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## Reliable Shared Memory Abstractions on Top of Asynchronous $t$ -Resilient Byzantine Message-passing Systems

Damien Imbs<sup>\*</sup> Sergio Rajsbaum<sup>\*</sup> Michel Raynal<sup>\*\* \*\*\*</sup> Julien Stainer<sup>\*\*\*</sup>

**Abstract:** This paper is on the construction and the use of a shared memory abstraction on top of an asynchronous message-passing system in which up to  $t$  processes may commit Byzantine failures. This abstraction consists of arrays of  $n$  single-writer/multi-reader atomic registers, where  $n$  is the number of processes. A distributed algorithm building such a shared memory abstraction is first presented. This algorithm assumes  $t < n/3$ , which is shown to be a necessary and sufficient condition for such a construction. Hence, the algorithm is resilient-optimal. Then the paper presents distributed algorithms built on top of this shared memory abstraction, which cope with up to  $t$  Byzantine processes. The simplicity of these algorithms constitutes a strong motivation for such a shared memory abstraction in the presence of Byzantine processes.

For a lot of problems, algorithms are more difficult to design and prove correct in a message-passing system than in a shared memory system. Using a protocol stacking methodology, the aim of the proposed abstraction is to allow an easier design (and proof) of distributed algorithm, when one has the underlying system is an asynchronous message-passing system prone to Byzantine failures.

**Key-words:** Approximate agreement, Asynchronous message-passing system, Atomic read/write register, Broadcast abstraction, Byzantine process, Distributed computing, Message-passing system, Quorum, Reliable broadcast, Reliable shared memory, Single-writer/multi-reader register,  $t$ -Resilience.

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*Abstraction d'une mémoire partagée fiable dans les systèmes asynchrones à passage de messages où jusqu'à  $t$  fautes Byzantines peuvent survenir*

**Résumé :** Cet article propose une abstraction de mémoire partagée adaptée aux systèmes asynchrones à passage de messages où jusqu'à  $t$  fautes Byzantines peuvent survenir. Il présente également une implémentation de cette abstraction ainsi que sa preuve.

**Mots clés :** Accord approché, systèmes asynchrones à passage de messages, registres atomiques read/write, diffusion fiable, processus Byzantins, calcul distribué, quorums, mémoire partagée fiable, tolérance à  $t$  fautes.

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# 1 Introduction

**Distributed computing** Distributed computing occurs when one has to solve a problem in terms of physically distinct entities (usually called nodes, processors, processes, agents, sensors, etc.) such that each entity has only a partial knowledge of the many parameters involved in the problem. In the following, we use the term *process* to denote a computing entity. From an operational point of view this means that the processes of a distributed system need to exchange information, and agree in some way or another, in order to cooperate to a common goal. If processes do not cooperate, the system is no longer a distributed system. Hence, a distributed system has to provide the processes with communication and agreement abstractions.

Understanding and designing distributed applications is not an easy task [3, 14, 22, 23, 24]. This is due to the fact that, due to its very nature, no process can capture instantaneously the global state of the application it is part of. More precisely, as processes are geographically localized at distinct places, distributed applications have to cope with the uncertainty created by asynchrony and failures. As a simple example, it is impossible to distinguish a crashed process from a very slow process in an asynchronous system prone to process crashes.

As in sequential computing, a simple approach to facilitate the design of distributed applications consists in designing appropriate abstractions. With such abstractions, the application designer can think about solutions to solve problems at a higher conceptual level than the one offered by the basic send/receive communication layer.

**Byzantine behavior** This failure type has first been introduced in the context of synchronous distributed systems (e.g., [12, 21, 23]), and then investigated in the context of asynchronous ones (e.g., [3, 14, 22]). A process has a *Byzantine* behavior when it arbitrarily deviates from its intended behavior; it then commits a Byzantine failure. Otherwise it is non-faulty (or non-Byzantine). This bad behavior can be intentional (malicious) or simply the result of a transient fault that altered the local state of a process, thereby modifying its behavior in an unpredictable way. Let us notice that process crashes (unexpected halting) and communication omissions, define a strict subset of Byzantine failures.

The major part of the papers on Byzantine failures considers (synchronous or asynchronous) message-passing systems, and mainly addresses agreement problems, such as consensus and total order broadcast, or the construction of a Byzantine-tolerant disk storage (e.g., [7, 11, 15, 16, 17]). Many of these papers consider registers built on top of duplicated disks (servers), which are accessed by clients, and where disks and clients may exhibit different type of failures (e.g., [2]). Moreover, in these client/server models, clients communicate only with servers and vice versa (the communication graph is bipartite).

The reason why different types of failures are addressed in [2, 11] comes from the fact that these papers consider “classical” read/write registers, namely, a register may be modified any number of times, and each read must return the last value that was written. In this context, a Byzantine client can overwrite a value it has previously written, and trick correct clients so that they believe that it wrote only once. To prevent this bad scenario, [11] considers clients that can fail only by crashing, while [2] restricts clients to be “semi-Byzantine”: they can issue a bounded number of faulty writes, but otherwise have to respect the code of their algorithm.

**Content of the paper: Construction of an atomic read/write memory** Differently, this paper is on the construction of a shared memory (atomic registers) on top of an asynchronous message-passing system where processes may exhibit a Byzantine behavior. Its first contribution is the definition of a shared memory (atomic registers) in the context of Byzantine processes, and the design of an algorithm that builds such a shared memory on top of an asynchronous message-passing systems where up to  $t$  processes may be Byzantine. These registers differ from classical registers in that each read returns the complete history of writes to the registers. This prevents the problem mentioned previously where a Byzantine process can make different correct processes read different values from the same register. This  $t$ -resilient shared memory is made up of  $n$  single-writer/multi-reader (SWMR) atomic registers (one per process). The paper also shows that  $t < n/3$  is a necessary and sufficient requirement for such a construction.

This construction and the associated upper bound  $t < n/3$  complement the previous result known on the construction of an atomic shared memory in asynchronous crash-prone message-passing systems, where it has been shown that  $t < n/2$  is an upper bound on the number of faulty processes [1]. Interestingly, the upper bound  $t < n/3$  is the same as the one for solving consensus in both Byzantine synchronous systems [12] and Byzantine asynchronous systems (enriched with an appropriate oracle such as a common coin) e.g. [6, 19].

**Content of the paper: From read/write registers to higher level abstractions** The second contribution of the paper is a set of algorithms that solve distributed computing problems on top of the previous  $t$ -resilient shared memory abstraction. These algorithms illustrate the versatility of these Byzantine-tolerant atomic registers. The two first algorithms, which are pretty simple, solve the “one-shot write-snapshot” problem, and the “correct-only” agreement problem (a weakened version of the consensus problem), respectively. Then the paper describes an algorithm that solves the multidimensional approximate agreement on top of the  $t$ -resilient shared memory abstraction. This algorithm can be seen as an adaptation to a Byzantine read/write shared memory system of

Mendes-Herlihy’s algorithm [18], which solves the same problem “directly” on top of an asynchronous Byzantine message-passing system.

As shown by these examples, the important feature of the proposed shared memory abstraction lies in the fact that it prevents Byzantine processes from corrupting synchronization among the correct processes. A Byzantine process can create inconsistency only on the values it writes, but any two correct processes see the same sequence of written values.

**Roadmap** The paper is composed of 6 sections. Section 2 introduces the underlying Byzantine asynchronous message-passing model. Section 3 defines the notion of Byzantine-tolerant atomic read/write registers, and presents an algorithm that builds such registers on top of the basic Byzantine asynchronous message-passing model. This section shows also that  $t < n/3$  is a necessary and sufficient requirement for such a construction. Then, Section 4 and Section 5 present algorithms that solve distributed computing problems on top of Byzantine-tolerant atomic registers. Finally, Section 6 concludes the paper.

## 2 Computation Model, Reliable Broadcast and Two Lemmas

### 2.1 Computation Model

**Computing entities** The system is made up of a set  $\Pi$  of  $n$  sequential processes, denoted  $p_1, p_2, \dots, p_n$ . These processes are asynchronous in the sense that each process progresses at its own speed, which can be arbitrary and remains always unknown to the other processes.

**Communication model** The processes cooperate by sending and receiving messages through bi-directional channels. The communication network is a complete network, which means that each process  $p_i$  can directly send a message to any process  $p_j$  (including itself). It is assumed that, when a process receives a message, it can unambiguously identify its sender. Each channel is reliable (no loss, corruption, or creation of messages), not necessarily first-in/first-out, and asynchronous (while the transit time of each message is finite, there is no upper bound on message transit times). Moreover, Byzantine processes are not prevented from reading all messages and reordering them.

**Byzantine failures** The model parameter  $t$  is an upper bound on the number of processes that can exhibit a Byzantine behavior [12, 21]. A Byzantine process is a process that behaves arbitrarily: it may crash, fail to send or receive messages, send arbitrary messages, start in an arbitrary state, perform arbitrary state transitions, etc. Hence, a Byzantine process, which is assumed to send the same message  $m$  to all the processes, can send a message  $m_1$  to some processes, a different message  $m_2$  to another subset of processes, and no message at all to the other processes. Moreover, Byzantine processes can collude to “pollute” the computation.

**Terminology and notation** A Byzantine process is also called a *faulty* process. A process that never commits a failure is called a *correct* (or *non-faulty*) process. Given an execution, let  $\mathcal{C}$  and  $\mathcal{F}$  denote the sets of correct and faulty processes, respectively.

This process model is denoted  $\mathcal{BAMP}_{n,t}[c(n,t)]$ , where  $c(n,t)$  is a constraint imposed on the model parameter  $t$ . As an example  $c(n,t) \equiv (n > 3t)$  means that less than one third of processes may be faulty.

### 2.2 Reliable Broadcast Abstraction

This section presents a reliable broadcast abstraction (denoted r-broadcast), that will be used to build atomic read/write registers. (Section 3). This abstraction is a simple generalization of a reliable broadcast due to Bracha [5]. While Bracha’s abstraction is for a single broadcast, the proposed abstraction considers that each process can issue a sequence of broadcasts. It is shown in [5] that  $t < n/3$  is a necessary requirement when one has to build such an abstraction in the presence of asynchrony and Byzantine failures.

**Specification** The reliable broadcast abstraction provides each process with the operations  $R\_broadcast()$  and  $R\_deliver()$ . When a process  $p_i$  invokes  $R\_broadcast()$  we say that “ $p_i$  r-broadcasts a value”. Similarly, when  $p_i$  returns from an invocation of  $R\_deliver()$  and obtains a value, we say “ $p_i$  r-delivers a value”.

The operation  $R\_broadcast()$  has two input parameters: a broadcast value  $v$ , and an integer  $sn$ , which is a local sequence number used to identify the successive r-broadcasts issued by the invoking process  $p_i$ . The sequence of numbers used by each (correct) process is the increasing sequence of consecutive integers.

- **RB-Validity.** If a correct process r-delivers a pair  $(v, sn)$  from a correct process  $p_x$ , then  $p_x$  invoked the operation  $R\_broadcast(v, sn)$ .

- **RB-Integrity.** Given any process  $p_i$  and any sequence number  $sn$ , a correct process  $r$ -delivers at most once a pair  $(-, sn)$  from  $p_i$ .
- **RB-Uniformity.** If a correct process  $r$ -delivers a pair  $(v, sn)$  from  $p_i$  (possibly faulty), then all the correct processes eventually  $r$ -deliver the same pair  $(v, sn)$  from  $p_i$ .
- **RB-Termination.** If the process that invokes  $R\_broadcast(v, sn)$  is correct, all the correct processes eventually  $r$ -deliver the pair  $(v, sn)$ .

RB-Validity relates the outputs to the inputs, namely what is  $r$ -delivered was  $r$ -broadcast. RB-Integrity states that there is no  $r$ -broadcast duplication. RB-Uniformity is an “all or none” property (it is not possible for a pair to be delivered by a correct process and to be never delivered by the other correct processes). RB-Termination is a liveness property: at least all the pairs  $r$ -broadcast by correct processes are  $r$ -delivered by them.

An algorithm implementing the reliable broadcast abstraction is presented in Appendix A. This algorithm is a simple variant of Bracha’s algorithm [5].

### 2.3 Two Preliminary Quorum-related Lemmas

This section states and proves two lemmas that will be central in the understanding and the proof of the construction of atomic SWMR registers described in the next section.

**Lemma 1** *Let  $m$ ,  $n$ , and  $t$  be positive integers.  $(m > \frac{n+t}{2}) \Leftrightarrow (m \geq \lfloor \frac{n+t}{2} \rfloor + 1)$ .*

**Proof**

- Direction  $\Leftarrow$ : As  $x - 1 < \lfloor x \rfloor$ , it follows that  $(m \geq \lfloor \frac{n+t}{2} \rfloor + 1) \implies (m > \frac{n+t}{2})$ .
- Direction  $\Rightarrow$ :  $(m > \frac{n+t}{2}) \Rightarrow (m \geq \lfloor \frac{n+t}{2} \rfloor + 1)$ .
  - Case  $(n + t) \bmod 2 = 0$ . We have then  $m > \frac{n+t}{2} \Rightarrow m \geq \frac{n+t}{2} + 1 = \lfloor \frac{n+t}{2} \rfloor + 1$ .
  - Case  $(n + t) \bmod 2 = 1$ . We have then  $m > \frac{n+t}{2} \Rightarrow m \geq \frac{n+t}{2} + \frac{1}{2} = \lfloor \frac{n+t}{2} \rfloor + 1$ .

□ Lemma 1

**Lemma 2** *Any two sets of processes  $Q_1$  and  $Q_2$  of size at least  $\lfloor \frac{n+t}{2} \rfloor + 1$  have at least one correct process in their intersection.*

**Proof** When considering integers, it follows from Lemma 1, that “strictly more than  $\frac{n+t}{2}$ ” is equivalent to “at least  $\lfloor \frac{n+t}{2} \rfloor + 1$ ”.

- $Q_1 \cup Q_2 \subseteq \{p_1, \dots, p_n\}$ . Hence,  $|Q_1 \cup Q_2| \leq n$ .
- $|Q_1 \cap Q_2| = |Q_1| + |Q_2| - |Q_1 \cup Q_2| \geq |Q_1| + |Q_2| - n \geq 2(\lfloor \frac{n+t}{2} \rfloor + 1) - n > 2(\frac{n+t}{2}) - n = t$ . Hence,  $|Q_1 \cap Q_2| \geq t + 1$ , from which it follows that  $Q_1 \cap Q_2$  contains at least one correct process.

□ Lemma 2

## 3 Construction of Single-Writer/Multi-Reader Atomic Registers

### 3.1 Atomic Read/Write Registers in the Presence of Byzantine Processes

**Single-writer/multi-reader (SWMR) registers** The fault-tolerant shared memory supplied to the upper abstraction layer is an array denoted  $REG[1..n]$ . For each  $i$ ,  $REG[i]$  is a single-writer/multi-reader (SWMR) register. This means that  $REG[i]$  can be written only by  $p_i$ . To that end,  $p_i$  invokes the operation  $REG[i].write(v)$  where  $v$  is the value it wants to write into  $REG[i]$ . Differently, any process  $p_j$  can read  $REG[i]$ . It invokes then the operation  $REG[i].read()$ .

Let us notice that the “single-writer” requirement is natural in the presence of Byzantine processes. If registers could be written by any process, it would be possible for the Byzantine processes to pollute the whole memory, and no non-trivial computation could be possible.

**The value returned by a read operation** A register  $REG[i]$  contains the sequence of values (also called a *history*) that have been written into it, and such a sequence is returned by the invocations of the operation  $REG[i].read()$ . Each register  $REG[i]$  is initialized to the empty sequence (denoted  $\epsilon$ ), which corresponds to the default value  $\perp$ . It is assumed that no process can write  $\perp$  into its register. Let us remark that returning a sequence of values is not a restriction, as, when a process obtains such a sequence  $h$ , it can always consider only its last value (or  $\perp$  if  $h = \epsilon$ ).

**Notations** Let  $p_i$  and  $p_j$  be correct processes. We use the following notations.

- $\text{op\_read}[j, i, h]$ : execution by a correct process  $p_j$  of  $REG[i].\text{read}()$ , returning the history  $h$ .
- $\text{op\_write}[i, wsn]$ :  $wsn^{\text{th}}$  execution by a correct process  $p_i$  of  $REG[i].\text{write}()$ .
- $|h|$ : length of the history  $h$  ( $|\epsilon| = 0$ ).

**Specification** The correct behavior of an SWMR register is defined by the following properties.

- R-Termination (liveness). Let  $p_i$  be a correct process.
  - Each invocation of  $REG[i].\text{write}()$  issued by  $p_i$  terminates.
  - $\forall j$ , each invocation of  $REG[j].\text{read}()$  issued by  $p_i$  terminates.
- R-Consistency (safety).
  - Single history. Let  $p_i$  be a (correct or faulty) process. There exists exactly one history  $H_i$  such that, for any correct process  $p_j$ , any  $\text{op\_read}[j, i, h]$  is such that  $h$  is a prefix of  $H_i$ .
  - Read followed by write. Let  $p_i$  and  $p_j$  be two correct processes. If  $\text{op\_read}[j, i, h]$  terminates before  $\text{op\_write}[i, wsn]$  starts, then  $|h| < wsn$ .
  - Write followed by read. Let  $p_i$  and  $p_j$  be two correct processes. If  $\text{op\_write}[i, wsn]$  terminates before  $\text{op\_read}[j, i, h]$  starts, then  $wsn \leq |h|$ .
  - No read inversion. Let  $p_i$  and  $p_j$  be two correct processes. If  $\text{op\_read}[j, i, h]$  terminates before  $\text{op\_read}[k, i, h']$  starts, then  $|h'| \geq |h|$ .

As, whatever  $i$ , the invocations of  $REG[i].\text{read}()$  by a faulty process  $p_j$  can return any value, the previous specification do not need to take them into account. Moreover, it is possible that, while executing a code different from the code of the write operation, a faulty process modifies the content of its register  $REG[j]$  (at the operational level, this happens when the messages it generates could have been sent by a correct implementation of the operation  $\text{write}()$ ). The specification of the consistency of such a register  $REG[j]$  takes this into account in the “no read inversion” property.

The previous properties state that each register is linearizable [10]. This means that it is possible to totally order the executions of the operations in such a way that (a) the execution of each operation appears as if it has been executed at a single point of the time line between its start event and its end event, (b) no two operations have been executed at the same time, and (c) each read by a correct process returns the sequence of values written before it in the sequence (when considering read/write registers, linearizability is equivalent to atomicity [13]).

An important theorem associated with linearizability is the following [10]: If each object (register) is linearizable, then the set of all the objects, considered as a single object, is linearizable. This means that linearizable objects compose for free.

## 3.2 The Construction

An algorithm constructing an SWMR atomic (linearizable) register in the presence of up to  $t$  Byzantine processes, is described in Figure 1. This algorithm requires  $t < n/3$ , hence it is suited to the computing model  $\mathcal{BAMP}_{n,t}[t < n/3]$ . This algorithm uses a **wait**(*condition*) statement. The corresponding process is blocked until *condition* becomes satisfied. While a process is blocked, it can process the messages it receives.

**Local variables** Each process  $p_i$  manages four local variables whose scope is the full computation (local variables are denoted with lower case letters, and subscripted by the process index  $i$ ).

- $reg_i[1..n]$  is the local representation of the array  $REG[1..n]$  of atomic SWMR registers. Each local register  $reg_i[j]$  is initialized to the empty sequence  $\epsilon$  whose size is 0 (the corresponding value being the default value  $\perp$ ). The content of  $reg_i[j]$  is called the local *history* of  $REG[j]$ , as known by  $p_i$ .
- $wsn_i$  is a sequence number generator (initialized to 0) for the writes of  $REG[i]$  (issued by  $p_i$ ).
- $rsn_i[1..n]$  is a local array such that  $rsn_i[j]$  is used to associate sequence numbers to the invocations of  $REG[j].\text{read}()$  issued by  $p_i$ . Each  $rsn_i[j]$  is initialized to 0.
- $approx\_rsn_i[1..n, 1..n]$  is a matrix of sequence numbers, each initialized to 0;  $approx\_rsn_i[k, j]$  is the last read sequence number ( $rsn_k[j]$ ) used by  $p_k$  to to read  $REG[j]$ , as known by  $p_i$ .

```

local variables initialization:
 $reg_i[1..n] \leftarrow [\epsilon, \dots, \epsilon]; wsn_i \leftarrow 0; rsn_i[1..n] \leftarrow [0, \dots, 0], approx\_rsn_i[1..n, 1..n] \leftarrow [0, \dots, 0].$ 
%-----

operation  $REG[i].write(v)$  is
(01)  $wsn_i \leftarrow wsn_i + 1;$ 
(02)  $R\_broadcast\ WRITE(v, wsn_i);$ 
(03) wait ( $WRITE\_DONE(wsn_i)$  received from strictly more than  $\frac{n+t}{2}$  different processes);
(04) return ().

operation  $REG[j].read()$  is
(05)  $rsn_i[j] \leftarrow rsn_i[j] + 1;$ 
(06)  $broadcast\ READ(j, rsn_i[j]);$ 
(07) wait ( $\exists h : READ\_VALUE(j, rsn_i[j], h)$  received from strictly more than  $\frac{n+t}{2}$  different processes);
(08) return ( $h$ ). % the last value in  $h$  can be returned if not interested in the history of  $REG[j]$  %
%-----

when a message  $WRITE(v, wsn)$  from  $p_j$  is  $R\_delivered$ :
(09) wait ( $|reg_i[j]| + 1 = wsn$ );
(10)  $reg_i[j] \leftarrow reg_i[j] \cdot v;$ 
(11)  $send\ WRITE\_DONE(wsn)$  to  $p_j$ ;
(12) for  $k \in [1..n]$  do  $send\ READ\_VALUE(j, approx\_rsn_i[k, j], reg_i[j])$  to  $p_k$  end for.

when a message  $READ(j, rsn)$  from  $p_k$  is received:
(13) if ( $approx\_rsn_i[k, j] < rsn$ ) then
(14)  $approx\_rsn_i[k, j] \leftarrow rsn;$ 
(15)  $send\ READ\_VALUE(j, rsn, reg_i[j])$  to  $p_k$ 
(16) end if.

```

Figure 1: Array of SWMR atomic registers in  $\mathcal{BAMP}_{n,t}[t < n/3]$  (code for  $p_i$ )

**The operation**  $REG[i].write(v)$  This operation is implemented by the client lines 01-04 and the server lines 09-12. Process  $p_i$  first increases  $wsn_i$  and r-broadcasts the message  $WRITE(v, wsn_i)$ . Let us remark that this is the only place where the algorithm uses the underlying reliable broadcast abstraction. The process  $p_i$  then waits for acknowledgments (message  $WRITE\_DONE(wsn_i)$ ) from a quorum including strictly more than  $\frac{n+t}{2}$  processes, and finally terminates the write operation. As we have seen (Lemmas 1 and 2), the intersection of any two quorums of such a size contains at least one correct process. This property will be used to prove the consistency of the register  $REG[i]$ .

When  $p_i$  is r-delivered a message  $WRITE(v, wsn)$  from a process  $p_j$ , it first waits until the previous write of  $REG[j]$  by  $p_j$  has locally terminated (line 09). Hence, all the writes of  $REG[j]$  are locally processed by  $p_i$  in the order they have been issued by  $p_j$ . When  $|reg_i[j]| + 1 = wsn$ ,  $p_i$  adds  $v$  at the tail of  $reg_i[j]$  (line 10), and sends back an acknowledgment to  $p_j$  (line 11).

Finally,  $p_i$  sends to each process  $p_k$  the customized message  $READ\_VALUE(j, approx\_rsn_i[k, j], reg_i[j])$  to inform  $p_k$  that this write from  $p_j$  has locally been taken into account at  $p_i$  (line 12). As we will see, this is to help terminate the invocations of  $REG[j].read()$  issued by the correct processes.

**The operation**  $REG[j].read()$  This operation is implemented by the client lines 05-08 and the server lines 12-16. When  $p_i$  wants to read  $REG[j]$ , it first broadcasts a read request appropriately identified (message  $READ(j, rsn_i[j])$ , lines 05-06), and waits for acknowledgment messages carrying the same history  $h$  (message  $READ\_VALUE(j, rsn_i[j], h)$ ), from a quorum of strictly more than  $\frac{n+t}{2}$  distinct processes (lines 07). When  $p_i$  stops waiting, it knows that, when they sent their acknowledgments, the history of  $REG[j]$  was equal to  $h$  for strictly more than  $\frac{n+t}{2}$  processes. When this occurs,  $p_i$  returns this history  $h$  as the result of the read operation (line 08).

On its server side, when a process  $p_i$  receives a read request message  $READ(j, rsn)$  from a process  $p_k$ , it first checks if its view of the read of  $REG[j]$  by  $p_k$  is “late”, i.e., if  $approx\_rsn_i[k, j] < rsn$  (line 13). If it is the case,  $p_i$  updates  $approx\_rsn_i[k, j]$  (line 14), and sends by return to  $p_k$  the message  $READ\_VALUE(j, rsn, reg_i[j])$ , to inform it of its current value of  $REG[j]$  (line 15). If  $approx\_rsn_i[k, j] \geq rsn$ , the read request is an old message and  $p_i$  ignores it.

**Comparing with the crash failure model** It is known that the algorithms implementing an atomic register on top of an asynchronous message-passing system prone to process crashes, require that “read have to write” [1]. More precisely, before returning a value, a process must write this value to ensure atomicity. Doing so, it is not possible that two sequential read invocations, both

concurrent with a write invocation, are such that the first read obtains the new value while the second read obtains the old value (demanding the reader to write the value it is about to return guarantees that there is no new/old inversion [22]).

As Byzantine failures are more severe than crash failures, the algorithm of Figure 1 needs to use a mechanism analogous to the “read have to write” to prevent new/old inversion from occurring. This is done by sending to each process  $p_k$  the customized message  $\text{READ\_VALUE}(j, \text{approx\_rsn}_i[j, k], \text{reg}_i[j])$  issued at line 12. These messages play the role of writes, that allow the wait statement of line 07 to always terminate with a correct history value for  $\text{REG}[j]$ .

**Message complexity** It follows from the previous discussion that, in addition to a reliable broadcast, each write generates  $O(n)$  messages  $\text{WRITE\_DONE}()$  and  $O(n^2)$  messages  $\text{READ\_VALUE}()$ , and each read generates at most  $2n$  messages. Hence, while the algorithm presented in [1] requires the assumption  $t < n/2$  (which is a necessary and sufficient requirement on the model parameter  $t$ ) and  $O(n)$  messages to implement an SWMR atomic register in the crash failure model, the proposed algorithm requires the assumption  $t < n/3$  (which is shown to be necessary and sufficient in Section 3.4), and  $O(n^2)$  messages plus a reliable broadcast, to implement SWMR atomic registers in the Byzantine asynchronous message-passing model.

### 3.3 Proof of the Construction

**Lemma 3** *Let  $n \geq 3t + 1$ . If  $p_i$  is correct and invokes  $\text{REG}[i].\text{write}()$ , its invocation terminates.*

**Proof** Let us first consider the first invocation of  $\text{REG}[i].\text{write}()$  by  $p_i$ . Due to the RB-termination property of the underlying reliable broadcast abstraction invoked by  $p_i$  at line 02, each correct process  $p_j$  r-delivers the message  $\text{WRITE}(-, 1)$ . When this occurs, the predicate of line 09 is satisfied, and  $p_j$  sends the message  $\text{WRITE\_DONE}(1)$  to  $p_i$ . As there are strictly more than  $\frac{n+t}{2}$  correct processes (since  $n > 3t$ ,  $\frac{n+t}{2} = n - t - \frac{n-3t}{2} < n - t$ ), it follows that  $p_i$  cannot remain blocked forever at line 03, and the write invocation terminates.

Let us now observe that each correct process  $p_j$  processes the messages  $\text{WRITE}(-, \text{wsn})$  r-broadcast by  $p_i$  in their sequence number order (line 09). As  $p_i$  uses the sequence of the consecutive integers to define its next sequence number  $\text{wsn}$ , it follows that the same reasoning as for  $\text{wsn} = 1$  applies iteratively to the subsequent invocations of  $\text{REG}[i].\text{write}()$  issued by  $p_i$ , which concludes the proof.  $\square$  Lemma 3

**Lemma 4** (Single history). *Let  $p_i$  be a (correct or faulty) process. It exists a history  $H_i$  such that any invocation of the operation  $\text{REG}[i].\text{read}()$  by a correct process returns a prefix of  $H_i$ .*

**Proof** Given a register  $\text{REG}[i]$ , let  $H_i$  be the the sequence of values defined by the sequence of messages  $\text{WRITE}(v, \text{wsn})$  r-delivered from  $p_i$  to the correct processes, where the order of the values in  $H_i$  is defined by the sequence numbers  $\text{wsn} = 1, 2$ , etc. It follows from the RB-integrity and RB-uniformity properties of the reliable broadcast, and the lines 09-10 of the algorithm, that the history  $\text{reg}_j[i]$  of any correct process  $p_j$  is a prefix of  $H_i$ . Consequently, the messages  $\text{READ\_VALUE}(-, -, \text{reg}_k[i])$  sent by any correct process  $p_k$  at line 12 or 15 are such that  $\text{reg}_k[i]$  is a prefix of  $H_i$ . It then follows that, when a correct process  $p_j$  computes  $h$  at line 07, this history comes from strictly more than  $\frac{n+t}{2} \geq t$  processes, thus at least from one correct process, and is consequently a prefix of  $H_i$ . Hence, any invocation of the operation  $\text{REG}[i].\text{read}()$  by a correct process returns a prefix of  $H_i$ .  $\square$  Lemma 4

**Lemma 5** *Let  $n \geq 3t + 1$ . If  $p_j$  is correct and invokes  $\text{REG}[i].\text{read}()$ , its invocation terminates.*

**Proof** The proof is by contradiction. let us assume that a correct process  $p_j$  invokes  $\text{REG}[i].\text{read}()$  and this invocation never terminates. This means that the predicate associated with the wait statement of line 07 remains false forever, namely,  $\nexists h$  such that the message  $\text{READ\_VALUE}(i, \text{rsn}_j[i], h)$  is received from strictly more than  $\frac{n+t}{2}$  different processes.

As  $p_j$  is correct, it broadcasts the request message  $\text{READ}(i, \text{sn})$  where  $\text{sn} = \text{rsn}_j[i]$  (line 06), and this message is received by all correct processes. Moreover,  $\text{sn}$  is the greatest sequence number ever used by  $p_j$  to read  $\text{REG}[i]$ , and, due to the contradiction assumption,  $\text{rsn}_j[i]$  keeps forever the value  $\text{sn}$ .

Let  $p_k$  be any correct process. When  $p_k$  receives  $\text{READ}(i, \text{sn})$  from  $p_j$ , the predicate  $\text{approx\_rsn}_k[j, i] < \text{sn}$  is satisfied (line 13). This is because  $\text{sn}$  is greater than all previous sequence numbers used by  $p_j$  to read  $\text{REG}[i]$ . It follows that  $p_k$  updates  $\text{approx\_rsn}_k[j, i]$  to  $\text{sn} = \text{rsn}_j[i]$ , and sends to  $p_j$  the message  $\text{READ\_VALUE}(i, \text{sn}, \text{reg}_k[i])$  (lines 14-15). Moreover, as the read of  $\text{REG}[i]$  by  $p_j$  never terminates,  $\text{approx\_rsn}_k[j, i]$  remains forever equal to  $\text{sn} = \text{rsn}_j[i]$ .

As the predicate of line 07 remains forever false at  $p_j$ , and  $p_j$  receives at least  $(n - t)$  messages  $\text{READ\_VALUE}(i, \text{sn}, -)$  (one from each correct process), it follows that  $p_j$  receives at least two messages  $\text{READ\_VALUE}(i, \text{sn}, h)$  and  $\text{READ\_VALUE}(i, \text{sn}, h')$  such that  $h$  is a strict prefix of  $h'$  or  $h'$  is a strict prefix of  $h$  (this is because, due to Lemma 4, both  $h$  and  $h'$  are prefixes of  $H_i$ ). Without loss of generality, let  $h'$  be the shortest history received, and  $h$  be the longest.



Due to the RB-uniformity property of the underlying broadcast abstraction, it follows that all the correct processes r-delivers the same messages  $\text{WRITE}()$  from  $p_i$ , and process them in the same order (line 09). Let  $p_k$  be a correct process. It follows directly from the code of the algorithm that, each time  $p_k$  adds a value to  $\text{reg}_k[i]$  (line 10), it sends a message  $\text{READ\_VALUE}(i, sn, \text{reg}_k[i])$  to  $p_j$  (line 12). It follows that there is a finite time after which  $p_j$  has received the very same message  $\text{READ\_VALUE}(j, sn, h)$  from strictly more than  $\frac{n+t}{2}$  different processes. The predicate of line 07 becomes then satisfied. This contradicts the initial assumption, and the lemma follows.  $\square$  *Lemma 5*

**Lemma 6** (Read followed by write). *Let  $p_i$  and  $p_j$  be two correct processes. If  $\text{op\_read}[j, i, h]$  terminates before  $\text{op\_write}[i, wsn]$  starts, then the returned history  $h$  is such that  $|h| < wsn$ .*

**Proof** As  $p_i$  is correct and has not yet invoked  $\text{op\_write}[i, wsn]$  when  $p_j$  terminates  $\text{op\_read}[j, i, h]$ , it follows that no correct process r-delivers a message  $\text{WRITE}(v, wsn)$  from  $p_i$  before  $\text{op\_read}[j, i, h]$  terminates. Hence, when they received from  $p_j$  the message  $\text{READ}(i, sn)$  generated by  $\text{op\_read}[j, i, -]$ , strictly more than  $\frac{n+t}{2}$  different processes  $p_x$  (i.e., strictly more than  $\frac{n-t}{2}$  correct processes) returned the same message  $\text{READ\_VALUE}(i, sn, h)$  where  $h = \text{reg}_x[i]$  and  $|h| < wsn$ . Consequently, the history  $h$  returned by  $p_j$  is smaller than  $wsn$ .  $\square$  *Lemma 6*

**Lemma 7** (Write followed by read). *Let  $n > 3t$ . Let  $p_i$  and  $p_j$  be two correct processes. If  $\text{op\_write}[i, wsn]$  terminates before  $\text{op\_read}[j, i, h]$  starts, then the returned history  $h$  is such that  $|h| \geq wsn$ .*

**Proof** It follows from line 03 and lines 10-11 that, when  $\text{op\_write}[i, wsn]$  terminates (which implies before  $\text{op\_read}[j, i, h]$  starts), there is a quorum  $Q_W$  of strictly more than  $\frac{n+t}{2}$  processes  $p_x$  such that  $|\text{reg}_x[i]| \geq wsn$ . Moreover, the invocation  $\text{op\_read}[j, i, h]$  obtains the same message  $\text{READ\_VALUES}(i, sn, h)$  from a quorum  $Q_R$  including strictly more than  $\frac{n+t}{2}$  correct processes. According to Lemmas 1 and 2, it follows that  $Q_W \cap Q_R$  contains at least one correct process  $p_y$ . As this process is such that  $|\text{reg}_y[i]| \geq wsn$ , it follows that  $|h| \geq wsn$ .  $\square$  *Lemma 7*

**Lemma 8** (No read inversion). *Let  $n > 3t$ . Let  $p_j$  and  $p_k$  be two correct processes. If  $\text{op\_read}[j, i, h]$  terminates before  $\text{op\_read}[k, i, h']$  starts, we have then  $|h| \leq |h'|$ .*

**Proof** To terminate,  $\text{op\_read}[j, i, h]$  received the same message  $\text{READ\_VALUE}(j, rsn, h)$  from a quorum  $Q_{R1}$  of strictly more than  $\frac{n+t}{2}$  different processes. Similarly, let  $Q_{R2}$  be the quorum of strictly more than  $\frac{n+t}{2}$  processes that allowed  $\text{op\_read}[k, i, h']$  to terminate.

According to Lemmas 1 and 2, there is a correct process  $p_x$  in  $Q_{R1} \cap Q_{R2}$ . As, (a)  $p_x$  is correct and sent the message  $\text{READ\_VALUE}(j, rsn, h)$  to  $p_j$ , and later sent the message  $\text{READ\_VALUE}(j, rsn, h')$  to  $p_k$ , and (b)  $\text{reg}_x[i]$  can only increase, we necessarily have  $|h| \leq |h'|$ , which concludes the proof of the lemma.  $\square$  *Lemma 8*

The following theorem follows from the previous lemmas 3-8.

**Theorem 1** *The algorithm described in Figure 1 implements an array of  $n$  SWMR atomic registers (one register per process) in the system model  $\mathcal{BAMP}_{n,t}[t < n/3]$ .*

To better understand and capture the behavior of the algorithm, the linearization points associated with the executions of (a) read and write operations issued by the correct processes, and (b) the writes by faulty processes, whose values have been read by correct processes, are described in Appendix B.

### 3.4 The Condition $t < n/3$ is Necessary and Sufficient

**Theorem 2** *The condition  $t < n/3$  is necessary and sufficient to built an SWMR atomic register in  $\mathcal{BAMP}_{n,t}[\emptyset]$ .*

**Proof** The algorithm presented in the previous section has shown that the condition  $t < n/3$  is sufficient to built an SWMR atomic register. So, the rest of the proof addresses the necessity part of the condition.

The proof is by contradiction. Let us assume that there is an algorithm  $A$  that builds an atomic register in  $\mathcal{BAMP}_{n,t}[n \leq 3t]$ , which means that it satisfies the R-consistency and R-termination properties stated in Section 3.1. For simplicity, and without loss of generality, we consider the largest possible value of  $t$ , i.e.,  $n = 3t$ . Let us first observe that to guarantee R-termination, a process cannot wait for messages from more than  $n - t = 2t$  processes.

Let us partition the set of processes into three sets  $Q_1$ ,  $Q_2$  and  $Q_3$ , each of size (at most)  $t$ . Moreover, let us consider two processes  $p_1 \in Q_1$  and  $p_2 \in Q_2$ .

Let us consider a first execution  $E_1$  defined as follows.

- The set of Byzantine processes is  $Q_1$ ; these processes do not send messages and appear as crashed.
- The process  $p_2 \in Q_2$  writes a value  $v$  in  $REG[2]$ . Due to the R-termination property of the algorithm  $A$ , the invocation of  $REG[2].write(v)$  by  $p_2$  terminates. Let  $\tau_w$  be the time instant at which this write terminates.

Let  $E_2$  be a second execution defined as follows.

- All processes are correct, but the processes of  $Q_2$  execute no step before  $\tau_r$  (defined below).
- After  $\tau_w$ , the process  $p_1 \in Q_1$  reads the register  $REG[2]$ . Due to the R-termination property of the algorithm  $A$ , and because the processes of  $Q_2$  could be Byzantine, the invocation of  $REG[2].read()$  by  $p_1$  terminates. Let  $\tau_r$  be the time instant at which this read terminates. As, no process of  $Q_2$  executes a step before the read terminates,  $p_2$  does not write  $REG[2]$  before  $\tau_r$ . Consequently, according to the R-consistency property *read followed by write*,  $REG[2]$  has still its initial value  $\epsilon$ . It follows that the read operation by  $p_1$  returns this initial value.

Let us finally consider  $E_3$ , a third execution defined as follows.

- The set of Byzantine processes is  $Q_3$ , and these processes behave exactly as in  $E_1$  with respect to the processes of  $Q_2$ , and exactly as in  $E_2$  with those of  $Q_1$ .
- The messages sent by the processes of  $Q_1$  to the processes of  $Q_2$  and by the processes of  $Q_2$  to the processes of  $Q_1$  are delayed until after  $\tau_r$ .
- The messages exchanged between themselves by the processes of  $Q_2 \cup Q_3$  are received at exactly the same time instants as in  $E_1$ . Similarly, the messages exchanged between themselves by the processes of  $Q_1 \cup Q_3$  are received at exactly the same time instants as in  $E_2$ .
- As the very same time instants as in  $E_1$ , process  $p_2 \in Q_2$  writes a value  $v$  in  $REG[2]$ . Since, from the point of view of the processes of  $Q_2$ , the executions  $E_1$  and  $E_3$  are indistinguishable, the invocation of  $REG[2].write(v)$  by  $p_2$  terminates at  $\tau_w$  too.
- As in execution  $E_2$ , after  $\tau_w$  the process  $p_1 \in Q_1$  reads the register  $REG[2]$ . Since, from the point of view of the processes of  $Q_1$ , the executions  $E_2$  and  $E_3$  are indistinguishable, the invocation of  $REG[2].read()$  by  $p_1$  terminates at  $\tau_r$  and returns  $\epsilon$ . But this violates the R-Consistency property *write followed by read*, which contradicts the existence of Algorithm  $A$ .

□*Theorem 2*

## 4 A Few Simple Abstractions on Top of SWMR Atomic Registers

This section presents two simple uses of the previous construction of an array of  $n$  SWMR atomic registers. Another example is given in Appendix C [4]. In these algorithms, the simplicity comes from the SWMR atomic registers built on top of a message-passing system. This SWMR atomic register abstractions provides us with an abstraction level that allows for simple solutions.

The classical notations for atomic registers are used in the following, namely, given an atomic register  $XX$ ,  $XX \leftarrow v$  stands for  $XX.write(v)$ , and  $x \leftarrow XX$  stands for  $x \leftarrow XX.read(v)$ . Moreover, only the last value of the sequence returned by  $XX.read(v)$  is considered.

### 4.1 One-Shot Write-Snapshot

This section describes a very simple one-shot write-snapshot object. This object provides the processes with a single operation denoted `write_snapshot()`. This operation has a single parameter, namely the value that the invoking process wants to write in the snapshot object. A process  $p_i$  can invoke `write_snapshot()` at most once. This operation returns to the invoking process  $p_i$  a set  $output_i$  made up of pairs  $\langle j, w \rangle$ , where  $w$  is the value written by the process  $p_j$ .

**Specification** A one-shot write-snapshot object is defined by the following properties.

- **Termination.** The invocation of `write_snapshot(v)` by a correct process  $p_i$  terminates.
- **Self-inclusion.** If  $p_i$  is correct and invokes `write_snapshot(v)`, then  $\langle i, v \rangle \in output_i$ .
- **Containment.** If both  $p_i$  and  $p_j$  are correct and invoke `write_snapshot()`, then  $output_i \subseteq output_j$  or  $output_j \subseteq output_i$ .
- **Validity.** If  $\langle j, w \rangle \in output_i$ , then  $p_j$  wrote  $w$  in the `write_snapshot` object.

```

operation write_snapshot( $v_i$ ):
(01)  $IN[i] \leftarrow v_i$ ;
(02) for  $x \in \{1, \dots, n\}$  do  $aux1[x] \leftarrow IN[x]$  end for;
(03) for  $x \in \{1, \dots, n\}$  do  $aux2[x] \leftarrow IN[x]$  end for;
(04) while ( $aux1 \neq aux2$ ) do
(05)    $aux1 \leftarrow aux2$ ;
(06)   for  $x \in \{1, \dots, n\}$  do  $aux2[x] \leftarrow IN[x]$  end for
(07) end while;
(08)  $output_i \leftarrow \{ \langle j, aux1[j] \rangle \mid aux1[j] \neq \perp \}$ ;
(09) return( $output_i$ ).

```

Figure 2: Write-snapshot in  $\mathcal{BAMP}_{n,t}[t < n/3]$  (code for  $p_i$ )

**The algorithm** The internal representation of the write-snapshot object is an array of SWMR atomic registers. This array, denoted  $IN[1..n]$ , is obtained from the construction of Figure 1 (slightly and easily modified at line 09 to ensure that a process writes at most once its register). It is assumed that  $IN[1..n]$  is initialized to  $[\perp, \dots, \perp]$ .

The algorithm implementing the operation `write_snapshot()` is very simple (Figure 2). The invoking process  $p_i$  first deposits its value in  $IN[i]$  (line 01), and issues an asynchronous “sequential double scan” (lines 02-03). If the sequential double scan is not successful (line 04), it executes other double scans (lines 02-03) until a pair of them is successful, i.e.,  $aux1[1..n] = aux2[1..n]$ . After the successful double scan,  $p_i$  computes its output  $output_i$ , namely, a set containing the pairs  $\langle j, w \rangle$  such that  $w$  is the value written by  $p_j$  (as known by the last successful double scan).

The termination of the algorithm is an immediate consequence of the bounded number of processes, and the fact that each register  $IN[i]$  is a one-write register. The validity and self-inclusion are trivial. The containment property follows from the fact that the number of non- $\perp$  entries can only increase.

## 4.2 Correct-Only Agreement

A correct-only agreement object is a one-shot object that provides the processes with a single operation denoted `correct_only_agreement()`. This operation is used by each process to propose a value and decide a set of values. A decided set contains only values proposed by correct processes and the decided sets satisfy the containment property. According to the topological bounds stated in [9], the problem captured by the correct-only agreement object can be solved if and only if  $n > (\dim(I) + 2)t$ , where  $\dim(I)$  is the dimension of the colorless input complex  $I$  with which the problem is instantiated. In our context, this necessary and sufficient condition boils down to  $n > (w + 1)t$ , where  $w > 1$  is the maximal number of distinct values that can be proposed by the correct processes in any execution (in the topology parlance,  $w - 1$  is the greatest dimension of a simplex of the colorless input complex  $I$ ).

**Specification** A correct-only agreement object is defined by the following properties. As in the previous section,  $output_i$  denotes the set of values output by a correct process  $p_i$ .

- **Termination.** The invocation of `correct_only_agreement()` by a correct process  $p_i$  terminates.
- **Containment.** If both  $p_i$  and  $p_j$  are correct and invoke `correct_only_agreement()`, then  $output_i \subseteq output_j$  or  $output_j \subseteq output_i$ .
- **Validity.** The set  $output_i$  of a correct process  $p_i$  is not empty and contains only values proposed by at least one correct process.

**The algorithm** The algorithm implementing the operation `correct_only_agreement()`, is described in Figure 3. As the reader can see, this algorithm is obtained from a simple adaptation of the algorithm implementing the operation `write_snapshot()`. The modified parts are the predicate used at line 04, and the computation of the output at line 08.

More precisely, a successful double scan is now necessary to exit the while loop, but is no longer sufficient. In addition, a process  $p_i$  must observe there is at least one value that has been proposed by  $(t + 1)$  processes (i.e., by at least one correct process). Finally, the output  $output_i$  contains all the values that, from  $p_i$ 's point of view, have been proposed by at least  $(t + 1)$  processes.

As in the previous section, the containment property is a consequence of the fact that the writes in the array  $IN[1..n]$  are atomic, and the number of non- $\perp$  entries can only increase. The termination property is a consequence of the following observations: (a) there is a bounded number of processes, (b) the atomic registers are one-write registers, and (c) the condition  $n > (w + 1)t$ . The validity follows from the condition  $n > (w + 1)t$  (hence there is at least one value that appears  $(t + 1)$  times), and the predicate of line 04.

```

operation correct_only_agreement( $v_i$ ):
(01)  $IN[i] \leftarrow v_i$ ;
(02) for  $x \in \{1, \dots, n\}$  do  $aux1[x] \leftarrow IN[x]$  end for;
(03) for  $x \in \{1, \dots, n\}$  do  $aux2[x] \leftarrow IN[x]$  end for;
(04) while [ $(aux1 \neq aux2) \vee (\#v : |\{j : aux1[j] = v\}| > t)$ ] do
(05)    $aux1 \leftarrow aux2$ ;
(06)   for  $x \in \{1, \dots, n\}$  do  $aux2[x] \leftarrow IN[x]$  end for
(07) end while;
(08)  $output_i \leftarrow \{v : |\{j : aux1[j] = v\}| > t\}$ ;
(09) return( $output_i$ ).

```

Figure 3: Correct-only agreement in  $\mathcal{BAMP}_{n,t}[t < n/(w+1)]$  (code for  $p_i$ )

**Remark** Both the one-write write-snapshot object and the correct-only agreement object share the same termination and containment properties. They can be seen as dual the one from the other in the following sense. One-write write-snapshot satisfies self-inclusion and a weak validity property, while correct-only agreement is not required to satisfy self-inclusion, but is constrained by a stronger validity property. As we have seen, both objects can be implemented by the same generic algorithm whose instances differ essentially in the predicate used to exit the while loop (line 04).

## 5 Solving Multidimensional Approximate Agreement

This section shows how an algorithm designed for the Byzantine asynchronous message-passing system model can be easily adapted to the Byzantine asynchronous shared memory model introduced in Section 3. The shared memory abstraction being at a higher abstraction level than message-passing, the shared memory version of the algorithm seems much easier to understand.

### 5.1 The Multidimensional Approximate Agreement Problem

**Approximate agreement** The  $\epsilon$ -approximate agreement problem has been introduced in the context of synchronous and asynchronous message-passing systems where processes can commit Byzantine failures [8]. Each process proposes a value in  $\mathbb{R}$ , and each correct process has to decide a value such that: (a) any decided value is in the range of the values proposed by the correct processes (validity), and (b) the distance between any two values decided by correct processes is at most  $\epsilon$  (agreement). It is shown in [8] that  $t < n/3$  is a necessary and sufficient condition on the number of Byzantine processes when one has to solve  $\epsilon$ -approximate agreement in both synchronous and asynchronous systems.

**Multidimensional approximate agreement** The  $\epsilon$ -approximate agreement problem has been generalized in [18] to the case where each input value is a point in  $\mathbb{R}^d$  (space of dimension  $d$ ). Such a point is defined by a size  $d$  vector (one entry per coordinate). The problem is then defined by the following properties. Let `multi_approx_agreement()` be the associated operation invoked by processes.

- Termination. The invocation of `multi_approx_agreement()` by a correct process  $p_i$  terminates.
- Validity. The value decided by any correct process is a point of  $\mathbb{R}^d$  within the convex hull of the points proposed by the correct processes.
- Agreement. The Euclidean distance between any two points decided by correct processes is at most  $\epsilon$ .

It is shown in [18] that  $n > (d+2)t$  is a sufficient and necessary requirement for the problem to be solved.

### 5.2 Solving Multidimensional Approximate Agreement

An algorithm solving the multidimensional approximate agreement problem on top of the shared memory abstraction is presented in Figure 4. This algorithm is an adaptation of Mendes-Herlihy's algorithm suited to asynchronous message-passing with up to  $t < n/3$  Byzantine processes [18].

**Shared memory: arrays of  $n$  SWMR atomic registers** The processes cooperate through the following arrays of SWMR atomic registers. Each register is written at most once by its writer. An input value is a  $d$ -dimensional vector (coordinates of the input of  $p_i$  in  $\mathbb{R}^d$ ).

- $VAL[1..n]$ : array such that  $VAL[i]$  is written by  $p_i$  to publish its input.

- $VIEW[1..n]$ : array such that  $VIEW[i]$  contains  $p_i$ 's view of the inputs.
- $EST[1..][1..n]$ : array such that  $EST[r][i]$  contains  $p_i$ 's estimate of its decision value at round  $r$ .
- $MAX[1..n]$ : array such that  $MAX[i] = r$  means that  $p_i$  has estimated that  $r$  rounds are enough to reach approximate agreement. Initially,  $MAX[i] = +\infty$ .

The value of any of these arrays is obtained with a `collect()` operation, which (differently from a snapshot) is an asynchronous and unordered read of the corresponding entries of the array.

**Procedures used in the algorithm** Let  $X$  denote a multiset of values of  $\mathbb{R}^d$  and  $\mathcal{CH}(X)$  the convex hull of such a multiset. Algorithm 4 uses the notion of *safe area* introduced in [18]. The safe area of  $X$  is defined by

$$\text{Safe}_t(X) = \bigcap_{X' \subseteq X, |X'|=|X|-t} \mathcal{CH}(X').$$

Informally, this captures the region of  $\mathbb{R}^d$  that is contained in all the convex hulls of the subsets of cardinal  $n - t$  of  $|X|$ . If the values of  $|X|$  are proposed by distinct processes, then this region is consequently contained in the convex hull of the subset of the values of  $X$  proposed by correct processes. It is shown in [18] that for any  $X$  such that  $|X| > t(d + 1)$ , the safe area  $\text{Safe}_t(X)$  is non-empty [18, Lemma 3.6].

The procedures `bary(S)`, `MultiDimMidpoint(S)`, and `SingleDimMaxDist(S)`, where  $S$  is a convex polytope of  $\mathbb{R}^d$ , are called locally by the processes; `bary(S)` denotes the barycenter of  $S$ , while the two later are defined as follows, by (where  $s[x]$  designate the  $x^{\text{th}}$  coordinate of  $s \in \mathbb{R}^d$ ):

$$\begin{aligned} \forall x \in \{1, \dots, d\} : \text{MultiDimMidpoint}(S)[x] &= (\max\{s[x], s \in S\} + \min\{s[x], s \in S\})/2, \\ \text{and} \quad \text{SingleDimMaxDist}(S) &= \max_{x \in \{1, \dots, d\}} (\max\{s[x], s \in S\} - \min\{s[x], s \in S\}). \end{aligned}$$

**Local variables at each process  $p_i$**  Each process  $p_i$  manages the following local data structures: two arrays  $my\_view_i[1..n]$  and  $views_i[1..n]$ , both initialized to  $[\perp, \dots, \perp]$ ; a set of points  $safe\_init_i$ ; a local estimate  $est_i$  of the point that will be locally decided; an array  $vals_i[1..n]$  of estimates values; an array  $max\_r_i[1..n]$  containing round number upper bounds; and  $r_i$  which contains  $p_i$ 's current round number.

**Behavior of a process  $p_i$**  A process  $p_i$  first writes its input value  $v_i$  (point in  $\mathbb{R}^d$ ) in  $VAL[i]$ , and collects a local view ( $my\_view_i[1..n]$ ) including at least  $(n - t)$  inputs (lines 01-02). Then, to make it public,  $p_i$  writes its view in  $VIEW[i]$ , and collects in  $views_i$  the views of at least  $(n - t)$  processes (lines 03-04). The process  $p_i$  then calculates in  $safe\_init_i$  the barycenter of the safe area of these views (line 05). The parameters  $d$  and  $\epsilon$  of the problem, and the set of barycenters  $safe\_inits_i$  of its current instance, allow  $p_i$  to locally compute an upper bound on the number of rounds to be executed, which is written into the atomic register  $MAX[i]$  (line 06). The set of barycenters  $safe\_inits_i$  is also used to compute  $p_i$ 's first estimate ( $est_i$ ) of its decision value (line 07).

Then, process  $p_i$  starts a sequence of asynchronous rounds whose aim is to refine its current estimate  $est_i$  (lines 09-13) until it returns its last estimate value. This occurs when  $p_i$  attains a round  $r_i$  during which it sees a set of more than  $t$  processes –i.e., at least one correct process– such that each process  $p_j$  of this set posted in its atomic register  $MAX[j]$  a last round upper bound smaller than  $r_i$ . This is captured by the outer termination predicate ( $|\{x : max\_r_i[x] < r_i\}| > t$ ) used at lines 12 and 13.

During a loop iteration  $r_i$ ,  $p_i$  first writes its current decision value estimate in  $EST[r_i][i]$  (line 09), and repeatedly collects both current estimates of decision values written in  $EST[r_i][1..n]$ , and upper bounds of the last round number from  $MAX[1..n]$ , until either the outer termination predicate is satisfied, or it sees at least  $(n - t)$  estimates computed at round  $r_i$  (first sub-predicate of line 11). The use of the outer termination predicate in the inner repeat loop (line 11) allows  $p_i$  not to remain blocked forever waiting for a correct process that has already terminated. Then, if it knows enough values deposited in  $EST[r_i]$ ,  $p_i$  collects again current estimates of decision values written in  $EST[r_i]$  (line 12), and computes a new estimate  $est_i$  (line 13). Finally, once a large enough number of rounds is reached (line 14),  $p_i$  returns (decides) its current estimate value (line 15).

### 5.3 Proof of the Algorithm

The proof of the correctness of the algorithm is similar to, and follows the structure of, the one of Mendes-Herlihy's algorithm [18].

The following lemmas prove that, at lines 03, 05, and 12, all the correct processes receive at least  $n - t$  values in common. This replaces the guarantee given by the module denoted `RBReceiveWitness()` in [18], which is in charge on the reception of messages sent with an underlying reliable broadcast on FIFO channels.

```

operation multi_approx_agreement( $v_i$ ):
(01)  $VAL[i] \leftarrow v_i; r_i \leftarrow 0$ ;
(02) repeat  $my\_view_i[1..n] \leftarrow VAL.collect()$  until ( $|\{x \mid my\_view_i[x] \neq \perp\}| \geq n - t$ ) end repeat;
(03)  $my\_view_i[1..n] \leftarrow VAL.collect()$ ;  $VIEW[i] \leftarrow my\_view_i[1..n]$ ;
(04) repeat  $views_i[1..n] \leftarrow VIEW.collect()$  until ( $|\{x \mid views_i[x] \neq \perp\}| \geq n - t$ ) end repeat;
(05)  $views_i[1..n] \leftarrow VIEW.collect()$ ;  $safe\_inits_i \leftarrow \{bary(Safe_t(X)) : X \in views_i\}$ ;
(06)  $MAX[i] \leftarrow \lceil \log_2(\sqrt{d}/\epsilon \cdot SingleDimMaxDist(safe\_inits_i)) \rceil$ ;
(07)  $est_i \leftarrow bary(Safe_t(safe\_inits_i))$ ;
(08) repeat  $r_i \leftarrow r_i + 1$ ;
(09)    $EST[r_i][i] \leftarrow est_i$ ;
(10)   repeat  $vals_i[1..n] \leftarrow EST[r_i].collect()$ ;  $max\_r_i[1..n] \leftarrow MAX.collect()$ 
(11)   until ( $|\{x : vals_i[x] \neq \perp\}| \geq n - t$ )  $\vee$  ( $|\{x : max\_r_i[x] < r_i\}| > t$ ) end repeat;
(12)   if ( $|\{x : vals_i[x] \neq \perp\}| \geq n - t$ ) then  $vals_i[1..n] \leftarrow EST[r_i].collect()$ ;
(13)    $est_i \leftarrow MultiDimMidpoint(Safe_t(vals_i))$  end if
(14) until ( $|\{x : max\_r_i[x] < r_i\}| > t$ ) end repeat;
(15) return( $est_i$ ).

```

Figure 4: Multidimensional approximate agreement in  $\mathcal{BAMP}_{n,t}[t < n/(d+2)]$  (code for  $p_i$ )

**Lemma 9** For any pair correct processes  $p_i$  and  $p_j$ ,  $|my\_view_i \cap my\_view_j| \geq n - t$ .

**Proof** A correct process performs a collect at line 03 only if it observes at least  $n - t$  values during its last collect at line 02. Without loss of generality, let  $p_i$  be the first process that starts its collect at line 03.

Let  $\tau$  be the time at which  $p_i$  starts its collect at line 03. At least  $(n - t)$  values have already been written in  $VAL$  by time  $\tau$  (otherwise,  $p_i$  would not perform this collect). Because of the linearizability (atomicity) property of the registers, any collect that starts after  $\tau$  will read the values already written in the array  $VAL$ , which concludes the proof of the lemma.  $\square_{Lemma 9}$

The reasoning is the same for the next two lemmas. Lemma 10 is on the sets  $view_i$  computed at line 05, while Lemma 11 is on the sets  $vals_i$  computed at line 12.

**Lemma 10** For any pair correct processes  $p_i$  and  $p_j$ ,  $|views_i \cap views_j| \geq n - t$ .

Let  $vals_{i,r}$  be the set of values observed at line 12 by process  $p_i$  at round  $r$ .

**Lemma 11** For any pair of correct processes  $p_i$  and  $p_j$  that perform a collect at line 12 during round  $r$ ,  $|vals_{i,r} \cap vals_{j,r}| \geq n - t$ .

The following lemma replaces Lemma 4.16 in [18]. It proves that all correct processes execute enough rounds. Let  $r_{min}$  be the minimum round estimate calculated by a correct process at line 06.

**Lemma 12** All correct processes run for at least  $r_{min}$  rounds.

**Proof** A correct process  $p_i$  may exit the loop at lines 08-14 only if it observes at least  $t + 1$  estimates of the number of rounds (values written in  $MAX$ ) lower than its round number. These  $t + 1$  values must include at least one value written by a correct process. Process  $p_i$  will thus execute at least  $r_{min}$  rounds.  $\square_{Lemma 12}$

The following lemma guarantees that no correct process can run for an infinite number of steps.

**Lemma 13** All correct processes terminate their invocation of `multi_approx_agreement()`.

**Proof** The algorithm does not contain any wait statement and contains four loops.

- The loop at line 02. All the correct processes write in  $VAL[1..n]$ , thus all correct processes will eventually observe at least  $(n - t)$  values during their collect and exit the loop.
- The loop at line 04. The reasoning is the same as previously. All the correct processes write in  $VIEW$ , thus all correct processes will eventually observe at least  $(n - t)$  values during their collect and exit the loop.
- The loop at lines 10-11. Let  $r_{min\_exit}$  be the smallest round after which a correct process  $p_i$  exits the loop. Thus, at any round  $r \leq r_{min\_exit}$ , each correct process writes a value in  $EST[r]$ . Because  $p_i$  exited the loop at round  $r_{min\_exit}$ , it observed strictly more than  $t$  finite values smaller than  $r_{min\_exit}$  in  $MAX[1..n]$ . Hence, any other correct process  $p_j$ , at round  $r \geq r_{min\_exit}$ , will then observe strictly more than  $t$  finite values smaller than  $r$  in  $MAX[1..n]$  and exit the loop.
- The loop at lines 08-14. As all the correct processes write a finite value in  $MAX[1..n]$ , it follows that each of them will eventually observe at least  $(n - t)t$  finite values during a collect at line 10. It will consequently execute the loop at lines 08-14 a finite number of times, which concludes the proof of the lemma.

□*Lemma 13*

As noticed in [18], for the decided values to be within  $\epsilon$  of each other, it is sufficient that they be within  $\epsilon/\sqrt{d}$  of each other in every coordinate. The reader can find in [18] the proof of Lemma 14 and Lemma 15 stated below. In [18], their proofs depend only on the guarantees given by the module `RBReceiveWitness()` devoted to the management of message reception. These guarantees are replaced here by the previous Lemmas 9, 10, and 11.

**Lemma 14** *For any correct process  $p_i$ , after  $R \geq \log_2(\sqrt{d}/\epsilon \cdot \text{SingleDimMaxDist}(\text{safe\_inits}_i))$  rounds, the values of the correct processes are within distance  $\epsilon/\sqrt{d}$  of each other at any dimension  $x$  [18, Lemma 4.10].*

**Lemma 15** *At any round, for any correct process  $p_i$ ,  $est_i$  is within the convex hull of the inputs of the correct processes [18, Lemma 4.17].*

Using the previous lemmas, we obtain the following theorem.

**Theorem 3** *The algorithm presented in Figure 4 is a correct implementation of multidimensional approximate agreement in the  $\mathcal{BAMP}_{n,t}[t < n/(d+2)]$  model.*

**Proof** The proof follows from Lemma 13 (termination), Lemmas 12 and 14 (agreement) and Lemma 15 (validity). □*Theorem 3*

## 6 Conclusion

This paper has first proposed a clean notion of atomic registers in the presence of Byzantine failures, and has then shown how to build it on top of a Byzantine asynchronous message-passing system. More precisely, an algorithm building a shared memory abstraction, made up of  $n$  single-writer/multi-reader atomic registers, has been presented and proved correct. The paper has also shown that  $t < n/3$  is a necessary and sufficient condition for such an algorithm.

The paper has then presented distributed algorithms suited to such a shared memory abstraction, which can cope with up to  $t$  Byzantine processes. The simplicity of these algorithms constitutes a strong motivation for the use of such a shared memory abstraction in the presence of Byzantine processes. As, for a lot of problems, algorithms are more difficult to design and prove correct in a message-passing system than in a shared memory system, the proposed abstraction should allow easier designs and proofs for other algorithms that have to cope with Byzantine failures.

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## A An Algorithm Implementing the Reliable Broadcast Abstraction

The r-broadcast algorithm presented in Figure 5 assumes  $t < n/3$ . It is a simple variant of Bracha's algorithm [5]. The operation broadcast  $XXX()$ , where  $XXX$  is a message tag, is a simple macro-operation standing for “**for**  $x \in \{1, \dots, n\}$  **do** send  $XXX()$  to  $p_x$  **end for**” (the sending order is arbitrary, which means that, if the sender crashes while executing this statement, an arbitrary subset of processes will receive the message).

Each process  $p_i$  manages a local array  $next_i[1..n]$ , where  $next_i[j]$  is the sequence number  $sn$  of the next application message (namely,  $APP(-, sn)$ ) from  $p_j$ , that  $p_i$  will process (line 03). Initially, for all  $i, j$ ,  $next_i[j] = 1$ .

When a process  $p_i$  invokes  $R\_broadcast\ APP(v, sn)$ , it broadcasts the message  $APP(v, sn)$  (line 01) where  $sn$  is its next sequence number. In its “server” role, the behavior of a process  $p_i$  is as follows.

- When a process  $p_i$  receives a message  $APP(v, sn)$  from a process  $p_j$  for the first time, it first waits until it can process this message (line 03). Process  $p_i$  then broadcasts a message  $ECHO(j, v, sn)$  (line 04). If the message just received is not the first message  $APP(-, sn)$ ,  $p_j$  is Byzantine and the message is discarded.



```

operation R_broadcast APP( $v, sn$ ):
(01) broadcast APP( $v, sn$ ).

when a message APP( $v, sn$ ) from  $p_j$  is received:
(02) if APP( $-, sn$ ) never received from  $p_j$ 
(03)   then wait ( $next_i[j] = sn$ );
(04)   broadcast ECHO( $j, v, sn$ )
(05) end if.

when a message ECHO( $j, v, sn$ ) is received:
(06) if ECHO( $j, v, sn$ ) received from strictly more than  $\frac{n+t}{2}$  different processes
(07)    $\wedge$  READY( $j, v, sn$ ) never sent
(08)   then broadcast READY( $j, v, sn$ )
(09) end if.

when a message READY( $j, v, sn$ ) is received:
(10) if READY( $j, v, sn$ ) received from at least  $t + 1$  different processes
(11)    $\wedge$  READY( $j, v, sn$ ) never sent
(12)   then broadcast READY( $j, v, sn$ )
(13) end if;
(14) if READY( $j, v, sn$ ) received from at least  $2t + 1$  different processes
(15)    $\wedge$  APP( $v, sn$ ) by  $p_j$  never R_delivered
(16)   then R_deliver APP( $v, sn$ ) from  $p_j$ ;
(17)    $next_i[j] \leftarrow next_i[j] + 1$ 
(18) end if.

```

Figure 5: Generalized Reliable Broadcast in  $\mathcal{BAMP}_{n,t}[t < n/3]$ , (code for process  $p_i$ )

- Then, when  $p_i$  has received the same message ECHO( $j, v, sn$ ) from “enough” processes (where “enough” means “strictly more than  $(n + t)/2$  different processes”), and has not yet broadcast a message READY( $j, v, sn$ ), it does so (lines 06-09).

The aim of (a) the messages ECHO( $j, v, sn$ ), and (b) the cardinality “strictly greater than  $(n + t)/2$  processes”, is to ensure that no two correct processes can r-deliver distinct messages from  $p_j$  (in the case where  $p_j$  is Byzantine). The aim of the messages READY( $j, v, sn$ ) is related to the liveness of the algorithm. Namely, its aim is to allow the r-delivery by the correct processes of the very same message APP( $v, sn$ ) from  $p_j$ , and this must always occur if  $p_j$  is correct. It is nevertheless possible that a message r-broadcast by a Byzantine process  $p_j$  be never r-delivered by the correct processes.

- Finally, when  $p_i$  has received the message READY( $j, v, sn$ ) from  $(t + 1)$  different processes, it broadcasts the same message READY( $j, v, sn$ ), it not yet done. This is required to ensure the RB-termination property. If  $p_i$  has received “enough” messages READY( $j, v, sn$ ) (as before “enough” means “from strictly more than  $(n + t)/2$  different processes”), it r-delivers the message APP( $v, sn$ ) r-broadcast by  $p_j$ .

Proofs that this algorithm satisfies the properties defining the reliable broadcast abstraction can be found in [5, 20].

## B Linearization Points Associated with SWMR Registers

To better understand and capture the behavior of the algorithm, we exhibit below a *linearization* order [10]. A linearization point is a point of the timeline (defined from an omniscient external observer point of view) at which a read or write operation issued by a correct process (or a write by a faulty process whose value has been read by a correct process) appears to have been instantaneously executed. For an operation issued by a correct process, this point has to lie between the beginning and the end of the corresponding execution.

Let us notice that a read by a Byzantine process may return any value. Hence, whatever the value it returns such a read invocation does not need to be linearized. We have the same for a write by a Byzantine process, whose value is never returned by a read from a correct process.

According to the Lemmas 4, 6, 7, and 8, we can define the following linearization points.

- An invocation of the operation  $REG[i].write(v)$  issued by a correct process  $p_i$  is linearized at the time  $\min(\tau_w, \tau_r)$  where
  - $\tau_w$  is the end of the wait statement executed by  $p_i$  at line 03, and
  - $\tau_r$  is the end of the wait statement of line 07 executed by the first correct process that returns a history including the value  $v$ .

- A write of a value  $v$  into  $REG[i]$  by a Byzantine process is linearized at the end of the wait statement of line 07 executed by the first correct process that returns the value  $v$ .
- Let  $op\_read[i, j, h]$  such that the last value of  $h$  is  $v$ . Let  $\tau_b$  and  $\tau_e$  be the times at which  $op\_read[i, j, h]$  starts and terminates, respectively. Let  $\tau_w$  be the linearization point at which  $v$  has been added to  $REG[j]$ .

Let us observe that  $\tau_w < \tau_e$  (otherwise, the history  $h$  returned by  $op\_read[i, j, h]$  could not end with  $v$ ). There are two cases.

- If  $\tau_w < \tau_b$ :  $op\_read[i, j, h]$  is linearized at time  $\tau_b$ .
- If  $\tau_b < \tau_w$ :  $op\_read[i, j, h]$  is linearized just after  $\tau_w$ . If two or more read invocations must be linearized just after the same time  $\tau$ , their linearization times are ordered according to their starting times.

## C Immediate Snapshot

The immediate snapshot object has been introduced in [4]. Such an object is a one-shot object that provides the processes with a single operation, denoted `immediate_snapshot()`. When a process invokes this operation, it proposes a value, and obtains an output which is a set of pairs similar to the one obtained in the one-shot write-snapshot.

**Specification in the crash failure model** When considering the asynchronous crash failure model, a one-shot immediate snapshot object is defined by the following properties. As in the previous section,  $output_i$  denotes the set of values output by a correct process  $p_i$ .

- Termination. The invocation of `immediate_snapshot()` by a correct process  $p_i$  terminates.
- Validity. If  $\langle i, v_i \rangle \in output_j$ , then  $p_i$  invoked `immediate_snapshot(v_i)`.
- Self-inclusion. If  $p_i$  returns from `immediate_snapshot()`, then  $\langle i, v_i \rangle \in output_i$ .
- Containment. If  $p_i$  and  $p_j$  return from `immediate_snapshot()`, then  $output_i \subseteq output_j$  or  $output_j \subseteq output_i$ .
- Immediacy.  $[(\langle i, v_i \rangle \in output_j) \wedge (\langle j, v_j \rangle \in output_i)] \Rightarrow (output_i = output_j)$ .

**Algorithm** The implementation of `immediate_snapshot()` in the crash failure model given in [4] is described in Figure 6 (the reader can consult Chapter 8 of [24], for a pedagogical presentation and a proof of this algorithm).

```

operation immediate_snapshot( $v_i$ ) is
(01)  $REG[j] \leftarrow v_i$ ;
(02) repeat  $LEVEL[i] \leftarrow LEVEL[i] - 1$ ;
(03)   for each  $j \in \{1, \dots, n\}$  do  $level_i[j] \leftarrow LEVEL[j]$  end for;
(04)    $view_i \leftarrow \{x \mid level_i[x] \leq level_i[i]\}$ 
(05) until  $(|view_i| \geq level_i[i])$  end repeat;
(06) let  $output_i = \{ \langle x, REG[x] \rangle : x \in view_i \}$ ;
(07) return( $output_i$ ).

```

Figure 6: Immediate snapshot algorithm (code for  $p_i$ ) [4]

The algorithm uses two arrays of SWMR atomic registers denoted  $REG[1..n]$  and  $LEVEL[1..n]$ , initialized to  $[\perp, \dots, \perp]$  and  $[n + 1, \dots, n + 1]$ , respectively. Only  $p_i$  can write  $REG[i]$  and  $LEVEL[i]$ .

A process  $p_i$  first writes its value in  $REG[i]$  (line 01). The array  $LEVEL[1..n]$  can be thought of as a ladder, where initially each process  $p_i$  is at the top of the ladder, namely, at level  $(n + 1)$ . Then  $p_i$  descends the ladder (line 02), one step after the other, according to predefined rules, until it stops at some level (or crashes). While descending the ladder,  $p_i$  registers its current position in the ladder in the atomic register  $LEVEL[i]$ . After it has stepped down from one ladder level to the next one,  $p_i$  computes a local view (denoted  $view_i$ ) of the progress of the other processes in their descent of the ladder. That view contains the processes  $p_x$  seen by  $p_i$  at the same or a lower ladder level (i.e., such that  $level_i[x] \leq LEVEL[i]$ , line 04). Then, if the current level  $\ell$  of  $p_i$  is such that  $p_i$  sees at least  $\ell$  processes in its view (i.e., processes that are at its level or a lower level),  $p_i$  stops at the level  $\ell$  of the ladder. Finally,  $p_i$  computes  $output_i$ , which is a set of pairs  $\langle x, v_x \rangle$  determined from the values of its last view  $view_i$ . This concise algorithm is such that the sets  $view_i$  of the processes that terminate the algorithm, satisfy the following main property: if  $|view_i| = \ell$ ,  $p_i$  stopped at the level  $\ell$ , and there are  $\ell$  processes whose current level is  $\leq \ell$ . From this property, follow the validity, self-inclusion, containment and immediacy properties that define the one-shot immediate snapshot object.

**From the crash failure model to the Byzantine failure model** Interestingly, the previous algorithm still works when considering that the arrays of SWMR atomic registers  $REG[1..n]$  and  $LEVEL[1..n]$  are implemented in  $\mathcal{BAMP}_{n,t}[t < n/3]$  with the construction of Figure 1.

In this case, the previous specification and line 03 have to be slightly modified as follows. Remembering that  $\mathcal{C}$  denote the set of correct processes, let  $c(output_i) = output_i \cap \mathcal{C}$  (projection of  $output_i$  on the correct processes). The specification of immediate snapshot in the Byzantine failure model is defined by the following properties.

- Termination. The invocation of `immediate_snapshot()` by a correct process  $p_i$  terminates.
- Validity. If  $p_i$  and  $p_j$  are correct and  $\langle i, v_i \rangle \in output_j$ , then  $p_i$  invoked `immediate_snapshot(v_i)`.
- Self-inclusion. If  $p_i$  is correct and invokes `immediate_snapshot()`, then  $\langle i, v_i \rangle \in output_i$ .
- Containment. If  $p_i$  and  $p_j$  are correct and invoke `immediate_snapshot()`, then  $c(output_i) \subseteq c(output_j)$  or  $c(output_j) \subseteq c(output_i)$ .
- Immediacy. If  $p_i$  and  $p_j$  are correct,  $[(\langle i, v_i \rangle \in output_j) \wedge (\langle j, v_j \rangle \in output_i)] \Rightarrow (c(output_i) = c(output_j))$ .

As far as line 03 is concerned, we have the following. If  $p_j$  is Byzantine,  $LEVEL[j]$  can contain an arbitrary value. Hence, instead of considering the content of  $LEVEL[j]$ ,  $p_i$  considers the number of times that  $p_j$  has written this register. Due to the algorithm of Figure 1, this number is equal to  $|LEVEL[j].read()|$  (the length of the sequence returned by  $LEVEL[j].read()$ ). It follows that  $p_i$  has to assign the value  $(n + 1) - |LEVEL[j].read()|$  to  $level_i[j]$ . It is easy to see that this quantity is the value of  $LEVEL[j]$  when only crash failures are committed.