Performance bounds for the pulse phase estimation of X-ray pulsars

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Performance Bounds for The Pulse Phase Estimation of X-Ray Pulsars

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Abstract

The use of X-ray pulsar signals appears to be a potential solution for autonomous deep space navigation. The main challenge in this kind of navigation is to estimate very precisely the initial phase of the pulse arriving at the detector. Consequently, a study of statistical performance is of the utmost interest. Previous studies indicate that in the performance of pulse phase estimators, the so-called threshold phenomenon arises when the observation time is below a critical limit. In this correspondence, to provide a prediction of the threshold position, the closed-form expressions of the lower bounds on the mean square error are derived and analyzed in both deterministic and Bayesian contexts. These bounds demand less computational cost than the classical Monte Carlo simulations. Numerical results show that the proposed bounds are able to predict the threshold location in both contexts.

Index Terms

X-ray Pulsar, Quinlan-Chaumette-Larzabal bound, Weiss-Weinstein bound , Bayesian bound, pulse phase estimation, spacecraft navigation.

I. Introduction

The development of deep space operation requires accurate and autonomous navigation solutions for the purpose of orienting and controlling a spacecraft. The current ground-based navigation is very accurate but highly depends on the communication with the ground station, and therefore, is not robust to a loss of contact. Besides, large errors can occur in shadowing areas or at large distance from the ground. While satellite navigation systems, such as the Global Positioning System (GPS), are helping devices operating inside the orbit of the GPS constellation to internally determine their location within a few meters or even less, a similar solution for spacecraft is still an open question. In this context, the celestial-based system that uses signals from celestial sources is a potential candidate

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for autonomous deep space navigation. Among various types of celestial sources, pulsars, discovered in 1967, are the subset that emits highly regular, stable, and periodic signals. Their behavior has been observed for years, so the shape and period of their pulse profile are known very accurately. This property could be of the utmost interest for navigation objectives. Therefore, in this contribution, we focus on pulsars among other celestial sources. In the literature, two kinds of pulsars were examined for navigation purposes: sources that emit in the radio band and sources that emit in the X-ray band. We here consider the X-ray pulsars for their feasibility in implementation (thanks to the smaller sized detectors compared to those of radio band) and better accuracy [1].

In pulsar-based navigation, the observed signal is the pulse time-of-arrival (TOA) (or the pulse phase) at the detector. Processing this signal with respect to the recorded database gives us the specific information of the location of the spacecraft. The main problem in this kind of navigation is to estimate very precisely the pulse initial phase, and this challenge has been examined in the literature. In [2], the statistical model of the pulse TOA has been developed and the pulse phase estimation is investigated by deriving and analyzing the maximum likelihood estimator (MLE) and the Cramér-Rao bound (CRB). In [3], the nonlinear least-squares (NLS) estimator of the pulse phase is proposed and compared to the MLE in terms of computational complexity and mean square error (MSE) over the observation time. In both papers, one can observe, in terms of MSE performance, the so-called threshold phenomenon which appears as the observation time is below a critical limit. This can be explained by the distorted cost function used by estimators whose global extremum appears at a far point from the true value [4]. This threshold phenomenon is very similar to the one observed in the classical array processing context. Typically, in the classical array processing context, the threshold value can be predicted by using other bounds tighter than the CRB. These bounds are useful to determine the ultimate performance of an estimator. Consequently, we are not proposing in this paper a new estimation scheme but a tool to analyze the performance of the aforementionned estimator. These bounds on the MSE can be divided into two categories depending on the parameter assumptions [5]. When the unknown parameters are assumed to be deterministic, the so-called deterministic bounds that evaluate the ”locally best” behavior of the estimators have been proposed. The other category, the so-called Bayesian bounds, deals with the case where the parameters are assumed to be random, and particularly, they take into account the support of the parameters throughout an *a priori* probability density function (pdf) so that they can evaluate the ”globally best” performance. The advantage of Bayesian bounds over the deterministic bounds is their capability to give the fundamental limits of an estimator in terms of MSE over all the MSE range. However, the usefulness of the deterministic bound still remains when the parameter is deterministic, and for threshold prediction. For these reasons, in this paper, we study the performance limits for both assumptions on the parameter of interest.

Note that, in classical array processing, observations are typically modeled as Gaussian random variables, while in X-ray pulse phase estimation, observations are modeled with a Poisson distribution. To the best of our knowledge, there are very few results on lower bounds related to this kind of scenario. We can cite here the works in [6] and in [7] where the behavior of the CRB and a simple approximation of the Barankin bound are studied respectively in emission tomography. However, those articles do not consider the Bayesian case which is studied in this paper. Particularly, we analyze the Quinlan-Chaumette-Larzabal Bound (QCLB) and the Weiss-Weinstein Bound (WWB) which are known to be tight and general (they include many other bounds of the literature as special cases) in the deterministic context and in the Bayesian context (respectively).

II. X-ray Signal Model

In this Section, we give a brief background about the mathematical observation model provided and justified in [2]. This will lead to the likelihood function which will be the cornerstone of our analysis. Let us call k the number of photons detected at the detectors in a fixed time interval (a, b). The photon TOAs are modeled as a non-homogeneous Poisson process (NHPP) with a time-varying rate $\lambda(t) \geq 0$. This means that k follows a Poisson distribution $p(k; (a, b))$ with associated parameter $\int_a^b \lambda(t)dt$:

$$p(k; (a, b)) = \frac{\left[\int_a^b \lambda(t)dt\right]^k \exp\left[-\int_a^b \lambda(t)dt\right]}{k!} \tag{1}$$

The rate function $\lambda(t)$ denotes the aggregate rate of all photons arriving at the detector from the X-ray pulsar and background, expressed in photons per second (ph/s). In practice, the rate function $\lambda(t)$ has the following form:

$$\lambda(t) = \lambda_b + \lambda_s h(\phi_{obs}(t)) \qquad \text{(ph/s)}, \tag{2}$$

where λ_s and λ_b are called the *effective source rate* and *effective background arrival rate*, respectively; $h(\phi(t))$ is the normalized pulse profile function, and $\phi_{obs}(t)$ is the phase observed at the detector. Note that, thanks to the database obtained from years, the shape and period of the pulse profile are known very accurately [8]. The pulse profile function $h(\phi(t))$ is defined as a periodic function with its period equal to one cycle, i.e., $h(\phi(t))$ is defined on the interval $\phi \in [0, 1]$, and we have $h(m + \phi(t)) = h(\phi(t))$ for all integers m. Besides, the function $h(\phi(t))$ is normalized, i.e, $\int_0^1 h(\phi)d\phi = 1$, and $\min_\phi h(\phi(t)) = 0$.

The observed phase at the detector is given by $\phi_{obs}(t) = \phi_0 + \int_{t_0}^t f(\tau)d\tau$, where ϕ_0 is the initial phase, where t_0 is the start of the observation interval, and where $f(t)$ is the observed signal frequency which depends on the constant source frequency and the variant Doppler frequency shift. Note that, in this paper, we concentrate on the initial phase estimation problem, then, we assume that the observed frequency is a known constant. This is the constant-frequency model as in [3] where the observed phase at the detector can be rewritten as $\phi_{obs} = \phi_0 + (t - t_0)f$. The Poisson rate function can, now, be considered as a function of the only unknown parameter, the initial phase, as below $\lambda(t; \phi_0) = \lambda_b + \lambda_s h(\phi_0 + (t - t_0)f)$. Since λ_b and λ_s are known from the database, then, the remaining challenge here is to estimate the initial phase ϕ_0. This is what has been done in [2] and [3] where two estimators, the MLE and the NLS, are studied and their performance has been compared to the CRB in terms of MSE. In this work, other bounds, such as QCLB and WWB, are exploited to have a better benchmark. For this reason, we derive below the likelihood function.

The observation interval $(t_0, t_0 + T_{obs})$ is partitioned into N equal-length segments. We define x_n, $n = 0, 1, \ldots, N-1$, as the number of photons detected in the n-th segment, and $\Delta t \equiv T_{obs}/N$ as the segment size. If N is large enough, the Poisson rate $\lambda(t, \phi_0)$ can be assumed constant in the n-th segment, i.e. $\lambda_n(\phi_0) = \lambda(t_n; \phi_0)$, where $t_n = t_0 + n\Delta t$. The conditional probability mass function (pmf) for each Poisson random variable x_n,

$n = 0, 1, \ldots, N-1$, can be written as: $p(x_n = x; \phi_0) = \frac{[\lambda_n(\phi_0)\Delta t]^x}{x!} \exp(-\lambda_n(\phi_0)\Delta t)$, where x is a non-negative integer. Under the assumption of independent observations, the likelihood of the full set of observations $\mathbf{x} = [x_0, x_1, \ldots, x_{N-1}]$ is given by

$$p(\mathbf{x}; \phi_0) = \prod_{n=0}^{N-1} \frac{[\lambda_n(\phi_0)\Delta t]^{x_n}}{x_n!} \exp(-\lambda_n(\phi_0)\Delta t). \tag{3}$$

In Section III and Section IV, we will derive the lower bounds based on (3) in the deterministic context and Bayesian context, respectively.

III. Deterministic Bound

In this Section, we consider the so-called deterministic bounds for pulse phase estimation. Mathematically, the Barankin bound (BB) [9] is known to be tighter than the CRB, however, it is not computable. In classical array processing, to obtain a computable BB, several approximations of BB were proposed [5]. Consequently, in this paper, we derive the QCLB [5] which is one of the tightest bounds w.r.t. other bounds of the Barankin family. This approximation is obtained following the search of an optimum over a set of test points, denoted as $[\theta_0, \ldots, \theta_{N-1}]$.

The Nth-order QCLB of the unknown parameter ϕ_0 satisfies the following relation $E_{\mathbf{x};\phi_0}\left[(\hat{\phi} - \phi_0)^2\right] \geq B_{QCL}^N(\phi_0)$, where $E_{\mathbf{x};\phi_0}\left[(\hat{\phi} - \phi_0)^2\right] = \sum_{x_0=0}^{\infty} \cdots \sum_{x_{N-1}=0}^{\infty} (\hat{\phi} - \phi_0)^2 p(\mathbf{x}; \phi_0)$ is the variance of any unbiased estimators $\hat{\phi}$ of ϕ_0. Hereafter, we use, for simplicity, $\sum_{\mathbf{x}=0}^{\infty}$ instead of $\sum_{x_0=0}^{\infty} \cdots \sum_{x_{N-1}=0}^{\infty}$.

The bound B_{QCL}^N is calculated as follows [5]: $B_{QCL}^N = \mathbf{v}^T \mathbf{M}_{QCL}^{-1} \mathbf{v}$ where

$$\begin{cases} \mathbf{v} = [\mathbf{\Phi}^T, 1, \ldots, 1]^T \in \mathbb{R}^{2N \times 1} \text{ where } \mathbf{\Phi} = [\xi_0 \ldots \xi_{N-1}]^T, \text{ where } \xi_n = \theta_n - \phi_0, \ n = 0 \ldots N-1, \\ \mathbf{M}_{QCL} = \begin{bmatrix} \mathbf{M}_{MS} & \mathbf{H}^T \\ \mathbf{H} & \mathbf{M}_{EFI} \end{bmatrix} \in \mathbb{R}^{2N \times 2N} \end{cases}, \tag{4}$$

where

$$\mathbf{M}_{MS} = E_{\mathbf{x};\phi_0}\left[\begin{pmatrix} \frac{p(\mathbf{x};\theta_0)}{p(\mathbf{x};\phi_0)} \\ \vdots \\ \frac{p(\mathbf{x};\theta_{N-1})}{p(\mathbf{x};\phi_0)} \end{pmatrix} \begin{pmatrix} \frac{p(\mathbf{x};\theta_0)}{p(\mathbf{x};\phi_0)} \\ \vdots \\ \frac{p(\mathbf{x};\theta_{N-1})}{p(\mathbf{x};\phi_0)} \end{pmatrix}^T \right], \tag{5}$$

$$\mathbf{M}_{EFI} = E_{\mathbf{x};\phi_0}\left[\begin{pmatrix} \frac{\partial \ln p(\mathbf{x};\theta_0)}{\partial \theta_0} \frac{p(\mathbf{x};\theta_0)}{p(\mathbf{x};\phi_0)} \\ \vdots \\ \frac{\partial \ln p(\mathbf{x};\theta_{N-1})}{\partial \theta_{N-1}} \frac{p(\mathbf{x};\theta_{N-1})}{p(\mathbf{x};\phi_0)} \end{pmatrix} \begin{pmatrix} \frac{\partial \ln p(\mathbf{x};\theta_0)}{\partial \theta_0} \frac{p(\mathbf{x};\theta_0)}{p(\mathbf{x};\phi_0)} \\ \vdots \\ \frac{\partial \ln p(\mathbf{x};\theta_{N-1})}{\partial \theta_{N-1}} \frac{p(\mathbf{x};\theta_{N-1})}{p(\mathbf{x};\phi_0)} \end{pmatrix}^T \right], \tag{6}$$

$$\mathbf{H} = E_{\mathbf{x};\phi_0}\left[\begin{pmatrix} \frac{\partial \ln p(\mathbf{x};\theta_0)}{\partial \theta_0} \frac{p(\mathbf{x};\theta_0)}{p(\mathbf{x};\phi_0)} \\ \vdots \\ \frac{\partial \ln p(\mathbf{x};\theta_{N-1})}{\partial \theta_{N-1}} \frac{p(\mathbf{x};\theta_{N-1})}{p(\mathbf{x};\phi_0)} \end{pmatrix} \begin{pmatrix} \frac{p(\mathbf{x};\theta_0)}{p(\mathbf{x};\phi_0)} \\ \vdots \\ \frac{p(\mathbf{x};\theta_{N-1})}{p(\mathbf{x};\phi_0)} \end{pmatrix}^T \right]. \tag{7}$$

The set $\theta_n, n = 0, \ldots, N-1$ is the so-called set of the test point which can be chosen to optimize the bound. After the calculation which is detailed in Appendix one obtains the closed-form expressions of elements (k, l) of matrix $\mathbf{M}_{MS}$ (see Appendix VII-A), $\mathbf{M}_{EFI}$ (see Appendix VII-B), and $\mathbf{H}$ (see Appendix VII-C) as follows:

$$\mathbf{M}_{MS}(k,l) = \exp\left\{ T_{obs} \int_0^1 \lambda(\phi) - \lambda(\xi_k + \phi) - \lambda(\xi_l + \phi) + \frac{\lambda(\xi_k + \phi)\lambda(\xi_l + \phi)}{\lambda(\phi)} d\phi \right\}, \tag{8}$$

$$\begin{aligned}\mathbf{M}_{EFI}(k,l) = \mathbf{M}_{MS}(k,l) \Bigg[& T_{obs}^2 \int_0^1 \frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k} \frac{\lambda(\phi+\xi_l)}{\lambda(\phi)} d\phi \int_0^1 \frac{\partial\lambda(\phi+\xi_l)}{\partial\xi_l} \frac{\lambda(\phi+\xi_k)}{\lambda(\phi)} d\phi \\ & + T_{obs}^2 \int_0^1 \frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k} d\phi \int_0^1 \frac{\partial\lambda(\phi+\xi_l)}{\partial\xi_l} d\phi - T_{obs}^2 \int_0^1 \frac{\partial\lambda(\phi+\xi_l)}{\partial\xi_l} d\phi \int_0^1 \frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k} \frac{\lambda(\phi+\xi_l)}{\lambda(\phi)} d\phi, \\ & - T_{obs}^2 \int_0^1 \frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k} d\phi \int_0^1 \frac{\partial\lambda(\phi+\xi_l)}{\partial\xi_l} \frac{\lambda(\phi+\xi_k)}{\lambda(\phi)} d\phi + T_{obs} \int_0^1 \frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k} \frac{\partial\lambda(\phi+\xi_l)}{\partial\xi_l} \frac{1}{\lambda(\phi)} d\phi \Bigg]\end{aligned} \tag{9}$$

and

$$\mathbf{H}(k,l) = \mathbf{M}_{MS}(k,l) \left[T_{obs} \int_0^1 \frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k} \frac{\lambda(\phi+\xi_l)}{\lambda(\phi)} d\phi - T_{obs} \int_0^1 \frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k} d\phi \right]. \tag{10}$$

In the expression of the QCLB, we can see the existence of an integral which can be computed easily and rapidly in a numerical way. Note that, it was also the case in the CRB calculus proposed in [2] and [3]. In Section V, we plot the QCLB versus several observation times and compare it to the CRB and the MSE of the MLE.

IV. Bayesian bound

As an alternative to the deterministic framework, we propose to handle the problem in the Bayesian framework which will provide a tight minimal bound over all the range of observation time and a good prediction of the observation time threshold. In particular, we assume that the parameter of interest ϕ_0 is random with an *a priori* uniform pdf over the support $[0,1]$. Note that not all the Bayesian bounds proposed in the literature are able to take into account the case when the parameters of interest are supposed to be uniformly distributed. Therefore, among various types of Bayesian bounds [10], we concentrate, in this Section, on the WWB (see [11] [12] [13]), which can deal with the uniformly distributed prior assumption and is one of the tightest bound of the Weiss and Weinstein family [14] [15].

The Weiss-Weinstein bound, denoted WWB, for the unknown parameter ϕ_0 satisfies the following relation $E_{\mathbf{x};\phi_0}\left[(\hat{\phi} - \phi_0)^2\right] \geq WWB$, where $E_{\mathbf{x};\phi_0}\left[(\hat{\phi} - \phi_0)^2\right] = \int_{\boldsymbol{\Theta}} \sum_{\mathbf{x}=0}^{\infty} (\hat{\phi} - \phi_0)^2 p(\mathbf{x}, \phi_0) d\phi_0$ is the variance of any estimators of ϕ_0, where $p(\mathbf{x}, \phi_0)$ being the joint pdf and $\boldsymbol{\Theta}$ is the parameter space. Note that, contrary to the deterministic bounds, no assumption is made on the estimator $\hat{\phi}$, e.g., $\hat{\phi}$ can be biased. The WWB is calculated by [11]

$$WWB = \sup_{u,s} \frac{u^2 \exp(2\eta(s,u))}{\exp(\eta(2s,u)) + \exp(\eta(2-2s,-u)) - 2\exp(\eta(s,2u))}, \tag{11}$$

where $s \in [0,1]$, where u is the test point chosen such that $\phi_0 + u \in [0,1]$, and $\eta(\alpha,\beta)$ is defined by

$$\begin{aligned}\eta(\alpha,\beta) &= \ln \int_{\boldsymbol{\Theta}} \sum_{\mathbf{x}=0}^{\infty} p(\mathbf{x},\phi_0+\beta)^{\alpha} p(\mathbf{x},\phi_0)^{1-\alpha} d\phi_0 \\ &= \ln \int_{\boldsymbol{\Theta}} \sum_{\mathbf{x}=0}^{\infty} p(\mathbf{x};\phi_0+\beta)^{\alpha} p(\phi_0+\beta)^{\alpha} p(\mathbf{x};\phi_0)^{1-\alpha} p(\phi_0)^{1-\alpha} d\phi_0 \\ &= \ln \int_{\boldsymbol{\Theta}} \eta'(\alpha,\beta) p(\phi_0+\beta)^{\alpha} p(\phi_0)^{1-\alpha} d\phi_0, \end{aligned} \tag{12}$$

where we define $\eta'(\alpha,\beta) = \sum_{\mathbf{x}=0}^{\infty} p(\mathbf{x};\phi_0+\beta)^{\alpha} p(\mathbf{x};\phi_0)^{1-\alpha}$. The closed-form expression of $\eta'(\alpha,\beta)$ is given by (see Appendix VII-D for details)

$$\eta'(\alpha,\beta) = \exp\left\{ T_{obs} \int_0^1 \left(-\alpha\lambda(\phi+\beta) - (1-\alpha)\lambda(\phi) + \lambda(\phi+\beta)^{\alpha}\lambda(\phi)^{1-\alpha}\right) d\phi \right\}. \tag{13}$$

Note that, the dependance of $\eta'(\alpha,\beta)$ on ϕ_0 is now removed, then, the integral in (12) can be calculated easily. Finally, the Weiss-Weinstein bound for the unknown parameter ϕ_0 is given by

$$WWB = \sup_{u,s} \frac{u^2(1-u)^2\eta'^2(s,u)}{(1-u)\eta'(2s,u) + (1-u)\eta'(2-2s,-u) - 2(1-2u)\eta'(s,2u)}. \tag{14}$$

As it appears in the QCLB [see (8)-(10)], integrals also exist in the expression of the WWB but they can be numerically integrated.

V. Numerical Results

To evaluate the proposed bounds, we compare them to the performance of the MLE of the pulse phase which is given by $\hat{\phi} = \arg\max_{\phi_0 \in \boldsymbol{\Theta}} \sum_{n=0}^{N-1} [x_n \ln[\lambda_n(\phi_0)\Delta t] - \lambda_n(\phi_0)\Delta t]$. The performance of the MLE is simulated using 1000 Monte Carlo runs. The observed frequency is $f = 29.85$ Hz and the pulsar period is 33.5 ms. The photon rates are $\lambda_b = 5$ (ph/s) and $\lambda_s = 15$ (ph/s). As the phase is defined on the $[0,1]$ interval, the phase error value is calculated modulo one cycle, i.e., $min[mod(\phi_0 - \hat{\phi}, 1), mod(\hat{\phi} - \phi_0, 1)]$. For example, the error between 0.9 cycle and 0.1 cycle should be 0.2 cycle, and not 0.8 cycle.

In Figure 1, we compare the QCLB to the McAulay-Seidman bound (MSB) [1], the CRB and the MSE of the MLE versus the observation time. The initial phase, $\phi_0 = 0.9$ cycle, is chosen arbitrarily. It can be seen that the QCLB provides a better prediction of the MLE MSE threshold location than the MSB. We also plot in Figure 2 the QCLB with various numbers of test point. One can see that using more test points achieves a slightly better bound but it increases the numerical complexity.

Figure 3 shows the WWB and the empirical global MSE of the MLE of ϕ_0 versus the observation time. It can be seen that the WWB predicts well not only the threshold position, but also the MSE of the MLE in all range of observation time (asymptotic and threshold regions).

[1] The MSB is also an approximation of Barankin bound but less tight than the QCLB. It can be calculated by [5] ($\boldsymbol{\Phi}^T \mathbf{M}_{MS}^{-1} \boldsymbol{\Phi}$) where $\boldsymbol{\Phi}$ and $\mathbf{M}_{MS}$ are given by (4) and (8).

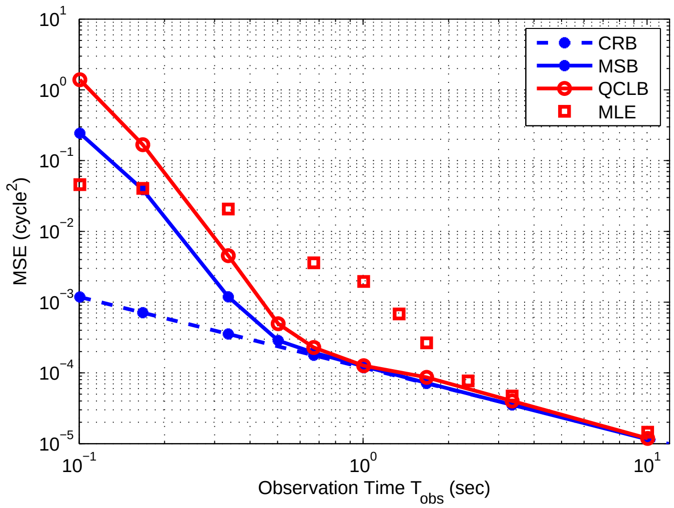

Fig. 1. QCLB, MSB and CRB and empirical MSE of the MLE of ϕ_0 versus observation time

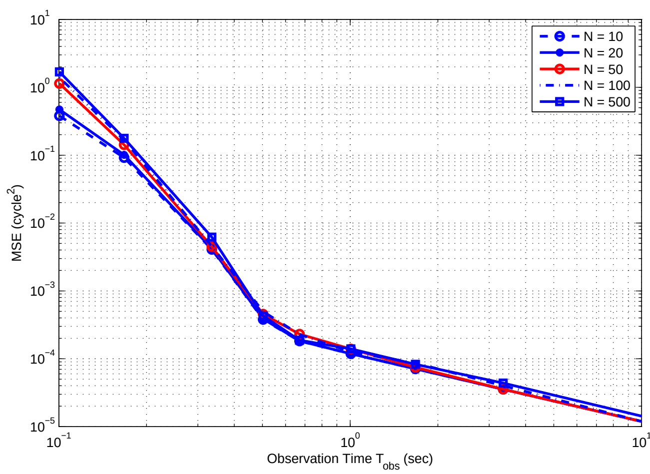

Fig. 2. QCLB of ϕ_0 versus observation time

VI. Conclusions

Accurate estimation of the initial phase of the pulse arriving at the detector appears to be the key challenge in a system using X-ray pulsars to perform the autonomous deep space navigating. Therefore, we have derived the closed-form expressions of the lower bounds on the MSE and analyzed their behavior for the problem of X-ray pulse phase estimation. Indeed, both deterministic (QCLB) and Bayesian bounds (WWB) have been considered. We have shown that both types of lower bound provide good prediction of the threshold location depending on the estimation framework.

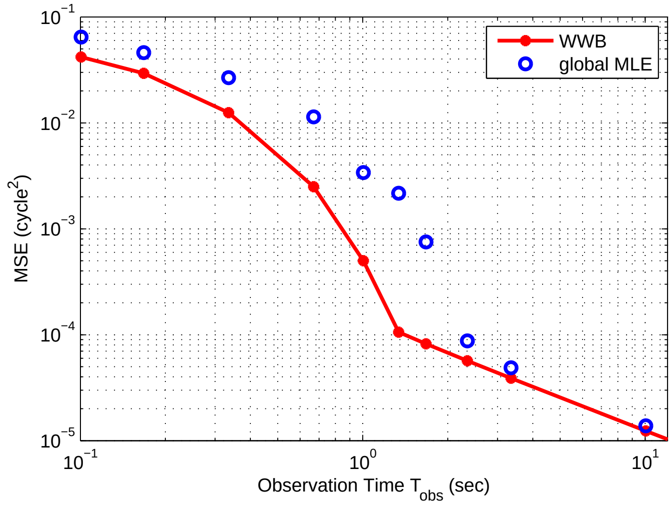

Fig. 3. WWB and empirical global MSE of MLE of ϕ_0 versus observation time

VII. Appendix

A. Derivation of $\mathbf{M}_{MS}(k,l)$

From (5), the element (k,l) of the matrix $\mathbf{M}_{MS}$ is given by

$$\begin{aligned}\mathbf{M}_{MS}(k,l) &= E_{\mathbf{x};\phi_0}\left[\frac{p(\mathbf{x};\theta_k)}{p(\mathbf{x};\phi_0)}\frac{p(\mathbf{x};\theta_l)}{p(\mathbf{x};\phi_0)}\right] \\ &= \sum_{\mathbf{x}=0}^{\infty}\frac{p(\mathbf{x};\theta_k)p(\mathbf{x};\theta_l)}{p(\mathbf{x};\phi_0)} \\ &= \prod_{n=0}^{N-1}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)\sum_{\mathbf{x}=0}^{\infty}\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!} \\ &= \exp\left\{\Delta t\sum_{n=0}^{N-1}\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)+\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)}{\lambda_n(\phi_0)}\right\}, \end{aligned} \tag{15}$$

since

$$\sum_{\mathbf{x}=0}^{\infty}\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}=\prod_{n=0}^{N-1}\exp\left(\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)}{\lambda_n(\phi_0)}\Delta t\right).$$

By taking the limit $\Delta t\to 0$, or equivalently $N\to\infty$, and by noting that $\lambda_n(.)=\lambda(t_n;.)$, we can convert the above summation to an integral:

$$\begin{aligned}\mathbf{M}_{MS}(k,l) &= \exp\left\{\int_{t_0}^{t_0+T_{obs}}\left[\lambda(t,\phi_0)-\lambda(t,\theta_k)-\lambda(t,\theta_l)+\frac{\lambda(t,\theta_k)\lambda(t,\theta_l)}{\lambda(t,\phi_0)}\right]dt\right\} \\ &= \exp\left\{\int_{\phi_0}^{\phi_0+fT_{obs}}\frac{1}{f}\left[\lambda(\phi)-\lambda(\phi+\xi_k)-\lambda(\phi+\xi_l)+\frac{\lambda(\phi+\xi_k)\lambda(\phi+\xi_l)}{\lambda(\phi)}\right]d\phi\right\}, \end{aligned} \tag{16}$$

where $\lambda(\phi+\alpha)=\lambda_b+\lambda_s h(\phi+\alpha)$. Note that we have used in the above derivation the change of variable formula $\phi=\phi_0+(t-t_0)f$. As the pulse profile function $h(\phi(t))$ is periodic with its period equal to one cycle, hence, when the observation time is an integer number of the pulsar period, i.e., $fT_{obs}\approx N_p$ (cycle), $\mathbf{M}_{MS}(k,l)$ can be rewritten as

$$\begin{aligned}
\mathbf{M}_{MS}(k,l) &= \exp\left\{\int\limits_0^{fT_{obs}} \frac{1}{f}\left[\lambda(\phi)-\lambda(\phi+\xi_k)-\lambda(\phi+\xi_l)+\frac{\lambda(\phi+\xi_k)\lambda(\phi+\xi_l)}{\lambda(\phi)}\right]d\phi\right\}\\
&= \exp\left\{\sum_{n=1}^{N_p}\int\limits_{n-1}^{n} \frac{1}{f}\left[\lambda(\phi)-\lambda(\xi_k+\phi)-\lambda(\xi_l+\phi)+\frac{\lambda(\xi_k+\phi)\lambda(\xi_l+\phi)}{\lambda(\phi)}\right]d\phi\right\}\\
&= \exp\left\{\frac{N_p}{f}\int\limits_0^{1} \left[\lambda(\phi)-\lambda(\xi_k+\phi)-\lambda(\xi_l+\phi)+\frac{\lambda(\xi_k+\phi)\lambda(\xi_l+\phi)}{\lambda(\phi)}\right]d\phi\right\}\\
&= \exp\left\{T_{obs}\int\limits_0^{1} \left[\lambda(\phi)-\lambda(\xi_k+\phi)-\lambda(\xi_l+\phi)+\frac{\lambda(\xi_k+\phi)\lambda(\xi_l+\phi)}{\lambda(\phi)}\right]d\phi\right\}. \qquad (17)
\end{aligned}$$

B. *Derivation of* $\mathbf{M}_{EFI}(k,l)$

From (6), the element (k,l) of the matrix $\mathbf{M}_{EFI}$ is given by

$$\begin{aligned}
\mathbf{M}_{EFI}(k,l) &= E_{\mathbf{x};\phi_0}\left[\frac{\partial \ln p(\mathbf{x};\theta_k)}{\partial\theta_k}\frac{p(\mathbf{x};\theta_k)}{p(\mathbf{x};\phi_0)}\frac{\partial \ln p(\mathbf{x};\theta_l)}{\partial\theta_l}\frac{p(\mathbf{x};\theta_l)}{p(\mathbf{x};\phi_0)}\right]\\
&= \sum_{\mathbf{x}=0}^{\infty}\left[\sum_{p=0}^{N-1}x_p\frac{\partial\ln[\lambda_p(\theta_k)\Delta t]}{\partial\theta_k}-\frac{\partial\lambda_p(\theta_k)\Delta t}{\partial\theta_k}\right]\left[\sum_{q=0}^{N-1}x_q\frac{\partial\ln[\lambda_q(\theta_k)\Delta t]}{\partial\theta_k}-\frac{\partial\lambda_q(\theta_k)\Delta t}{\partial\theta_k}\right]\\
&\qquad \prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)\\
&= C_1+C_2+C_3+C_4, \qquad (18)
\end{aligned}$$

whose components C_i, $i=1,2,3,4$, are calculated as follows

$$\begin{aligned}
C_1 &= \sum_{\mathbf{x}=0}^{\infty}\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}x_p x_q\frac{\partial\ln[\lambda_p(\theta_k)\Delta t]}{\partial\theta_k}\frac{\partial\ln[\lambda_q(\theta_l)\Delta t]}{\partial\theta_l}\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)\\
&= \sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\frac{\partial\ln[\lambda_p(\theta_k)\Delta t]}{\partial\theta_k}\frac{\partial\ln[\lambda_q(\theta_k)\Delta t]}{\partial\theta_k}\sum_{\mathbf{x}=0}^{\infty}x_p x_q\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right). \qquad (19)
\end{aligned}$$

Let us define

$$\begin{aligned}
A &= \sum_{\mathbf{x}=0}^{\infty}x_p x_q\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)\\
&= A(p=q)+A(p\neq q), \qquad (20)
\end{aligned}$$

where

$$A(p=q)=\sum_{\mathbf{x}=0}^{\infty}x_p^2\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)$$
$$=\sum_{\mathbf{x}=0}^{\infty}x_p(\mathbf{x}_p-1)\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)$$
$$+\sum_{\mathbf{x}=0}^{\infty}x_p\prod_{m=0}^{N-1}\left[\frac{\lambda_m(\theta_k)\lambda_m(\theta_l)\Delta t}{\lambda_m(\phi_0)}\right]^{x_m}\frac{1}{x_m!}\exp\left(\Delta t[\lambda_m(\phi_0)-\lambda_m(\theta_k)-\lambda_m(\theta_l)]\right)$$
$$=\sum_{\mathbf{x}'=0}^{\infty}\left[\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\right]^2\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x'_n}\frac{1}{x'_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)$$
$$+\sum_{\mathbf{x}''=0}^{\infty}\left[\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\right]\prod_{m=0}^{N-1}\left[\frac{\lambda_m(\theta_k)\lambda_m(\theta_l)\Delta t}{\lambda_m(\phi_0)}\right]^{x''_m}\frac{1}{x''_m!}\exp\left(\Delta t[\lambda_m(\phi_0)-\lambda_m(\theta_k)-\lambda_m(\theta_l)]\right)$$
$$=\left[\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\right]^2\mathbf{M}_{MS}(k,l)+\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\mathbf{M}_{MS}(k,l),\tag{21}$$

where $\mathbf{x}'=[x_0,\ldots,x_{p-1},x_p-2,x_{p+1},\ldots,x_{N-1}]$, and $\mathbf{x}''=[x_0,\ldots,x_{p-1},x_p-1,x_{p+1},\ldots,x_{N-1}]$, and where

$$A(p\neq q)=\sum_{\mathbf{x}=0}^{\infty}x_px_q\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n}\frac{1}{x_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)$$
$$=\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\frac{\lambda_q(\theta_k)\lambda_q(\theta_l)\Delta t}{\lambda_q(\phi_0)}\sum_{\mathbf{x}=0}^{\infty}\prod_{n=0}^{N-1}\left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x'''_n}\frac{1}{x'''_n!}\exp\left(\Delta t[\lambda_n(\phi_0)-\lambda_n(\theta_k)-\lambda_n(\theta_l)]\right)$$
$$=\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\frac{\lambda_q(\theta_k)\lambda_q(\theta_l)\Delta t}{\lambda_q(\phi_0)}\mathbf{M}_{MS}(k,l),\tag{22}$$

where $\mathbf{x}'''=[x_0,\ldots,x_p-1,\ldots,x_q-1,\ldots,x_{N-1}]$. Note that in the above derivation, we assumed , without loss of generality, that $p<q$. Consequently, we get

$$C_1=\mathbf{M}_{MS}(k,l)\Bigg\{\sum_{p=0}^{N-1}\sum_{q=0}^{N-1}\frac{\partial\ln[\lambda_p(\theta_k)\Delta t]}{\partial\theta_k}\frac{\partial\ln[\lambda_q(\theta_l)\Delta t]}{\partial\theta_l}\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\frac{\lambda_q(\theta_k)\lambda_q(\theta_l)\Delta t}{\lambda_q(\phi_0)}$$
$$+\sum_{p=0}^{N-1}\frac{\partial\ln[\lambda_p(\theta_k)\Delta t]}{\partial\theta_k}\frac{\partial\ln[\lambda_p(\theta_l)\Delta t]}{\partial\theta_l}\frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)}\Bigg\}$$
$$=\mathbf{M}_{MS}(k,l)\Bigg\{\sum_{p=0}^{N-1}\frac{\partial\lambda_p(\theta_k)}{\partial\theta_k}\frac{\lambda_p(\theta_l)}{\lambda_p(\phi_0)}\Delta t\sum_{q=0}^{N-1}\frac{\partial\lambda_q(\theta_l)}{\partial\theta_l}\frac{\lambda_q(\theta_k)}{\lambda_q(\phi_0)}\Delta t+\sum_{p=0}^{N-1}\frac{\partial\lambda_p(\theta_k)}{\partial\theta_k}\frac{\partial\lambda_p(\theta_l)}{\partial\theta_l}\frac{1}{\lambda_p(\phi_0)}\Delta t\Bigg\}.\tag{23}$$

Using again the same calculating technique as in the derivation of the MS matrix, we get

$$C_1=\mathbf{M}_{MS}(k,l)\left[T_{obs}^2\int_0^1\frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k}\frac{\lambda(\phi+\xi_l)}{\lambda(\phi)}d\phi\int_0^1\frac{\partial\lambda(\phi+\xi_l)}{\partial\xi_l}\frac{\lambda(\phi+\xi_k)}{\lambda(\phi)}d\phi+T_{obs}\int_0^1\frac{\partial\lambda(\phi+\xi_k)}{\partial\xi_k}\frac{\partial\lambda(\phi+\xi_l)}{\partial\xi_l}\frac{1}{\lambda(\phi)}d\phi\right].\tag{24}$$

Similarly, we derive the other components

$$C_2 = \sum_{\mathbf{x}=0}^{\infty} \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{\partial \lambda_p(\theta_k)\Delta t}{\partial \theta_k} \frac{\partial \lambda_q(\theta_l)\Delta t}{\partial \theta_l} \prod_{n=0}^{N-1} \left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n} \frac{1}{x_n!} \exp\left(\Delta t[\lambda_n(\phi_0) - \lambda_n(\theta_k) - \lambda_n(\theta_l)]\right)$$
$$= \mathbf{M}_{MS}(k,l) \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{\partial \lambda_p(\theta_k)\Delta t}{\partial \theta_k} \frac{\partial \lambda_q(\theta_l)\Delta t}{\partial \theta_l}$$
$$= \mathbf{M}_{MS}(k,l) T_{obs}^2 \int_0^1 \frac{\partial \lambda(\phi+\xi_k)}{\partial \xi_k} d\phi \int_0^1 \frac{\partial \lambda(\phi+\xi_l)}{\partial \xi_l} d\phi, \tag{25}$$

$$C_3 = \sum_{\mathbf{x}=0}^{\infty} x_p \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{\partial \ln[\lambda_p(\theta_k)\Delta t]}{\partial \theta_k} \frac{\partial \lambda_q(\theta_l)\Delta t}{\partial \theta_l} \prod_{n=0}^{N-1} \left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n} \frac{1}{x_n!} \exp\left(\Delta t[\lambda_n(\phi_0) - \lambda_n(\theta_k) - \lambda_n(\theta_l)]\right)$$
$$= \sum_{q=0}^{N-1} \frac{\partial \lambda_q(\theta_l)\Delta t}{\partial \theta_l} \sum_{p=0}^{N-1} \frac{\partial \ln[\lambda_p(\theta_k)\Delta t]}{\partial \theta_k} \frac{\lambda_p(\theta_k)\lambda_p(\theta_l)\Delta t}{\lambda_p(\phi_0)} \mathbf{M}_{MS}(k,l)$$
$$= \mathbf{M}_{MS}(k,l) T_{obs}^2 \int_0^1 \frac{\partial \lambda(\phi+\xi_l)}{\partial \xi_l} d\phi \int_0^1 \frac{\partial \lambda(\phi+\xi_k)}{\partial \xi_k} \frac{\lambda(\phi+\xi_l)}{\lambda(\phi)} d\phi, \tag{26}$$

$$C_4 = \sum_{\mathbf{x}=0}^{\infty} x_q \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} \frac{\partial \lambda_p(\theta_k)\Delta t}{\partial \theta_k} \frac{\partial \ln[\lambda_q(\theta_l)\Delta t]}{\partial \theta_l} \prod_{n=0}^{N-1} \left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n} \frac{1}{x_n!} \exp\left(\Delta t[\lambda_n(\phi_0) - \lambda_n(\theta_k) - \lambda_n(\theta_l)]\right)$$
$$= \sum_{p=0}^{N-1} \frac{\partial \lambda_p(\theta_k)\Delta t}{\partial \theta_k} \sum_{q=0}^{N-1} \frac{\partial \ln[\lambda_q(\theta_l)\Delta t]}{\partial \theta_l} \frac{\lambda_q(\theta_k)\lambda_q(\theta_l)\Delta t}{\lambda_q(\phi_0)} \mathbf{M}_{MS}(k,l)$$
$$= \mathbf{M}_{MS}(k,l) T_{obs}^2 \int_0^1 \frac{\partial \lambda(\phi+\xi_k)}{\partial \xi_k} d\phi \int_0^1 \frac{\partial \lambda(\phi+\xi_l)}{\partial \xi_l} \frac{\lambda(\phi+\xi_k)}{\lambda(\phi)} d\phi. \tag{27}$$

Finally, plugging (24), (25), (26), and (27) into (18) we get (9).

C. *Derivation of* $\mathbf{H}(k,l)$

From (7), the element (k,l) of the matrix $\mathbf{H}$ is given by

$$\mathbf{H}(k,l) = E_{\mathbf{x};\phi_0}\left[\frac{\partial \ln p(\mathbf{x};\theta_k)}{\partial \theta_k} \frac{p(\mathbf{x};\theta_k)}{p(\mathbf{x};\phi_0)} \frac{p(\mathbf{x};\theta_l)}{p(\mathbf{x};\phi_0)}\right]$$
$$= \sum_{\mathbf{x}=0}^{\infty} \left[\sum_{p=0}^{N-1} x_p \frac{\partial \ln[\lambda_p(\theta_k)\Delta t]}{\partial \theta_k} - \frac{\partial \lambda_p(\theta_k)\Delta t}{\partial \theta_k}\right] \prod_{n=0}^{N-1} \left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n} \frac{1}{x_n!} \exp\left(\Delta t[\lambda_n(\phi_0) - \lambda_n(\theta_k) - \lambda_n(\theta_l)]\right)$$
$$= \sum_{p=0}^{N-1} \frac{\partial \ln[\lambda_p(\theta_k)\Delta t]}{\partial \theta_k} \sum_{\mathbf{x}=0}^{\infty} x_p \prod_{n=0}^{N-1} \left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n} \frac{1}{x_n!} \exp\left(\Delta t[\lambda_n(\phi_0) - \lambda_n(\theta_k) - \lambda_n(\theta_l)]\right)$$
$$+ \sum_{p=0}^{N-1} \frac{\partial \lambda_p(\theta_k)\Delta t}{\partial \theta_k} \prod_{n=0}^{N-1} \left[\frac{\lambda_n(\theta_k)\lambda_n(\theta_l)\Delta t}{\lambda_n(\phi_0)}\right]^{x_n} \frac{1}{x_n!} \exp\left(\Delta t[\lambda_n(\phi_0) - \lambda_n(\theta_k) - \lambda_n(\theta_l)]\right)$$
$$= \mathbf{M}_{MS}(k,l)\left[T_{obs} \int_0^1 \frac{\partial \lambda(\phi+\xi_k)}{\partial \xi_k} \frac{\lambda(\phi+\xi_l)}{\lambda(\phi)} d\phi - T_{obs} \int_0^1 \frac{\partial \lambda(\phi+\xi_k)}{\partial \xi_k} d\phi\right]. \tag{28}$$

D. Derivation of $\eta'(\alpha,\beta)$

We have

$$\eta'(\alpha,\beta) = \sum_{\mathbf{x}=0}^{\infty} \left\{ \prod_{n=0}^{N-1} \frac{[\lambda_n(\phi_0+\beta)\Delta t]^{x_n}}{x_n!}\; exp(-\lambda_n(\phi_0+\beta)\Delta t) \right\}^{\alpha} \left\{ \prod_{m=0}^{N-1} \frac{[\lambda_m(\phi_0)\Delta t]^{x_m}}{x_m!}\; exp(-\lambda_m(\phi_0)\Delta t) \right\}^{1-\alpha}$$
$$= \sum_{\mathbf{x}=0}^{\infty} \prod_{n=0}^{N-1} \frac{[\lambda_n(\phi_0+\beta)^{\alpha}\lambda_n(\phi_0)^{1-\alpha}\Delta t]^{x_n}}{x_n!} \exp\left[-\alpha\lambda_n(\phi_0+\beta)\Delta t - (1-\alpha)\lambda_n(\phi_0)\Delta t\right]$$
$$= \exp\left\{ T_{obs} \int_0^1 \left(-\alpha\lambda(\phi+\beta) - (1-\alpha)\lambda(\phi) + \lambda(\phi+\beta)^{\alpha}\lambda(\phi)^{1-\alpha}\right) d\phi \right\}. \tag{29}$$

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