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# The Erdős-Sós Conjecture for Geometric Graphs<sup>†</sup>

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Let  $f(n, k)$  be the minimum number of edges that must be removed from some complete geometric graph  $G$  on  $n$  points, so that there exists a tree on  $k$  vertices that is no longer a planar subgraph of  $G$ . In this paper we show that  $(\frac{1}{2}) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n, k) \leq 2 \frac{n(n-2)}{k-2}$ . For the case when  $k = n$ , we show that  $2 \leq f(n, n) \leq 3$ . For the case when  $k = n$  and  $G$  is a geometric graph on a set of points in convex position, we completely solve the problem and prove that at least three edges must be removed.

**Keywords:** extremal graph theory, geometric graph, spanning tree

## 1 Introduction

One of the most notorious problems in extremal graph theory is the Erdős-Sós Conjecture, which states that every simple graph with average degree greater than  $k - 2$  contains every tree on  $k$  vertices as a subgraph. This conjecture was recently proved true for all sufficiently large  $k$  (unpublished work of Ajtai, Komlós, Simonovits, and Szemerédi).

In this paper we investigate a variation of this conjecture in the setting of geometric graphs. Recall that a *geometric graph*  $G$  consists of a set  $S$  of points in the plane (these are the vertices of  $G$ ), plus a set of straight line segments, each of which joins two points in  $S$  (these are the edges of  $G$ ). In particular, any set  $S$  of points in the plane in *general position* (no three of its points are collinear) naturally induces a complete geometric graph. For brevity, we often refer to the edges of this graph simply as edges of  $S$ . If

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$S$  is in convex position then  $G$  is a *convex geometric graph*. A geometric graph is *planar* if no two of its edges cross each other. An *embedding* of an abstract graph  $H$  into a geometric graph  $G$  is an isomorphism from  $H$  to a planar geometric subgraph of  $G$ . For  $r \geq 0$ , an  *$r$ -edge* is an edge of  $G$  such that in one of the two open semi-planes defined by the line containing it, there are exactly  $r$  points of  $G$ . The convex hull of  $S$  is the intersection of all convex sets containing  $S$ . We will frequently need to refer to the vertices and edges at the boundary of the convex hull of  $S$ , which for brevity we will denote simply as *convex hull vertices* and *convex hull edges* of  $S$ .

In this paper all point sets are in general position and  $G$  is a complete geometric graph on  $n$  points. It is well known that for every integer  $1 \leq k \leq n$ ,  $G$  contains every tree on  $k$  vertices as a planar subgraph [3]. Even more, it is possible to embed any such tree into  $G$ , when the image of a given vertex is prespecified [5].

Let  $F$  be a subset of edges of  $G$ , which we call *forbidden edges*. If  $T$  is a tree for which every embedding into  $G$  uses an edge of  $F$ , then we say that  $F$  *forbids*  $T$ . In this paper we study the question of what is the minimum size of  $F$  so that there is a tree on  $k$  vertices that is forbidden by  $F$ . Let  $f(n, k)$  be the minimum of this number taken over all complete geometric graphs on  $n$  points. As  $f(2, 2) = 1$ ,  $f(3, 3) = 2$ ,  $f(4, 3) = 3$ ,  $f(4, 4) = 2$  and  $f(n, 2) = \binom{n}{2}$ , we assume through out the paper that  $n \geq 5$  and  $k \geq 3$ .

We show the following bounds on  $f(n, k)$ .

**Theorem 1.1** 
$$\left(\frac{1}{2}\right) \frac{n^2}{k-1} - \frac{n}{2} \leq f(n, k) \leq 2 \frac{n(n-2)}{k-2}$$

**Theorem 1.2** 
$$2 \leq f(n, n) \leq 3$$

In the case when  $G$  is a convex complete geometric graph, we show that the minimum number of edges needed to forbid a tree on  $n$  vertices is three.

An equivalent formulation of the problem studied in this paper is to ask how many edges must be removed from  $G$  so that it no longer contains *every* planar subtree on  $k$  vertices. A related problem is to ask how many edges must be removed from  $G$  so that it no longer contains *any* planar subtree on  $k$  vertices. For the case of  $k = n$ , in [6], it is proved that if any  $n - 2$  edges are removed from  $G$ , it still contains a planar spanning subtree. Note that if the  $n - 1$  edges incident to any vertex of  $G$  are removed, then  $G$  no longer contains a spanning subtree. In general, for  $2 \leq k \leq n - 1$ , in [1], it is proved that if any set of  $\left\lceil \frac{n(n-k+1)}{2} \right\rceil - 1$  edges are removed from  $G$ , it still contains a planar subtree on  $k$  vertices. In the same paper it is also shown that this bound is tight—a geometric graph on  $n$  vertices and a subset of  $\left\lceil \frac{n(n-k+1)}{2} \right\rceil$  of its edges are shown, so that when these edges are removed, every planar subtree has at most  $k - 1$  vertices. In [4] the authors study the seemingly unrelated (non-geometric) problem of packing two trees into planar<sup>(i)</sup> (abstract) graphs. That is, given two trees on  $n$  vertices, the authors consider the question of when it is possible to find a planar graph having both of them as spanning trees and in which the trees are edge disjoint. However, although theirs is a combinatorial question rather than

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<sup>(i)</sup> A *planar* (abstract) graph is an *abstract* graph that can be embedded in the plane; the embedding may not be unique.

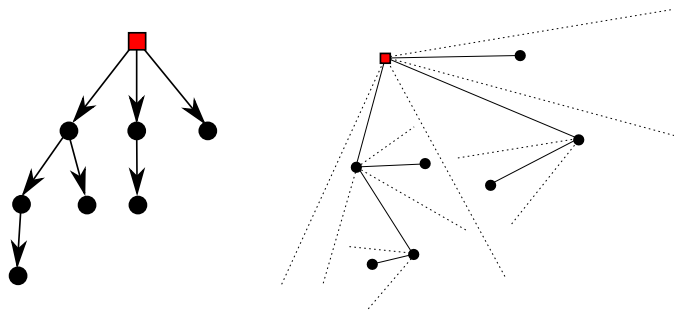


Fig. 1: An embedding of a tree using the algorithm.

geometric, their Theorem 2.1 implies our Lemma 2.2. We provide a self contained proof of Lemma 2.2 for completeness.

A previous version of this paper appeared in the conference proceedings of EUROCG'12 [2].

## 2 Spanning Trees

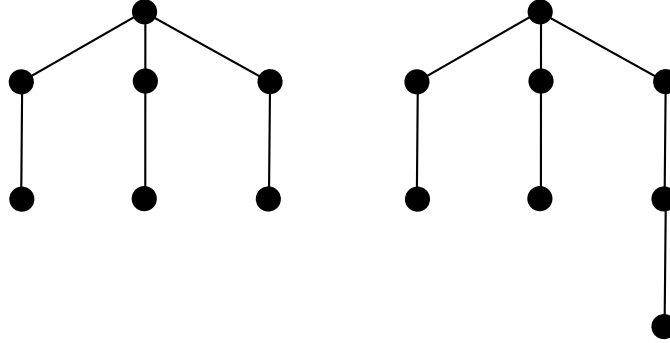
In this section we consider the case when  $k = n$ . Let  $T$  be a tree on  $n$  vertices. Consider the following algorithm to embed  $T$  into  $G$ . Choose a vertex  $v$  of  $T$  and root  $T$  at  $v$ . For every vertex of  $T$  choose an arbitrary order of its children. Suppose that the neighbors of  $v$  are  $u_1, \dots, u_m$ , and let  $n_1, \dots, n_m$  be the number of nodes in their corresponding subtrees. Choose a convex hull point  $p$  of  $G$  and embed  $v$  into  $p$ . Sort the remaining points of  $G$  counter-clockwise by angle around  $p$ . A *wedge* is a region of the plane bounded by two infinite rays sharing a common apex. Choose  $m + 1$  rays centered at  $p$  so that the wedge between two consecutive rays is convex and between the  $i$ -th ray and the  $(i + 1)$ -th ray there are exactly  $n_i$  points of  $G$ . Let  $S_i$  be this set of points. A convex hull vertex  $q$  of  $S_i$  is *visible* from  $p$  if the line segment with endpoints  $p$  and  $q$  intersects the convex hull of  $S$  only at  $q$ . For each  $u_i$  choose a convex hull vertex of  $S_i$  visible from  $p$  and embed  $u_i$  into this point. Recursively embed the subtrees rooted at each  $u_i$  into  $S_i$ . Note that this algorithm provides an embedding of  $T$  into  $G$ . We will use this embedding frequently throughout the paper. See Figure 1.

For every integer  $n \geq 2$  we define a tree  $T_n$  as follows: If  $n = 2$ , then  $T_n$  consists of only one edge; if  $n$  is odd, then  $T_n$  is constructed by subdividing once every edge of a star on  $\frac{n+1}{2}$  vertices; if  $n$  is even and greater than 2, then  $T_n$  is constructed by subdividing an edge of  $T_{n-1}$ . See Figure 2.

We prove the lower bound of  $f(n, n) \geq 2$  of Theorem 1.2.

**Theorem 2.1** *If  $G$  has only one forbidden edge, then any tree on  $n$  vertices can be embedded into  $G$ , without using the forbidden edge.*

**Proof:** Let  $e$  be the forbidden edge of  $G$ . Let  $T$  be a tree on  $n$  vertices. Choose a root for  $T$ . Sort the children of each node of  $T$ , by increasing size of their corresponding subtrees. Embed  $T$  into  $G$  with the embedding algorithm, choosing at all times the *rightmost* point (the first point when sorting clockwise around the root) as the root of the next subtree. Suppose that  $e$  is used in this embedding. Let  $e := pq$  so that  $u$  is embedded into  $p$  and  $v$  is embedded into  $q$  (note that  $u$  is the parent of  $v$  in  $T$ ).

Fig. 2:  $T_7$  and  $T_8$ .

Suppose that the subtree rooted at  $v$  has  $m \geq 2$  nodes. In the algorithm, we embedded this subtree into a set of exactly  $m$  points. We chose a convex hull point ( $q$ ), of this set visible from  $p$  to embed  $v$ . In this case we may choose another convex hull point visible from  $p$  to embed  $v$  and continue with the algorithm. Note that  $pq$  is no longer used in the final embedding.

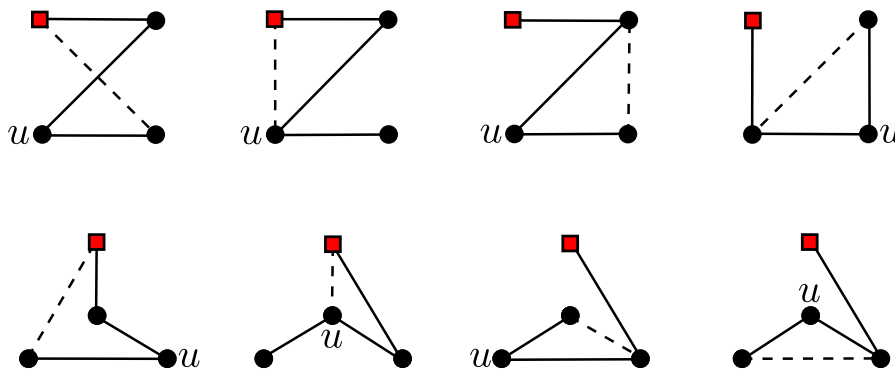
Suppose that  $v$  is a leaf, and that  $v$  has a sibling  $v'$  whose subtree has at least two nodes. Then we may interchange  $v$  and  $v'$  in the order of the children of  $u$ , so that  $e$  is no longer used in the embedding, or if it is, then  $v'$  is embedded into  $q$ , but then we proceed as above.

Suppose that  $v$  is a leaf, has at least one sibling and all its siblings are leaves. The subtree rooted at  $u$  is a star. We choose a point distinct from  $p$  and  $q$  in the point set where this subtree is embedded, and embed  $u$  into this point. Afterward we join it to the remaining points. This produces an embedding that avoids  $e$ .

Assume then, that  $v$  is a leaf and that it has no siblings. We distinguish the following cases:

1.  **$u$  has no siblings.** In this case, the subtree rooted at the parent of  $u$  is a path of length two. If  $u$  has no grandparent then  $n = 3$  and  $T$  can be trivially embedded into  $G$  without using  $e$ . Suppose  $u$  has a grandparent. In this case there are only four vertices to consider:  $v$ ,  $u$ , the parent of  $u$  and the grandparent of  $u$ . We keep the current location of the grandparent of  $u$ , and change the points into which the remaining vertices are embedded. This can always be done so that  $e$  is not used in the embedding. All possible cases are shown in Figure 3.
2.  **$u$  has a sibling  $u'$  whose subtree is not an edge.** We may change the order of the siblings of  $u$ , with respect to their parent, so that the subtree rooted at  $u'$  will be embedded into the point set containing  $p$  and  $q$ . In the initial order—increasing by size of their corresponding subtrees— $u'$  is after  $u$ . We may assume that in the new ordering, the order of the siblings of  $u$  before it, stays the same. Therefore  $p$  is the rightmost point of the set into which the subtree rooted at  $u'$  will be embedded. Embed  $u'$  into  $p$ . Either we find an embedding not using  $e$ , or this embedding falls into one of the cases considered before.
3.  **$u$  has at least one sibling, and the subtree at every sibling of  $u$  is an edge.**

Suppose that  $u$  has no grandparent; then  $T$  is equal to  $T_n$  and  $n$  is odd. Let  $w$  be the parent of  $u$ . Embed  $w$  into  $p$ . Let  $p_1, \dots, p_{n-1}$  be the points of  $G$  different from  $p$  sorted counter-clockwise



**Fig. 3:** The embedding of a path of length three. The grandparent of  $u$  is highlighted and the forbidden edge is dashed.

by angle around  $p$ ; choose  $p_1$  so that the angle between two consecutive points is less than  $\pi$ . Let  $u_1, \dots, u_{(n-1)/2}$  be the neighbors of  $w$ . Embed each  $u_i$  into  $p_{2i-1}$  and its child into  $p_{2i}$ . If  $q$  equals  $p_{2j-1}$  for some  $j$  then embed  $u_j$  into  $p_{2j}$  and its child into  $p_{2j-1}$ . This embedding avoids  $e$ .

Suppose that  $w$  is the grandparent of  $u$  and let  $p'$  be the point into which  $w$  is embedded. Let  $S$  be the point set into which the subtree rooted at the parent of  $u$  is embedded. Note that  $S$  has an odd number of points. We replace the embedding as follows. Sort  $S$  counter-clockwise by angle around  $p'$ . Call a point *even* if it has an even number of points before it in this ordering. Call a point *odd* if it has an odd number of points before it in this ordering. If  $e$  is incident to an odd point, then we embed the parent of  $u$  into this point. The remaining subtree rooted at  $u$  can be embedded without using  $e$ . If the endpoints of  $e$  are both even, between them there is an odd point. We embed the parent of  $u$  into this point. The remaining vertices can be embedded without using  $e$  (see Figure 4).

□

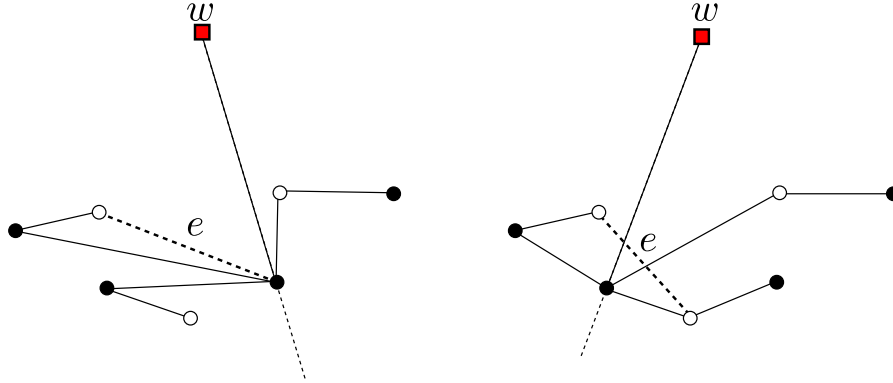
The upper bound of  $f(n, n) \leq 3$  of Theorem 1.2 follows directly from Lemma 3.2. Now we prove in Lemma 2.2 and Theorem 2.3, that if  $G$  is a convex geometric graph, at least three edges are needed to forbid some tree on  $n$  vertices.

**Lemma 2.2** *Let  $T$  be a tree on  $n$  vertices. If  $G$  is a convex geometric graph, then  $T$  can be embedded into  $G$  using less than  $\frac{n}{2}$  convex hull edges of  $G$ .*

**Proof:** If  $T$  is a star, then any embedding of  $T$  into  $G$  uses only two convex hull edges. If  $T$  is a path then it can be embedded into  $G$  using at most two convex hull edges. Therefore, we may assume that  $T$  is neither a star nor a path.

Since  $T$  is not a path, it has a vertex of degree at least three. Choose this vertex as the root. Since  $T$  is not a star, the root has a child whose subtree has at least two nodes. Order the children of  $T$  so that this node is first. Embed  $T$  into  $G$  with the embedding algorithm.

Let  $u$  and  $v$  be vertices of  $T$ , so that  $u$  is the parent of  $v$ . Suppose that the subtree rooted at  $v$  has at least two nodes. Then in the embedding algorithm we have at least two choices to embed  $v$  once the ordering



**Fig. 4:** The two sub-cases, when  $u$  has a grandparent  $w$ , and all the subtrees of its children are edges. Odd points are painted in black and even points in white. The forbidden edges are dashed.

of the children of  $u$  has been chosen. At least one of the choices is such that  $uw$  is not embedded into a convex hull edge. Therefore, we may assume that the embedding is such that each convex hull edge used, is incident to a leaf.

Note that every vertex of  $T$ , distinct from the root, is incident to at most one convex hull edge in the embedding. Since the first child of the root is not a leaf, no convex hull edge is used to embed this child. Only in the embedding of the last child of the root a convex hull may have been used. Therefore every vertex of  $T$  is incident to at most one convex hull edge. Thus the set of convex hull edges used in the embedding is a matching. Therefore at most  $n/2$  convex hull edges are used in the embedding.

Suppose that exactly  $n/2$  convex hull edges are used. One of these edges must be incident to the root. Since the root was chosen of degree at least three it has a child which is not a leaf nor the first child; we place this vertex last in the ordering of the children of the root. The leaf adjacent to the root can no longer be a convex hull edge and the embedding uses less than  $n/2$  convex hull edges.  $\square$

**Theorem 2.3** *If  $G$  is a convex geometric graph and has at most two forbidden edges, then any tree on  $n$  vertices can be embedded into  $G$ , without using a forbidden edge.*

**Proof:** Let  $f_0$  be an embedding given by Lemma 2.2, of  $T$  into  $G$ . For  $0 \leq i \leq n$ , let  $f_i$  be the embedding produced by rotating  $f_0$ ,  $i$  places to the right. Assume that in each of these rotations at least one forbidden edge is used, as otherwise we are done. Let  $e_1, \dots, e_m$  be the edges of  $T$  that are mapped to a forbidden edge in some rotation. Assume that the two forbidden edges are an  $l$ -edge and an  $r$ -edge respectively.

Suppose that  $l \neq r$ . Then, each edge of  $T$  can be embedded into a forbidden edge at most once in all of the  $n$  rotations. Thus  $m \geq n$ . This is a contradiction, since  $T$  has  $n - 1$  edges.

Suppose that  $l = r$ . Then, each of the  $e_i$  is mapped twice to a forbidden edge. Thus  $m \geq n/2$ . By Lemma 2.2,  $f_0$  uses less than  $n/2$  convex hull edges. Therefore,  $l = r > 0$ . But a set of  $n/2$  or more  $r$ -edges, with  $r > 0$ , must contain a pair of edges that cross. And we are done, since  $f_0$  is an embedding.  $\square$

### 3 Bounds on $f(n, k)$

In this section we prove Theorem 1.1. First we show the upper bound.

**Lemma 3.1** *If  $T_n$  is embedded into  $G_n$  then every edge incident to a leaf of  $T_n$  must be embedded into a convex hull edge.*

**Proof:** Let  $e := uv$  be an edge of  $T_n$  incident to leaf. Suppose that  $u$  is the leaf vertex. Then  $v$  is of degree two. Suppose that  $e$  is not embedded into a convex hull edge of  $G$ . Then  $e$  divides  $S \setminus \{u, v\}$  into two non-empty subsets  $S_1$  and  $S_2$ , so that  $S_1$  lies on the opposite side of  $S_2$  with respect to  $e$ . Assume that the parent of  $v$  is embedded into  $S_1$ . Then no vertex of  $T_n$  can be embedded into  $S_2$  without crossing  $e$ . Therefore  $e$  must be a convex hull edge of  $G$ .  $\square$

**Lemma 3.2** *If  $G$  is a convex geometric graph, then forbidding three consecutive convex hull edges of  $G$  forbids the embedding of  $T_n$ .*

**Proof:** Recall that  $T_n$  comes from subdividing a star, let  $v$  be the non leaf vertex of this star. Let  $p_1p_2, p_2p_3, p_3p_4$  be the forbidden edges, in clockwise order around the convex hull of  $G$ . Note that by Lemma 3.1, in any embedding of  $T_n$  into  $G$ , an edge incident to a leaf of  $T_n$ , must be embedded into a convex hull edge. Neither the leaves of  $T_n$  nor its neighbors can be embedded into  $p_2$  or  $p_3$ , without using a forbidden edge. Thus,  $v$  must be embedded into  $p_2$  or  $p_3$ . Without loss of generality assume that  $v$  is embedded into  $p_2$ . But then, the embedding must use  $p_2p_3$  or  $p_3p_4$ .  $\square$

**Lemma 3.3** *If  $G$  is a convex geometric graph, then forbidding the convex hull edges incident to any three vertices  $p_1, p_2$  and  $p_3$  of  $G$ , forbids the embedding of  $T_n$ .*

**Proof:** Note that by Lemma 3.1, neither a leaf of  $T_n$ , nor its neighbor can be embedded into  $p_1, p_2$  or  $p_3$ , without using a forbidden edge. But at most two points do not fall into this category.  $\square$

**Lemma 3.4**

$$f(n, k) \leq 2 \frac{n(n-2)}{k-2}$$

**Proof:** Let  $G$  be a complete convex geometric graph. We forbid every  $r$ -edge of  $G$  for  $r = 0, \dots, \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil$ . Note that, in total we are forbidding at most  $n \left( \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil + 1 \right) \leq 2 \frac{n(n-2)}{k-2}$  edges. As every subset of points of  $G$  is in convex position, it suffices to show that every induced subgraph  $H$  of  $G$  on  $k$  vertices is in one of the two configurations of Lemmas 3.2 and 3.3.

Assume then, that  $H$  does not contain three consecutive forbidden edges in its convex hull nor three vertices, each with its two convex hull edges forbidden.  $H$  has at most two (non-adjacent) pairs of consecutive forbidden edges in its convex hull. Therefore every forbidden edge of  $H$  in its convex hull—with the exception of at most two—must be preceded by an  $\ell$ -edge (of  $G$ ), with  $\ell > \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil$ . The number of these  $\ell$ -edges contained in  $H$  is at least  $\frac{k-2}{2}$ . The points separated by these edges amount to more than  $\frac{k-2}{2} \left\lceil 2 \frac{n-2}{k-2} - 2 \right\rceil \geq n - k$  points of  $G$ . This is a contradiction, since together with the  $k$  points of  $H$  this is strictly more than  $n$ .  $\square$



Now, we show the lower bound of Theorem 1.1.

**Lemma 3.5**

$$f(n, k) \geq \binom{1}{2} \frac{n^2}{k-1} - \frac{n}{2}$$

**Proof:** Let  $F$  be a set of edges whose removal from  $G$  forbids some  $k$ -tree. Let  $H := G \setminus F$ . Note that  $H$  contains no complete  $K_k$  as a subgraph, otherwise any  $k$ -tree can be embedded into this subgraph. By Turán's Theorem [7],  $H$  cannot contain more than  $\binom{k-2}{k-1} \frac{n^2}{2}$  edges. Thus  $F$  must have size at least

$$\binom{n}{2} - \binom{k-2}{k-1} \frac{n^2}{2} = \binom{1}{2} \frac{n^2}{k-1} - \frac{n}{2}$$

□

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