Algebraic properties of copulas defined from matrices

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Introduction

- Extension of the bivariate family [Amblard Girard 2002]

$$c(u,v) = 1 + \theta\phi(u)\phi(v),$$

- $c(u,v) = {}^t\phi(u)A\phi(v)$ where :
  - $\phi(u) = {}^t\{1, \phi_2(u), \cdots, \phi_p(u)\}$,
  - $\{\phi_i\}$ is an orthonormal family of functions,
  - $A \in \mathbb{R}^{p\times p}$ is a symmetric matrix such that $Ae_1 = e_1$, with ${}^te_1 = (1, 0, \cdots, 0)$.
- For which A and ϕ, $c(u,v)$ is a density of copula ?

Plan

- Definition of the **family of copulas** $\mathcal{C}_\phi$,
- **Algebraic** properties of the set of convenient matrices $\mathcal{A}_\phi$ and of the copulas family $\mathcal{C}_\phi$,
- **Dependence** properties of the family $\mathcal{C}_\phi$,
- **Projection** on $\mathcal{C}_\phi$,
- Examples.

A new family of copulas

Defnition :

$$c(u,v) = {}^t\phi(u)A\phi(v)$$

- $\phi(u) = {}^t\{1, \phi_2(u), \cdots, \phi_p(u)\}$,
- $\{\phi_i\}$ is an orthonormal family of functions, using $L^2(\mathbb{R})$ scalar product :

$$<\phi_i, \phi_j> = \int_0^1 \phi_i(t)\phi_j(t)dt,$$

- $A \in \mathbb{R}^{p\times p}$ is a symmetric matrix such that $Ae_1 = e_1$, with $e_1 = {}^t(1, 0 \cdots, 0)$.

A new family of copulas

$$\mathcal{A}_\phi = \left\{ \begin{array}{l} A \in \mathbb{R}^{p\times p},\ {}^tA = A,\ Ae_1 = e_1, \\ \forall (u,v) \in [0,1]^2,\ {}^t\phi(u)A\phi(v) \geq 0 \end{array} \right\}$$

$$\mathcal{C}_\phi = \left\{ \begin{array}{l} c : [0,1]^2 \longrightarrow \mathbb{R}, \\ c(u,v) = {}^t\phi(u)A\phi(v),\ A \in \mathcal{A}_\phi \end{array} \right\}$$

A new family of copulas

Properties :

- $\int_0^1 \phi(t)dt = e_1$:

$$\int_0^1 \phi(v)dv = \begin{pmatrix} \int_0^1 1dv \\ \int_0^1 \phi_2(v)dv \\ \dots \\ \int_0^1 \phi_p(v)dv \end{pmatrix},$$

$$= e_1 \text{ because } \{\phi_i\} \text{ is orthonormal.}$$

A new family of copulas

- $\mathcal{A}_\phi$ is not empty, $A_1 = e_1^t e_1 \in \mathcal{A}_\phi$
- $\mathcal{C}_\phi$ is a set of copulas density.
  - **Positivity :** $\forall (u,v) \in \mathbb{R}^2,\ c(u,v) \geq 0$,
  - **Uniform marginals :**

$$\begin{aligned} \int_0^1 c(u,v)dv &= {}^t\phi(u)A\int_0^1 \phi(v)dv, \\ &= {}^t\phi(u)e_1, \\ &= 1 \end{aligned}$$

A new family of copulas

Each copula of $\mathcal{C}_\phi$ is defined by one unique matrix :

$$
\begin{aligned}
& {}^t\phi(u)A\phi(v) = {}^t\phi(u)B\phi(v) \\
\Rightarrow\ & \phi(u)\,{}^t\phi(u)A\phi(v)\,{}^t\phi(v) = \phi(u)\,{}^t\phi(u)B\phi(v)\,{}^t\phi(v) \\
\Rightarrow\ & \iint_0^1 \phi(u)\,{}^t\phi(u)A\phi(v)\,{}^t\phi(v)dudv \\
& = \iint_0^1 \phi(u)\,{}^t\phi(u)B\phi(v)\,{}^t\phi(v)dudv \\
\Rightarrow\ & \int_0^1 \phi(u)\,{}^t\phi(u)du A \int_0^1 \phi(v)\,{}^t\phi(v)dv \\
& = \int_0^1 \phi(u)\,{}^t\phi(u)du B \int_0^1 \phi(v)\,{}^t\phi(v)dv \\
\Rightarrow\ & A = B \text{ because } \{\phi_i\} \text{ is an orthonormal family.}
\end{aligned}
$$

Examples

- The copula associated to $A_1 = e_1\,{}^t e_1$ is the independent copula Π,
- If $p = 2$, necessarily $A = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix}$ and $c(u, v) = 1 + \theta\phi(u)\phi(v)$ [Amblard Girard 2002],
- The cubic family [Nelsen et al. 1997] can be written in our formalism (p=3),
- If $\{\phi_i\}$ is an orthonormal family and $\forall(u, v) \in [0, 1]^2\ {}^t\phi(u)\phi(v) \geq 0$, $I_p \in \mathcal{A}_\phi$ and ${}^t\phi(u)\phi(v) \in \mathcal{C}_\phi$

Algebraic properties of $\mathcal{A}_\phi$

- $\mathcal{A}_\phi$ is a convex set,
- $A_1 = e_1\, {}^t e_1 \in \mathcal{A}_\phi$,
- $(\mathcal{A}_\phi, \times)$ is a semi group :

$$\begin{aligned} {}^t\phi(u)AB\phi(v) &= {}^t\phi(u)A\int_0^1 \phi(y)\, {}^t\phi(y)dy\, B\phi(v), \\ &= \int_0^1 \left({}^t\phi(u)A\phi(y)\right)\left({}^t\phi(y)B\phi(v)\right)dy, \\ &\geq 0. \end{aligned}$$

  - $ABe_1 = e_1$,
  - the product $\times$ is an associative operator.
- If $I_p \in \mathcal{A}_\phi$, $(\mathcal{A}_\phi, \times)$ is a monoid.

Algebraic properties of $\mathcal{C}_\phi$

- $\mathcal{C}_\phi$ is a convex set,
- $\Pi \in \mathcal{C}_\phi$,
- $(\mathcal{C}_\phi, \star)$ is a semi group :

$$\begin{aligned} c_A \star c_B(u,v) &\triangleq \int_0^1 c_A(u,s) c_B(s,v) ds, \\ &= \int_0^1 {}^t\phi(u) A \phi(s) {}^t\phi(s) B \phi(v) ds, \\ &= {}^t\phi(u) A \int_0^1 \phi(s) {}^t\phi(s) ds B \phi(v), \\ &= {}^t\phi(u) A I_p B \phi(v). \end{aligned}$$

  - the operator $\star$ is associative,
- If ${}^t\phi(u)\phi(v) \in \mathcal{C}_\phi$, $(\mathcal{C}_\phi, \star)$ is a monoid .

Isomorphism between $\mathcal{A}_\phi$ and $\mathcal{C}_\phi$

- **Definition :**

$$T_\phi : \{copulas\} \longrightarrow \mathbb{R}^{p\times p}$$
$$c \longmapsto \iint_0^1 \phi(x)c(x,y)\,{}^t\phi(y)dxdy$$

- $T_\phi(c)e_1 = e_1$.
- T_ϕ is an isomorphism between $(\mathcal{A}_\phi, \times)$ and $(\mathcal{C}_\phi, \star)$:
  - Each matrix of $\mathcal{A}_\phi$ defines a copula of $\mathcal{C}_\phi$,
  - T_ϕ associates to a copula of $\mathcal{C}_\phi$ its matrix A,
  - $T_\phi(c_A \star c_B) = A \times B$.

Isomorphism between $\mathcal{A}_\phi$ and $\mathcal{C}_\phi$

$T_\phi(c)e_1 = e_1$:

$$\begin{aligned}
T_\phi(c)e_1 &= \iint \phi(x)c(x,y)\ {}^t\phi(y)e_1 dydx \\
&= \iint \phi(x)c(x,y)\phi_1(y)dydx \\
&= \int \int_0^1 \phi(x)c(x,y)1dy\,dx \\
&= \int_0^1 \phi(x)1dx \\
&= e_1
\end{aligned}$$

Dependence coefficients

- **Spearman 's Rho :**

$$\rho_\phi \triangleq 12\iint_0^1 C(u,v)dudv - 3$$
$$= 12\,{}^t\mu A\mu - 3,$$
$$\text{where } \mu = \int_0^1 x\phi(x)dx.$$

If $A = \mathrm{diag}\{a_{i,i}\}$, $\rho_\phi = 12\sum_{i=2}^{p} a_{i,i}\mu_i^2$

- **Tail dependence coefficient :**

$$\lambda_\phi = P(V \geq u | U \geq u) = \frac{\bar{C}(u,u)}{1-u} = 0.$$

Projection on $\mathcal{C}_\phi$

- **Definition** :

$$P(c)(u,v) = {}^t\phi(u) T_\phi(c)\phi(v)$$

- If $I_p \in \mathcal{A}_\phi$, $P(c)(u,v)$ is a copula :

$$\begin{aligned} P(c)(u,v) &= {}^t\phi(u) T_\phi(c)\phi(v) \\ &= {}^t\phi(u) \iint_0^1 {}^t\phi(x) c(x,y)\phi(y)dxdy\,\phi(v) \\ &= \iint_0^1 {}^t\phi(u)\phi(x) c(x,y)\, {}^t\phi(y)\phi(v) dxdy \\ &= c_{I_p} \star c \star c_{I_p}(u,v) \text{ if } I_p \in \mathcal{A}_\phi \end{aligned}$$

- If $I_p \in \mathcal{A}_\phi$; $P(c)(u,v) \in \mathcal{C}_\phi$.

Projection on $\mathcal{C}_\phi$

- $\int_0^1 c_1 \star c_2(u,u)du$ defines a **Scalar product**.
- P is an orthogonal projection on $(\mathcal{C}_\phi, <>)$:

$P(P(c)) = P(c)$:

$$\begin{aligned} P(P(c)(u,v) &=^t \phi(u) T_\phi(P(c))\phi(v) \\ &= {}^t\phi(u) \int\int_0^1 \phi(x) P(c)(x,y)\ {}^t\phi(y) dxdy \phi(v), \\ &= {}^t\phi(u) \int\int \phi(x)\ {}^t\phi(x) T_\phi(c)\phi(y)\, {}^t\phi(y) dxdy \phi(v) \\ &= {}^t\phi(u) \int \phi(x)\ {}^t\phi(x) dx\, T_\phi(c) \int\ {}^t\phi(y)\phi(y) dy \phi(v) \\ &= {}^t\phi(u) T_\phi(c)\phi(v), \\ &= P(c). \end{aligned}$$

$$\forall s \in \mathcal{C}_\phi,\ < c - P(c), s >= 0.$$

Projection on $\mathcal{C}_\phi$

$$\forall s \in \mathcal{C}_\phi, \; < c - P(c), s >= 0$$

$$\begin{aligned}
< c, s > &= \iint c(u,t)\ {}^t\phi(t) A \phi(u) dt du \\
&= \iint c(u,t) tr({}^t\phi(t) A \phi(u)) dt du \\
&= tr(\iint c(u,t) \phi(u)\ {}^t\phi(t) dt du A) \\
&= tr(T_\phi(c) A).
\end{aligned}$$

$$\begin{aligned}
< P(c), s > &= \iint {}^t\phi(u) T_\phi(c) \phi(t)\ {}^t\phi(t) A \phi(u) dt du \\
&= tr(T_\phi(c) A).
\end{aligned}$$

Example : FGM family

$$\phi(x) = \sqrt{3}(1-2x),\ A = \mathrm{diag}\{1,\theta\},\ |\theta| \leq 1/3,$$

- **Copula** :
  $c(u,v) = 1 + 3\theta(1-2u)(1-2v),\ I_p \notin \mathcal{A}_\phi.$
- **"Projection "on** $\mathcal{C}_\phi$: $T_\phi(c) = \mathrm{diag}\{1,\widetilde{\theta}\}$

$$\begin{aligned}\widetilde{\theta} &= 3\iint c(x,y)(1-2x)(1-2y)dxdy \\ &= 3[4\iint xyc(x,y)dxdy - 1] = \rho_c.\end{aligned}$$

- If $|\rho_c| \leq 1/3,\ P(c)$ is a FGM copula and $\rho_{P(c)} = \rho_c$,
- If $|\rho_c| > 1/3,\ P(c)$ is not a copula.

Example : trigonometric family

$$\begin{cases} \phi_0(x) = 1, \\ \phi_{2j-1}(x) = \sqrt{2}\sin(2\pi jx), \\ \phi_{2j}(x) = \sqrt{2}\cos(2\pi jx) \end{cases} \qquad A = diag\{1, \theta, \theta, \cdots\}$$

- **Copula** : $c_k(x, y) = 1 + 2\theta[H_k(x - y) - 1]$,

  $H_k(t) = \frac{\sin((2k+1)\pi t)}{\sin(\pi t)}$ the Dirichlet Kernel.

- **Spearman's rho** : $\rho_k(\theta) = \frac{6\theta}{\pi^2} \sum_{j=1}^{k} \frac{1}{j^2}$

  $\rho_1(1/2) = \frac{3}{\pi^2} \simeq 0.3$,
  $\rho_2(1/2) = \frac{15}{4\pi^2} \simeq 0.38$,
  $\rho_3(4/9) = \frac{98}{27\pi^2} \simeq 0.37$.

Example : cosinus family

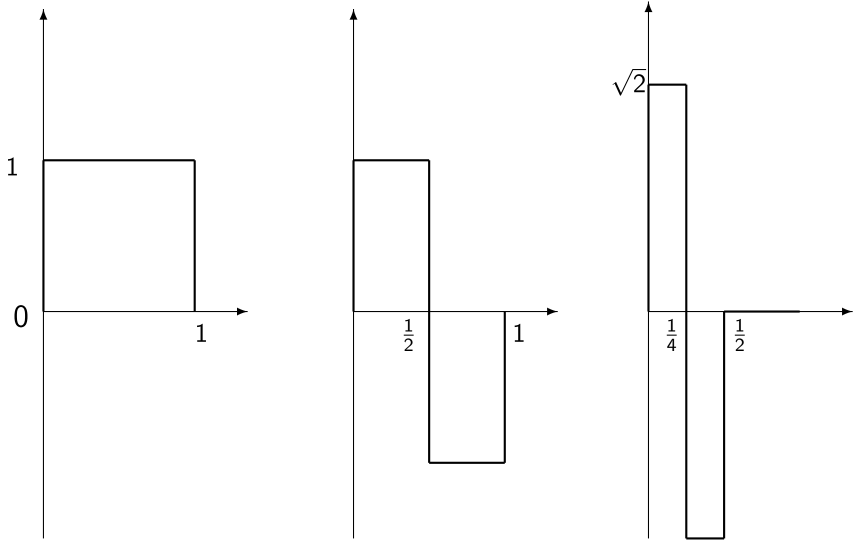
$$\left\{ \begin{array}{lcl} \phi_0(x) & = & 1, \\ \phi_j(x) & = & \sqrt{2}\cos(\pi j x), \end{array} \right. \qquad A = diag\{1, \theta, \theta, \cdots\}$$

- **Copula** :
  $c_k(x,y) = 1 + 2\theta[H_k(\frac{x-y}{2}) - 1][H_k(\frac{x+y}{2}) - 1].$
- **Spearman's rho** :

$$\rho_k(\theta) = \frac{48\theta}{\pi^4} \sum_{j=1}^{k} \frac{(1 + (-1)^{j+1})}{j^4},$$

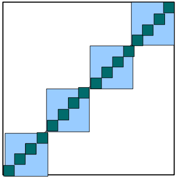
$\rho_1(1/2) = \frac{48}{\pi^4} \simeq 0.49.$

Example : The Haar basis



Example : The Haar basis

- **Copula** : $c_k(x,y) = 1 + \theta[K_k(x,y) - 1]$.



$K_k(x,y)$ is always positive, $I_{2^k} \in \mathcal{A}_\phi$

- **Spearman's rho** :

$$\rho_k(\theta) = 1 - \frac{\theta}{2^{2k}}$$

$\rho_k(1) \rightarrow 1$ as $k \rightarrow \infty$.

Conclusion and future work

- We proposed a new family of copulas defined from matrices,
- The family includes some known families (FGM, cubic sections,...),
- The family contains high dependence copulas (Haar),
- Other orthonormal families should be studied (polynomial...),
- Dependence properties have to be more deeply studied.

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