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List Edge and List Total Colorings of Planar Graphs without non-induced 7-cycles[†]

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Giving a planar graph G, let $\chi'_l(G)$ and $\chi''_l(G)$ denote the list edge chromatic number and list total chromatic number of G respectively. It is proved that if G is a planar graph without non-induced 7-cycles, then $\chi'_l(G) \leq \Delta(G) + 1$ and $\chi''_l(G) \leq \Delta(G) + 2$ where $\Delta(G) \geq 7$.

Keywords: List coloring; Planar graph; Choosability.

1 Introduction

The terminology and notation used but undefined in this paper can be found in [1]. Let G be a graph and we use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, face set, maximum degree, and minimum degree of G, respectively. Let $d_G(x)$ or simply $d(x)$, denote the degree of a vertex (resp. face) x in G. A vertex (resp. face) x is called a k-$vertex$ (resp. k-$face$), k^+-$vertex$ (resp. k^+-$face$), or k^--$vertex$, if $d(x) = k$, $d(x) \geq k$, or $d(x) \leq k$. We use $(d_1, d_2, \cdots, d_n)$ to denote a face f if $d_1, d_2, \cdots, d_n$ are the degrees of vertices which are incident with the face f. If u_1, u_2, $\cdots$, u_n are the vertices on the boundary walk of a face f, then we write $f = u_1u_2 \cdots u_n$. Let $\delta(f)$ denote the minimal degree of vertices which are incident with f. We use $f_i(v)$ to denote the number of i-faces which are incident with v for each $v \in V(G)$. Let $n_i(f)$ denote the number of i-vertices which are incident with f for each $f \in F(G)$. A cycle C of length k is called *k-cycle*, and if there is at least one edge $xy \in E(G)\backslash E(C)$ and $x, y \in V(C)$, the cycle C is called *non-induced k-cycle*.

The mapping L is said to be a $total\ assignment$ for a graph G if it assigns a list $L(x)$ of possible colors to each element $x \in V(G) \cup E(G)$. If G has a proper total coloring $\phi(x) \in L(x)$ for all $x \in V(G) \cup E(G)$, then we say that G is *total-L-colorable*. Let $f : V(G) \cup E(G) \to N$ where f is a function into the positive integers. We say that G is *total-f-choosability* if it is total-L-colorable for every total assignment L

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satisfying $|L(x)| = f(x)$ for all $x \in V(G) \cup E(G)$. The *list total coloring number* $\chi_l''(G)$ of G is the smallest integer k such that G is total-f-choosability when $f(x) = k$ for each $x \in V(G) \cup E(G)$. The *list edge coloring number* $\chi_l'(G)$ of G is defined similarly in terms of coloring edges alone; and so is the concept of *edge-f-choosability*. On the list coloring number of a graph G, there is a famous conjecture known as the List Coloring Conjecture.

Conjecture 1 *For a multigraph G,*

$$(a)\ \chi_l'(G) = \chi'(G); \qquad (b)\ \chi_l''(G) = \chi''(G).$$

Part (a) of Conjecture 1 was formulated independently by Vizing, by Gupta, by Alberson and Collins, and by Bollobás and Harris [6, 11]. It is well known as the *List Coloring Conjecture*. Part (b) was formulated by Borodin, Kostochka and Woodall [2]. Part (a) has been proved for bipartite multigraphs [5]. Part (a) and Part (b) have been proved for outerplanar graphs [15], and graphs with $\Delta \geq 12$ which can be embedded in a surface of nonnegative characteristic [2]. There are several related results for planar graphs, such as planar graphs without 4-cycles by Hou et al.[9], planar graphs without 4- and 5-cycles or planar graphs without intersecting 4-cycles by Liu et al.[13], planar graphs without triangles adjacent 4-cycles by Li et al.[14], planar graphs without intersecting triangles by Sheng et al.[18].

To confirm Conjecture 1 is a challenging work. From the Vizing Theorem and the Total Coloring Conjecture, the following weak conjecture is presented.

Conjecture 2 *For a multigraph G,*

$$(a)\ \chi_l'(G) \leq \Delta(G) + 1; \qquad (b)\ \chi_l''(G) \leq \Delta(G) + 2.$$

Part (a) of Conjecture 2 has been proved for complete graphs of odd order [7]. Wang et al. confirmed part (a) of Conjecture 2 for planar graphs without 6-cycles or without 5-cycles [17, 16]. Zhang et al. proved part (a) of Conjecture 2 for planar graphs without triangles [19]. Hou et al. proved part (a) of Conjecture 2 for planar graphs without adjacent triangles or 7-cycles [8]. Cai et al. confirmed part (a) of Conjecture 2 for planar graphs without chordal 5-cycles [3]. Part (b) of Conjecture 2 was proved by Hou et al. for planar graphs G with $\Delta(G) \geq 9$ [10]. Dong et al. confirmed Conjecture 2 for planar graphs without 6-cycles with chord [4].

In this paper, we shall show the following result.

Theorem Let G be a planar graph without non-induced 7-cycles, if $\Delta(G) \geq 7$, then $\chi_l'(G) \leq \Delta(G) + 1$ and $\chi_l''(G) \leq \Delta(G) + 2$.

2 Planar graphs without non-induced 7-cycles

First let us introduce some important lemmas.

Lemma 3 *Let G be a planar graph without non-induced 7-cycles. Then there is an edge $uv \in E(G)$ such that* $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta(G)+1}{2} \rfloor$ *and* $d(u) + d(v) \leq \max\{9, \Delta(G) + 2\}$.

Proof: Suppose to the contrary that G is a minimal counterexample to Lemma 3 in terms of the number of vertices and edges. Then we have $\delta(G) \geq 3$.

By Euler's formula $|V|-|E|+|F|=2$ and $\sum_{v\in V(G)} d(v)=\sum_{f\in F(G)} d(f)=2|E|$, we have

$$\sum_{v\in V(G)}(2d(v)-6)+\sum_{f\in F(G)}(d(f)-6)=-6(|V|-|E|+|F|)=-12.$$

Define an initial charge function w on $V(G)\cup F(G)$ by setting $w(v)=2d(v)-6$ if $v\in V(G)$ and $w(f)=d(f)-6$ if $f\in F(G)$, so that $\sum_{x\in V(G)\cup F(G)} w(x)=-12$. Now redistribute the charge according to the following discharging rules.

For convenience, let $\bar{w}(v)$ denote the total charge transferred from a vertex v to all its incident 4- and 5-faces where $d(v)=5$.

$D1$ Let f be a 3-face incident with a vertex v. Then v gives f charge $\frac{4-\bar{w}(v)}{f_3(v)}$ if $d(v)=5$, $\frac{3}{2}$ if $d(v)\geq 6$.

$D2$ Let f be a 4-face incident with a vertex v. Then v gives f charge $\frac{1}{2}$ if $d(v)=4$, 5 and 6, 1 if $d(v)\geq 7$.

$D3$ Let f be a 5-face incident with a vertex v. Then v gives f charge $\frac{1}{5}$ if $d(v)=4$, 5 and 6, $\frac{1}{3}$ if $d(v)\geq 7$.

Let the new charge of each element x be $w'(x)$ for each $x\in V(G)\cup F(G)$.

In the following, let us check the new charge $w'(x)$ of each element $x\in V(G)\cup F(G)$.

Suppose $d(v)=3$. Then $w'(v)=w(v)=0$.

Suppose $d(v)=4$. Then $w(v)=2$, $f_4(v)\leq 4$. If $2\leq f_4(v)\leq 4$, then $f_5(v)=0$ for G contains no non-induced 7-cycles. We have $w'(v)\geq 2-\frac{1}{2}\times 4=0$ by $D2$. Otherwise, i.e. $f_4(v)\leq 1$, then $f_5(v)\leq 4$. Thus we have $w'(v)>2-\frac{1}{2}-\frac{1}{5}\times 4=\frac{7}{10}>0$ by $D2$ and $D3$.

Suppose $d(v)=5$. Then $w(v)=4$, $f_3(v)\leq 5$. If $1\leq f_3(v)$, then $w'(v)\geq 4-\frac{4-\bar{w}(v)}{f_3(v)}f_3(v)-\bar{w}(v)=0$ by $D1$. Otherwise, i.e. $f_3(v)=0$, then $f_4(v)+f_5(v)\leq 5$. It is clear that $w'(v)>4-\frac{1}{2}\times 5=\frac{3}{2}>0$ by $D2$ and $D3$.

Suppose $d(v)=6$. Then $w(v)=6$, $f_3(v)\leq 4$ for G contains no non-induced 7-cycles. If $f_3(v)=4$, then $f_4(v)=0$ and $f_5(v)=0$ for G contains no non-induced 7-cycles. We have $w'(v)\geq 6-\frac{3}{2}\times 4=0$ by $D1$. If $f_3(v)\leq 3$, then it is clear that $w'(v)>6-\frac{3}{2}\times 3-\frac{1}{2}\times 3=0$ by $D1$, $D2$ and $D3$.

Suppose $d(v)=7$. Then $w(v)=8$, $f_3(v)\leq 5$ for G contains no non-induced 7-cycles.

Suppose $f_3(v)=5$. Then $f_4(v)=0$ and $f_5(v)=0$ for G contains no non-induced 7-cycles. We can get $w'(v)\geq 8-\frac{3}{2}\times 5=\frac{1}{2}>0$ by $D1$.

Suppose $f_3(v)=4$. Then $f_4(v)\leq 2$. If $f_4(v)=2$, then $f_5(v)=0$ for G contains no non-induced 7-cycles. We have $w'(v)\geq 8-\frac{3}{2}\times 4-1\times 2=0$ by $D1$ and $D2$. If $f_4(v)\leq 1$, then $f_5(v)\leq 1$ for G contains no non-induced 7-cycles. We have $w'(v)\geq 8-\frac{3}{2}\times 4-1-\frac{1}{3}=\frac{2}{3}>0$ by $D1$, $D2$ and $D3$.

Suppose $f_3(v)=3$. Then $f_4(v)\leq 2$ and $f_5(v)\leq 2$ for G contains no non-induced 7-cycles. It is clear that $w'(v)>8-\frac{3}{2}\times 3-1\times 2-\frac{1}{3}\times 2=\frac{5}{6}>0$ by $D1$, $D2$ and $D3$.

Suppose $f_3(v)\leq 2$. Then it is clear that $w'(v)>8-\frac{3}{2}\times 2-1\times 5=0$ by $D1$, $D2$ and $D3$.

Suppose $d(v)=8$. Then $w(v)=10$, $f_3(v)\leq 6$ for G contains no non-induced 7-cycles. If $f_3(v)=6$, then $f_4(v)=0$ and $f_5(v)=0$ for G contains no non-induced 7-cycles. We have $w'(v)\geq 10-\frac{3}{2}\times 6=1>0$ by $D1$. If $f_3(v)=5$, then $f_4(v)\leq 1$ and $f_5(v)\leq 1$ for G contains no non-induced 7-cycles. We can get $w'(v)\geq 10-\frac{3}{2}\times 5-1-\frac{1}{3}=\frac{7}{6}>0$ by $D1$, $D2$ and $D3$. If $f_3(v)\leq 4$, then it is clear that $w'(v)\geq 10-\frac{3}{2}\times 4-1\times 4=0$ by $D1$, $D2$ and $D3$.

Suppose $d(v) = 9$. Then $w(v) = 12$, $f_3(v) \leq 7$ for G contains no non-induced 7-cycles. If $f_3(v) = 7$, then $f_4(v) = 0$ and $f_5(v) = 0$ for G contains no non-induced 7-cycles. We can get $w'(v) \geq 12 - \frac{3}{2} \times 7 = \frac{3}{2} > 0$ by $D1$. If $f_3(v) \leq 6$, then it is clear that $w'(v) > 12 - \frac{3}{2} \times 6 - 1 \times 3 = 0$ by $D1$, $D2$ and $D3$.

Suppose $d(v) \geq 10$. Then $w(v) = 2d(v) - 6$, $f_4(v) + f_5(v) \leq d(v) - f_3(v)$. Thus we have $w'(v) \geq 2d(v) - 6 - \frac{3}{2} f_3(v) - f_4(v) - \frac{1}{3} f_5(v) \geq d(v) - 6 - \frac{1}{2} f_3(v)$ by $D1$, $D2$ and $D3$. Since $f_3(v) \leq \frac{4}{5} d(v)$, we have $w'(v) \geq \frac{3}{5} d(v) - 6 \geq 0$.

Suppose $d(f) = 3$. Then $w(f) = -3$.

Suppose $\delta(f) = 3$. Then f is a $(3, 7^+, 7^+)$-face by assumption. We have $w'(f) = -3 + \frac{3}{2} \times 2 = 0$ by $D1$.

Suppose $\delta(f) = 4$. Then f is a $(4, 6^+, 6^+)$-face by assumption. We have $w'(f) = -3 + \frac{3}{2} \times 2 = 0$ by $D1$.

Suppose $\delta(f) = 5$. Then f is a $(5, 5^+, 5^+)$-face.

Suppose f is a $(5, 5, 5)$-face. For convenience, let $f = uvw$. Of the three vertices u, v and w, there is at most one vertex which is incident with at least four 3-faces for the reason that G contains no non-induced 7-cycles. Without loss of generality, let $f_3(u) \geq 4$. Then $f_3(u) + f_4(u) + f_5(u) \leq 5$, $f_3(v) + f_4(v) + f_5(v) \leq 3$ and $f_3(w) + f_4(w) + f_5(w) \leq 3$ for G contains no non-induced 7-cycles. We have $\frac{4-\bar{w}(v)}{f_3(u)} \geq \frac{4}{5}$, $\frac{4-\bar{w}(v)}{f_3(v)} \geq \frac{4}{3}$, $\frac{4-\bar{w}(v)}{f_3(w)} \geq \frac{4}{3}$ by $D2$ and $D3$. Thus $w'(f) \geq -3 + \frac{4}{5} + \frac{4}{3} \times 2 = \frac{7}{15} > 0$ by $D1$. Now we assume that $f_3(u) \leq 3$, $f_3(v) \leq 3$, $f_3(w) \leq 3$. Then we have $\frac{4-\bar{w}(v)}{f_3(v)} \geq 1$ for G contains no non-induced 7-cycles and by $D1$, $D2$ and $D3$. Thus $w'(f) \geq -3 + 1 \times 3 = 0$ by $D1$.

Suppose f is a $(5, 5, 6^+)$-face. For convenience, let $f = uvw$ where $d(u) = d(v) = 5$. Since $f_3(u) + f_4(u) + f_5(u) \leq 5$, $f_3(v) + f_4(v) + f_5(v) \leq 5$, we have $\frac{4-\bar{w}(v)}{f_3(u)} \geq \frac{4}{5}$, $\frac{4-\bar{w}(v)}{f_3(v)} \geq \frac{4}{5}$ by $D2$ and $D3$. Thus $w'(f) \geq -3 + \frac{4}{5} \times 2 + \frac{3}{2} = \frac{1}{10} > 0$ by $D1$.

Suppose f is a $(5, 6^+, 6^+)$-face. Then we have $w'(f) > -3 + \frac{3}{2} \times 2 = 0$ by $D1$.

Suppose $\delta(f) \geq 6$. Then we have $w'(f) = -3 + \frac{3}{2} \times 3 = \frac{3}{2} > 0$ by $D1$.

Suppose $d(f) = 4$. Then $w(f) = -2$. If $\delta(f) = 3$, then f is a $(3, 3^+, 7^+, 7^+)$-face by assumption. We have $w'(f) \geq -2 + 1 \times 2 = 0$ by $D2$. If $\delta(f) \geq 4$, then f is a $(4^+, 4^+, 4^+, 4^+)$-face. We have $w'(f) \geq -2 + \frac{1}{2} \times 4 = 0$ by $D2$.

Suppose $d(f) = 5$. Then $w(f) = -1$.

Suppose $\delta(f) = 3$. Then $n_3(f) \leq 2$ by assumption. If $n_3(f) = 2$, then f is a $(3, 3, 7^+, 7^+, 7^+)$-face by assumption. We have $w'(f) \geq -1 + \frac{1}{3} \times 3 = 0$ by $D3$. If $n_3(f) = 1$, then f is a $(3, 4^+, 4^+, 7^+, 7^+)$-face by assumption. We have $w'(f) \geq -1 + \frac{1}{3} \times 2 + \frac{1}{5} \times 2 = \frac{1}{15} > 0$ by $D3$.

Suppose $\delta(f) \geq 4$. Then we have $w'(f) \geq -1 + \frac{1}{5} \times 5 = 0$ by $D3$.

Suppose $d(f) \geq 6$. Then $w'(f) = w(f) \geq 0$.

From the above discussion, we obtain $-12 = \sum_{x \in V(G) \cup F(G)} w'(x) \geq 0$, a contradiction. □

Lemma 4 *Let G be a planar graph without non-induced 7-cycles. Then $\chi'_l(G) \leq k+1$ and $\chi''_l(G) \leq k+2$ where $k = \max\{\Delta(G), 7\}$.*

Proof: Suppose to the contrary that G' and G'' are minimal counterexamples to the conclusions for χ'_l and χ''_l respectively. Let L' and L'' be list assignments such that $|L'(e)| = k+1$ for each $e \in E(G)$, G' is not edge-L'-colorable, and $|L''(x)| = k+2$ for each $x \in V(G) \cup E(G)$, G'' is not total-L''-colorable. By Lemma 3, G' and G'' contain an edge $uv \in E(G)$ such that $\min\{d(u), d(v)\} \leq \lfloor \frac{\Delta(G)+1}{2} \rfloor$ and $d(u) + d(v) \leq \max\{9, \Delta(G)+2\} = k+2$.

Let $\bar{G'} = G' - uv$. Then $\bar{G'}$ is edge-L'-colorable by assumption. For $d(u) + d(v) \leq k + 2$, there are at most k edges which are adjacent to uv in $\bar{G'}$. Thus there is at lest one color in $L'(uv)$ which we can use to color uv. Then G' is edge-L'-colorable, a contradiction.

Let $\bar{G''} = G'' - uv$. Then $\bar{G''}$ is total-L''-colorable by assumption. With loss of generality, let $d(u) = \min\{d(u), d(v)\}$. Erase the color on u, then there is at least one color in $L''(uv)$ which we can use to color uv for $d(u) + d(v) \leq k + 2$. For $d(u) \leq \lfloor \frac{\Delta(G)+1}{2} \rfloor \leq \lfloor \frac{k+1}{2} \rfloor$, then u is adjacent to at most $\lfloor \frac{k+1}{2} \rfloor$ vertices and is incident with at most $\lfloor \frac{k+1}{2} \rfloor$ edges. Thus there is at least one color in $L''(u)$ which we can use to color u. Then G'' is total-L''-colorable, a contradiction. From the above discussion, we have $\chi'_l(G) \leq k + 1$ and $\chi''_l(G) \leq k + 2$ where $k = \max\{\Delta(G), 7\}$. □

By Lemma 4, it is easy to obtain the main theorem.

Theorem Let G be a planar graph without non-induced 7-cycles, if $\Delta(G) \geq 7$, then $\chi'_l(G) \leq \Delta(G) + 1$ and $\chi''_l(G) \leq \Delta(G) + 2$.

References

[1] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, North-Holland, New York, 1976.

[2] O. Borodin, A. Kostochka, D. Woodall, List edge and list total colourings of multigraphs, J. Combin. Theory Ser. B 71 (1997) 184–204.

[3] J. Cai, J. Hou, X. Zhang, G. Liu, Edge-choosability of planar graphs without non-induced 5-cycles, Information Processing Letters, 109 (7) (2009) 343–346.

[4] A. DONG, G. Liu, G. Li, List edge and list total colorings of planar graphs without 6-cycles with chord, Bull. Korean Math. Soc. 49 (2012) 359–365.

[5] F. Galvin, The list chromatic index of a bipartite multigraph, J. Combin. Theory Ser. B 63 (1995) 153–158.

[6] R. Hägkvist, A. Chetwynd, Some upper bounds on the total and list chromatic numbers of multigraphs, J. Graph Theory 16 (1992) 503–516.

[7] R. Hägkvist, J, Janssen, New bounds on the list-chromatic index of the complete graph and other simple graphs, Combin. Probab. Comput. 6 (1997) 295–313.

[8] J. Hou, G. Liu, J. Cai, Edge-choosability of planar graphs without adjacent triangles or without 7-cycle, Discrete Mathematics, 309 (2009) 77–84.

[9] J. Hou, G. Liu, J. Cai, List Edge and List Total Colorings of Planar Graphs without 4-cycles, Theoretical Computer Science, 369 (2006) 250–255.

[10] J. Hou, G. Liu, J. Wu, Some results on list total colorings of planar graphs, Lecture Note in Computer Science, 4489 (2007) 320–328.

[11] T. Jensen, B. Toft, Graph Coloring Problem, Wiley-Interscience, New York, 1995.

[12] A. Kostochka, List edge chromatic number of graphs with large girth, Discrete Math. 101 (1992) 189–201.

[13] B. Liu, J. Hou, G. Liu, List edge and list total colorings of planar graphs without short cycles, Information Processing Letters, 108 (2008) 347–351.

[14] R. Li, B. Xu, Edge choosability and total choosability of planar graphs without no 3-cycles adjacent 4-cycles, Discrete Mathematics, 311 (2011) 2158–2163.

[15] W. Wang, K. Lih, Choosability, edge-choosability and total choosability of outerplane graphs, European J. Combin. 22 (2001) 71–78.

[16] W. Wang, K. Lih, Choosability and edge choosability of planar graphs without five cycles, Appl. Math. Lett. (2002) 15 561–565.

[17] W. Wang, K. Lih, Structure properties and edge choosability of planar graphs without 6-cycles, Combin. Probab. Comput. 10 (2001) 267–276.

[18] H. Sheng, Y. Wang, A structural theorem for planar graphs with some applications, Discrete Applied Mathematics, 159 (2011) 1183–1187.

[19] L. Zhang, B. Wu, Edge choosability of planar graphs without small cycles, Discrete Math. 283 (2004) 289–293.