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The Erdős-Sós Conjecture for Geometric Graphs[†]

Luis Barba^{1,2} Ruy Fabila-Monroy^{3‡} Dolores Lara^{4†} Jesús Leañós⁵
 Cynthia Rodríguez^{6,7} Gelasio Salazar^{8§} Francisco J. Zaragoza⁶

¹Boursier FRIA du FNRS, Département d'Informatique, ULB, Belgium

²School of Computer Science, Carleton University, Canada

³Departamento de Matemáticas, CINVESTAV, Mexico

⁴Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Barcelona, Spain.

⁵Unidad Académica de Matemáticas, UAZ, Mexico

⁶Departamento de Sistemas, UAM Azcapotzalco, Mexico

⁷Department of Combinatorics and Optimization, University of Waterloo, Canada

⁸Instituto de Física, UASLP, Mexico

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Let $f(n, k)$ be the minimum number of edges that must be removed from some complete geometric graph G on n points, so that there exists a tree on k vertices that is no longer a planar subgraph of G . In this paper we show that $(\frac{1}{2}) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n, k) \leq 2 \frac{n(n-2)}{k-2}$. For the case when $k = n$, we show that $2 \leq f(n, n) \leq 3$. For the case when $k = n$ and G is a geometric graph on a set of points in convex position, we completely solve the problem and prove that at least three edges must be removed.

Keywords: extremal graph theory, geometric graph, spanning tree

1 Introduction

One of the most notorious problems in extremal graph theory is the Erdős-Sós Conjecture, which states that every simple graph with average degree greater than $k - 2$ contains every tree on k vertices as a subgraph. This conjecture was recently proved true for all sufficiently large k (unpublished work of Ajtai, Komlós, Simonovits, and Szemerédi).

In this paper we investigate a variation of this conjecture in the setting of geometric graphs. Recall that a *geometric graph* G consists of a set S of points in the plane (these are the vertices of G), plus a set of straight line segments, each of which joins two points in S (these are the edges of G). In particular, any set S of points in the plane in *general position* (no three of its points are collinear) naturally induces a complete geometric graph. For brevity, we often refer to the edges of this graph simply as edges of S . If

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S is in convex position then G is a *convex geometric graph*. A geometric graph is *planar* if no two of its edges cross each other. An *embedding* of an abstract graph H into a geometric graph G is an isomorphism from H to a planar geometric subgraph of G . For $r \geq 0$, an *r-edge* is an edge of G such that in one of the two open semi-planes defined by the line containing it, there are exactly r points of G . The convex hull of S is the intersection of all convex sets containing S . We will frequently need to refer to the vertices and edges at the boundary of the convex hull of S , which for brevity we will denote simply as *convex hull vertices* and *convex hull edges* of S .

In this paper all point sets are in general position and G is a complete geometric graph on n points. It is well known that for every integer $1 \leq k \leq n$, G contains every tree on k vertices as a planar subgraph [3]. Even more, it is possible to embed any such tree into G , when the image of a given vertex is prespecified [5].

Let F be a subset of edges of G , which we call *forbidden edges*. If T is a tree for which every embedding into G uses an edge of F , then we say that F *forbids* T . In this paper we study the question of what is the minimum size of F so that there is a tree on k vertices that is forbidden by F . Let $f(n, k)$ be the minimum of this number taken over all complete geometric graphs on n points. As $f(2, 2) = 1$, $f(3, 3) = 2$, $f(4, 3) = 3$, $f(4, 4) = 2$ and $f(n, 2) = \binom{n}{2}$, we assume through out the paper that $n \geq 5$ and $k \geq 3$.

We show the following bounds on $f(n, k)$.

Theorem 1.1
$$\left(\frac{1}{2}\right) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n, k) \leq 2 \frac{n(n-2)}{k-2}$$

Theorem 1.2
$$2 \leq f(n, n) \leq 3$$

In the case when G is a convex complete geometric graph, we show that the minimum number of edges needed to forbid a tree on n vertices is three.

An equivalent formulation of the problem studied in this paper is to ask how many edges must be removed from G so that it no longer contains *every* planar subtree on k vertices. A related problem is to ask how many edges must be removed from G so that it no longer contains *any* planar subtree on k vertices. For the case of $k = n$, in [6], it is proved that if any $n - 2$ edges are removed from G , it still contains a planar spanning subtree. Note that if the $n - 1$ edges incident to any vertex of G are removed, then G no longer contains a spanning subtree. In general, for $2 \leq k \leq n - 1$, in [1], it is proved that if any set of $\left\lceil \frac{n(n-k+1)}{2} \right\rceil - 1$ edges are removed from G , it still contains a planar subtree on k vertices. In the same paper it is also shown that this bound is tight—a geometric graph on n vertices and a subset of $\left\lceil \frac{n(n-k+1)}{2} \right\rceil$ of its edges are shown, so that when these edges are removed, every planar subtree has at most $k - 1$ vertices. In [4] the authors study the seemingly unrelated (non-geometric) problem of packing two trees into planar⁽ⁱ⁾ (abstract) graphs. That is, given two trees on n vertices, the authors consider the question of when it is possible to find a planar graph having both of them as spanning trees and in which the trees are edge disjoint. However, although theirs is a combinatorial question rather than

⁽ⁱ⁾ A *planar* (abstract) graph is an *abstract* graph that can be embedded in the plane; the embedding may not be unique.

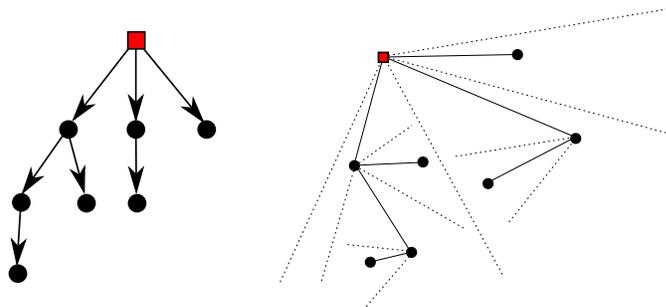


Fig. 1: An embedding of a tree using the algorithm.

geometric, their Theorem 2.1 implies our Lemma 2.2. We provide a self contained proof of Lemma 2.2 for completeness.

A previous version of this paper appeared in the conference proceedings of EUROCG'12 [2].

2 Spanning Trees

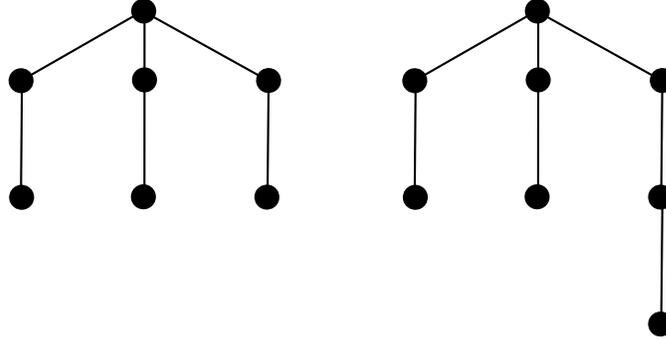
In this section we consider the case when $k = n$. Let T be a tree on n vertices. Consider the following algorithm to embed T into G . Choose a vertex v of T and root T at v . For every vertex of T choose an arbitrary order of its children. Suppose that the neighbors of v are u_1, \dots, u_m , and let n_1, \dots, n_m be the number of nodes in their corresponding subtrees. Choose a convex hull point p of G and embed v into p . Sort the remaining points of G counter-clockwise by angle around p . A *wedge* is a region of the plane bounded by two infinite rays sharing a common apex. Choose $m + 1$ rays centered at p so that the wedge between two consecutive rays is convex and between the i -th ray and the $(i + 1)$ -th ray there are exactly n_i points of G . Let S_i be this set of points. A convex hull vertex q of S_i is *visible* from p if the line segment with endpoints p and q intersects the convex hull of S only at q . For each u_i choose a convex hull vertex of S_i visible from p and embed u_i into this point. Recursively embed the subtrees rooted at each u_i into S_i . Note that this algorithm provides an embedding of T into G . We will use this embedding frequently throughout the paper. See Figure 1.

For every integer $n \geq 2$ we define a tree T_n as follows: If $n = 2$, then T_n consists of only one edge; if n is odd, then T_n is constructed by subdividing once every edge of a star on $\frac{n+1}{2}$ vertices; if n is even and greater than 2, then T_n is constructed by subdividing an edge of T_{n-1} . See Figure 2.

We prove the lower bound of $f(n, n) \geq 2$ of Theorem 1.2.

Theorem 2.1 *If G has only one forbidden edge, then any tree on n vertices can be embedded into G , without using the forbidden edge.*

Proof: Let e be the forbidden edge of G . Let T be a tree on n vertices. Choose a root for T . Sort the children of each node of T , by increasing size of their corresponding subtrees. Embed T into G with the embedding algorithm, choosing at all times the *rightmost* point (the first point when sorting clockwise around the root) as the root of the next subtree. Suppose that e is used in this embedding. Let $e := pq$ so that u is embedded into p and v is embedded into q (note that u is the parent of v in T).

Fig. 2: T_7 and T_8 .

Suppose that the subtree rooted at v has $m \geq 2$ nodes. In the algorithm, we embedded this subtree into a set of exactly m points. We chose a convex hull point (q), of this set visible from p to embed v . In this case we may choose another convex hull point visible from p to embed v and continue with the algorithm. Note that pq is no longer used in the final embedding.

Suppose that v is a leaf, and that v has a sibling v' whose subtree has at least two nodes. Then we may interchange v and v' in the order of the children of u , so that e is no longer used in the embedding, or if it is, then v' is embedded into q , but then we proceed as above.

Suppose that v is a leaf, has at least one sibling and all its siblings are leaves. The subtree rooted at u is a star. We choose a point distinct from p and q in the point set where this subtree is embedded, and embed u into this point. Afterward we join it to the remaining points. This produces an embedding that avoids e .

Assume then, that v is a leaf and that it has no siblings. We distinguish the following cases:

1. **u has no siblings.** In this case, the subtree rooted at the parent of u is a path of length two. If u has no grandparent then $n = 3$ and T can be trivially embedded into G without using e . Suppose u has a grandparent. In this case there are only four vertices to consider: v , u , the parent of u and the grandparent of u . We keep the current location of the grandparent of u , and change the points into which the remaining vertices are embedded. This can always be done so that e is not used in the embedding. All possible cases are shown in Figure 3.
2. **u has a sibling u' whose subtree is not an edge.** We may change the order of the siblings of u , with respect to their parent, so that the subtree rooted at u' will be embedded into the point set containing p and q . In the initial order—increasing by size of their corresponding subtrees— u' is after u . We may assume that in the new ordering, the order of the siblings of u before it, stays the same. Therefore p is the rightmost point of the set into which the subtree rooted at u' will be embedded. Embed u' into p . Either we find an embedding not using e , or this embedding falls into one of the cases considered before.
3. **u has at least one sibling, and the subtree at every sibling of u is an edge.**

Suppose that u has no grandparent; then T is equal to T_n and n is odd. Let w be the parent of u . Embed w into p . Let p_1, \dots, p_{n-1} be the points of G different from p sorted counter-clockwise

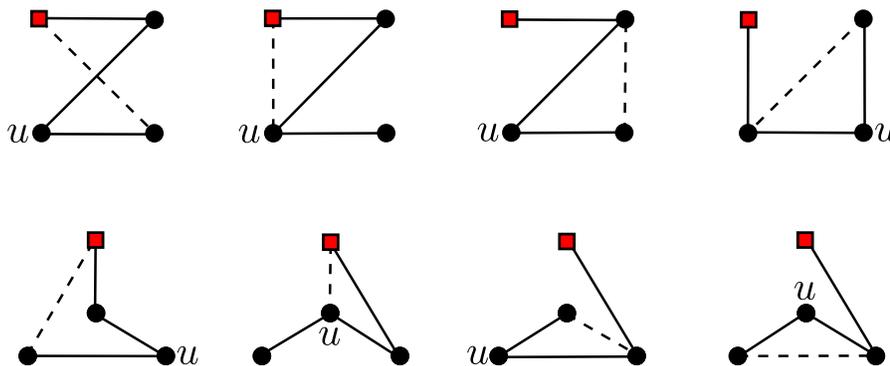


Fig. 3: The embedding of a path of length three. The grandparent of u is highlighted and the forbidden edge is dashed.

by angle around p ; choose p_1 so that the angle between two consecutive points is less than π . Let $u_1, \dots, u_{(n-1)/2}$ be the neighbors of w . Embed each u_i into p_{2i-1} and its child into p_{2i} . If q equals p_{2j-1} for some j then embed u_j into p_{2j} and its child into p_{2j-1} . This embedding avoids e .

Suppose that w is the grandparent of u and let p' be the point into which w is embedded. Let S be the point set into which the subtree rooted at the parent of u is embedded. Note that S has an odd number of points. We replace the embedding as follows. Sort S counter-clockwise by angle around p' . Call a point *even* if it has an even number of points before it in this ordering. Call a point *odd* if it has an odd number of points before it in this ordering. If e is incident to an odd point, then we embed the parent of u into this point. The remaining subtree rooted at u can be embedded without using e . If the endpoints of e are both even, between them there is an odd point. We embed the parent of u into this point. The remaining vertices can be embedded without using e (see Figure 4).

□

The upper bound of $f(n, n) \leq 3$ of Theorem 1.2 follows directly from Lemma 3.2. Now we prove in Lemma 2.2 and Theorem 2.3, that if G is a convex geometric graph, at least three edges are needed to forbid some tree on n vertices.

Lemma 2.2 *Let T be a tree on n vertices. If G is a convex geometric graph, then T can be embedded into G using less than $\frac{n}{2}$ convex hull edges of G .*

Proof: If T is a star, then any embedding of T into G uses only two convex hull edges. If T is a path then it can be embedded into G using at most two convex hull edges. Therefore, we may assume that T is neither a star nor a path.

Since T is not a path, it has a vertex of degree at least three. Choose this vertex as the root. Since T is not a star, the root has a child whose subtree has at least two nodes. Order the children of T so that this node is first. Embed T into G with the embedding algorithm.

Let u and v be vertices of T , so that u is the parent of v . Suppose that the subtree rooted at v has at least two nodes. Then in the embedding algorithm we have at least two choices to embed v once the ordering

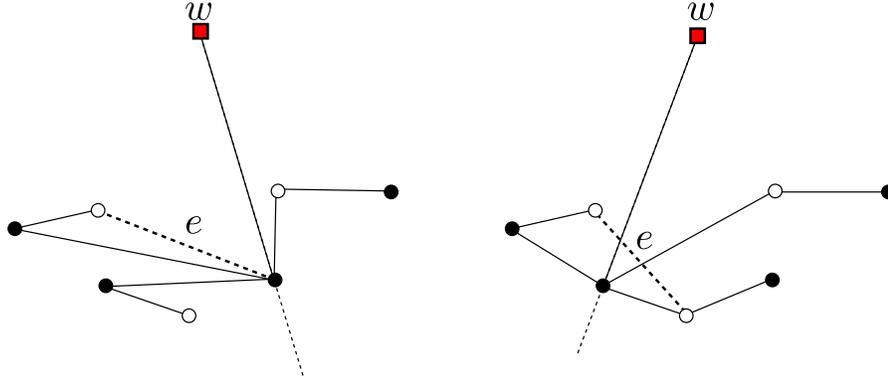


Fig. 4: The two sub-cases, when u has a grandparent w , and all the subtrees of its children are edges. Odd points are painted in black and even points in white. The forbidden edges are dashed.

of the children of u has been chosen. At least one of the choices is such that uw is not embedded into a convex hull edge. Therefore, we may assume that the embedding is such that each convex hull edge used, is incident to a leaf.

Note that every vertex of T , distinct from the root, is incident to at most one convex hull edge in the embedding. Since the first child of the root is not a leaf, no convex hull edge is used to embed this child. Only in the embedding of the last child of the root a convex hull may have been used. Therefore every vertex of T is incident to at most one convex hull edge. Thus the set of convex hull edges used in the embedding is a matching. Therefore at most $n/2$ convex hull edges are used in the embedding.

Suppose that exactly $n/2$ convex hull edges are used. One of these edges must be incident to the root. Since the root was chosen of degree at least three it has a child which is not a leaf nor the first child; we place this vertex last in the ordering of the children of the root. The leaf adjacent to the root can no longer be a convex hull edge and the embedding uses less than $n/2$ convex hull edges. \square

Theorem 2.3 *If G is a convex geometric graph and has at most two forbidden edges, then any tree on n vertices can be embedded into G , without using a forbidden edge.*

Proof: Let f_0 be an embedding given by Lemma 2.2, of T into G . For $0 \leq i \leq n$, let f_i be the embedding produced by rotating f_0 , i places to the right. Assume that in each of these rotations at least one forbidden edge is used, as otherwise we are done. Let e_1, \dots, e_m be the edges of T that are mapped to a forbidden edge in some rotation. Assume that the two forbidden edges are an l -edge and an r -edge respectively.

Suppose that $l \neq r$. Then, each edge of T can be embedded into a forbidden edge at most once in all of the n rotations. Thus $m \geq n$. This is a contradiction, since T has $n - 1$ edges.

Suppose that $l = r$. Then, each of the e_i is mapped twice to a forbidden edge. Thus $m \geq n/2$. By Lemma 2.2, f_0 uses less than $n/2$ convex hull edges. Therefore, $l = r > 0$. But a set of $n/2$ or more r -edges, with $r > 0$, must contain a pair of edges that cross. And we are done, since f_0 is an embedding. \square

3 Bounds on $f(n, k)$

In this section we prove Theorem 1.1. First we show the upper bound.

Lemma 3.1 *If T_n is embedded into G_n then every edge incident to a leaf of T_n must be embedded into a convex hull edge.*

Proof: Let $e := uv$ be an edge of T_n incident to leaf. Suppose that u is the leaf vertex. Then v is of degree two. Suppose that e is not embedded into a convex hull edge of G . Then e divides $S \setminus \{u, v\}$ into two non-empty subsets S_1 and S_2 , so that S_1 lies on the opposite side of S_2 with respect to e . Assume that the parent of v is embedded into S_1 . Then no vertex of T_n can be embedded into S_2 without crossing e . Therefore e must be a convex hull edge of G . \square

Lemma 3.2 *If G is a convex geometric graph, then forbidding three consecutive convex hull edges of G forbids the embedding of T_n .*

Proof: Recall that T_n comes from subdividing a star, let v be the non leaf vertex of this star. Let p_1p_2, p_2p_3, p_3p_4 be the forbidden edges, in clockwise order around the convex hull of G . Note that by Lemma 3.1, in any embedding of T_n into G , an edge incident to a leaf of T_n , must be embedded into a convex hull edge. Neither the leaves of T_n nor its neighbors can be embedded into p_2 or p_3 , without using a forbidden edge. Thus, v must be embedded into p_2 or p_3 . Without loss of generality assume that v is embedded into p_2 . But then, the embedding must use p_2p_3 or p_3p_4 . \square

Lemma 3.3 *If G is a convex geometric graph, then forbidding the convex hull edges incident to any three vertices p_1, p_2 and p_3 of G , forbids the embedding of T_n .*

Proof: Note that by Lemma 3.1, neither a leaf of T_n , nor its neighbor can be embedded into p_1, p_2 or p_3 , without using a forbidden edge. But at most two points do not fall into this category. \square

Lemma 3.4

$$f(n, k) \leq 2 \frac{n(n-2)}{k-2}$$

Proof: Let G be a complete convex geometric graph. We forbid every r -edge of G for $r = 0, \dots, \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil$. Note that, in total we are forbidding at most $n \left(\left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil + 1 \right) \leq 2 \frac{n(n-2)}{k-2}$ edges. As every subset of points of G is in convex position, it suffices to show that every induced subgraph H of G on k vertices is in one of the two configurations of Lemmas 3.2 and 3.3.

Assume then, that H does not contain three consecutive forbidden edges in its convex hull nor three vertices, each with its two convex hull edges forbidden. H has at most two (non-adjacent) pairs of consecutive forbidden edges in its convex hull. Therefore every forbidden edge of H in its convex hull—with the exception of at most two—must be preceded by an ℓ -edge (of G), with $\ell > \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil$. The number of these ℓ -edges contained in H is at least $\frac{k-2}{2}$. The points separated by these edges amount to more than $\frac{k-2}{2} \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil \geq n - k$ points of G . This is a contradiction, since together with the k points of H this is strictly more than n . \square

Now, we show the lower bound of Theorem 1.1.

Lemma 3.5

$$f(n, k) \geq \binom{1}{2} \frac{n^2}{k-1} - \frac{n}{2}$$

Proof: Let F be a set of edges whose removal from G forbids some k -tree. Let $H := G \setminus F$. Note that H contains no complete K_k as a subgraph, otherwise any k -tree can be embedded into this subgraph. By Turán's Theorem [7], H cannot contain more than $\binom{k-2}{k-1} \frac{n^2}{2}$ edges. Thus F must have size at least

$$\binom{n}{2} - \binom{k-2}{k-1} \frac{n^2}{2} = \binom{1}{2} \frac{n^2}{k-1} - \frac{n}{2}$$

□

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