coloring 2-Edge-Weightings of Graphs. *European J. Combin.*, 32(1):21–27, 2011.
- [WY08] Tao Wang and Qinglin Yu. On vertex-coloring 13-edge-weighting. *Front. Math. China*, 3(4):581–587, 2008.